Local center cutting plane methods (LCCPM)

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May 17, 2023, **Column Generation 2023**

Linear programming

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\begin{aligned}
(LP) \quad z^{LP} = \min_{x} \quad & c^{\top} x & (1)\\
\text{s.t.:} \quad & Ax = b & (2)\\
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where $c \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$ (has a very large number of columns). The dual variable vector associated with constraint set (2) is denoted by π.

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- Large-scale optimization problems from industrial applications, especially in the field of transportation, are often solved by the method of **Column Generation (CG)** (Desrosiers et al., 1984; Barnhart et al., 1998; Desaulniers et al., 2002).

Standard Column Generation (SCG) scheme

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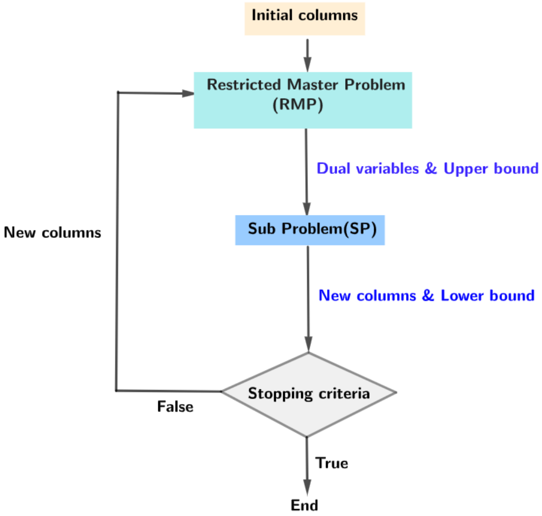

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- **Primal-dual interior point method**
  - → Pros:
    - ✓ Get stable dual solution.
    - ✓ No degeneracy issues.
  - → Cons:
    - ✓ Highly fractional solutions which have a negative impact on the branching process.
    - ✓ Crossover is exponential for large problems.

Cutting plane methods

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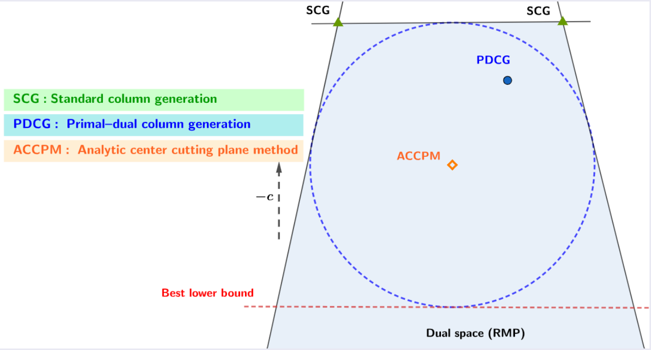
- Several methods have been proposed to improve the convergence of the Kelley method, the most popular ones are known as **center methods**.
- **Global vs local center method? unified framework?**



Center methods

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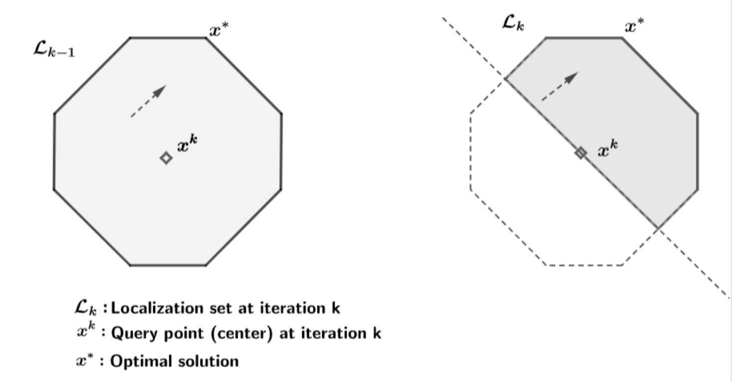

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1. The Elzinga-Moore method (Elzinga and Moore, 1975). The query point x^{k+1} is chosen as the **Chebyshev center** of $\mathcal{L}_k$, i.e., the center of the largest Euclidean ball that lies in $\mathcal{L}_k$ (simple linear program).

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3. The center of gravity method (Levin, 1965). The query point x^{k+1} is chosen as the **center of gravity** of $\mathcal{L}_k$.
4. The volumetric center method (Vaidya, 1989).The query point x^{k+1} is chosen as the **volumetric center**, i.e., the point that minimizes the determinant of the Hessian of the logarithmic barrier of $\mathcal{L}_k$.

Method	Analytical complexity	Issues
Kelley	$\frac{1}{\epsilon^n}$	✓ convergence rate is disastrous (worst-case).
Center of gravity	$nln(\frac{1}{\epsilon})$	✓ not practical.
Volumetric center	$nln(\frac{1}{\epsilon})$	✓ not practical.
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The big issue : not guarantee a "significant" improvement of the (primal) objective value at each iteration (or even in a fixed number of iterations).

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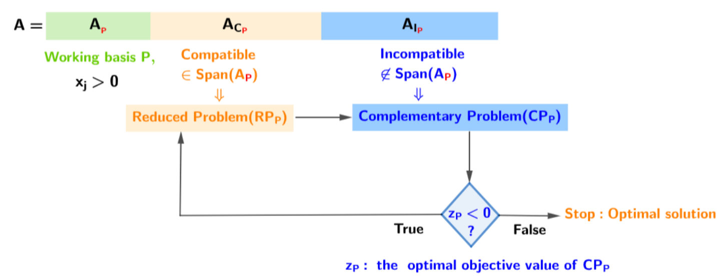

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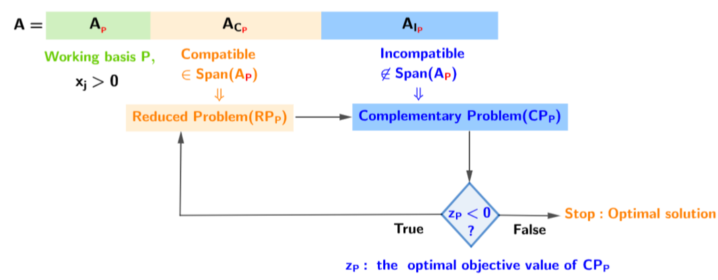

- **A variant of IPS, called Polynomial IPS** (Emine et al., 2021), can ensure a significant improvement of the objective function at each iteration and finds an $\epsilon-$approximation of the optimal solution in a polynomial number of iterations.
- Is IPS a center method?

Preliminaries

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- **Complementary problem (IPS)**

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\begin{aligned}
(CP_P) \quad z_P = \max_{\pi, y} \quad & y && (4)\\
\text{s.t.:} \quad & c_j - \pi^T A_j = 0 \quad \forall j \in P && (5)\\
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$$c_j - \pi^T A_j \geq y \quad \forall j \in I_P \tag{6}$$

- **We introduce the local dual polyhedron approximation**

$$\mathcal{D} := \{\pi \in \mathbb{R}^m \mid A_P^\top \pi = c_P \text{ and } A_{I_P}^\top \pi \leq c_{I_P}\}$$

- If π satisfies (5), any compatible column will have non negative reduced cost (Elhallaoui et al. (2011)).

  $\Rightarrow$ The set $\mathcal{D}$ is a subset of feasible dual solutions.

Local center cutting plane method (LCCPM): motivation

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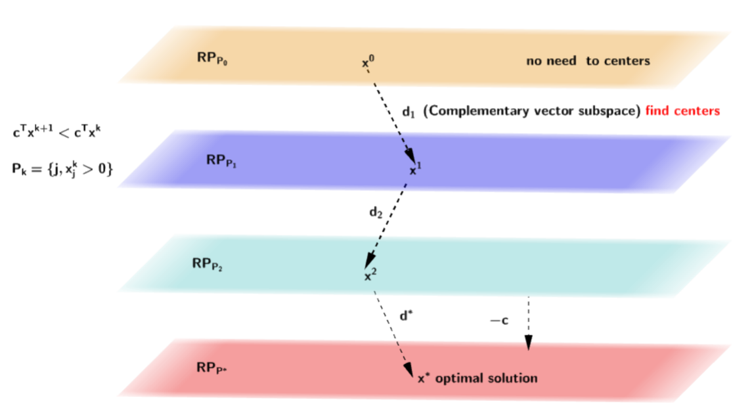
- **Geometrical insight,**



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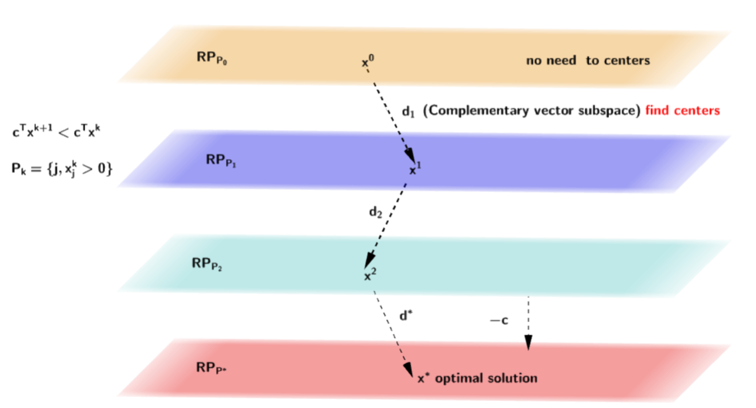

- **Advantages**: primal exact approach, we reduce the number of calls to the procedure that finds centers, avoids degeneracy.

Center Methods Unified Framework

- Mathematically speaking, center methods solve the following family of local center programs:

$$\max_{\pi,s} \quad g(s) \tag{7}$$
$$\text{s.t.:} \quad A_q^\top \pi + F(s) = c_q, \tag{8}$$
$$\pi \in \mathbb{S}, \tag{9}$$
$$s_j > 0, \quad j \in \{1, \ldots, q\} \tag{10}$$

where the submatrix A_q is composed of q columns of A, c_q is the subvector of c composed q dimension, the function $g : \mathbb{R}^q \to \mathbb{R}$ is concave in s, $F : \mathbb{R}^q \to \mathbb{R}^q$ is linear in s, and the set $\mathbb{S}$ is a polyhedron defined by some linear equalities.

Some functions g and F

- Analytic center : $g(s) = \sum_{j=1}^{n} \ln(s_j)$ and $F(s) = s$.
- Chebyshev center: $g(s) = \min_{j \in \{1,\ldots,n\}} s_j$ and $F_j(s) = \| A_j \|_* s_j$ ([1]).
- Harmonic center : $g(s) = \frac{n}{\sum_{j=1}^{n} \frac{1}{s_j}}$ (or $-\sum_{j=1}^{n} \frac{1}{s_j}$) et $F(s) = s$ (new center).

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Proposition

- If $q = n$ and $\mathbb{S} = \mathbb{R}^m$, we find the classical center methods.
- If $q = |\mathcal{I}_P|$ and $\mathbb{S} = \{\pi \in \mathbb{R}^m \mid A_P^\top \pi = c_P\}$, **we find the local center cutting plane methods: local ACCPM, local Chebyshev, ...**

[1] $\| A_j \|_*$: any norm of vector A_j

Duality of the local center problem

- **For F(s) = s**, the Lagrangian dual of the local center problem is:

$$\textit{reduced cost} \rightarrow \min_{v,w} \quad \tilde{c}_{I_P}^\top v + w \tag{11}$$

$$\textit{compatibility constraints} \rightarrow \text{s.t.:} \quad (MA_{I_P})v = 0, \tag{12}$$

$$\textit{normalization constraint} \rightarrow \quad -f(v) - \frac{w}{|I_P|} \leq 1 \tag{13}$$

où $f(v) = \min_{s>0} \frac{(-g(s)+v^T s+|I_P|)}{|I_P|}$.

Lemma

1. *If* $g(s) = \sum_{j\in\mathcal{I}_P} \ln(s_j)$ *then* $f(v) = \sum_{j\in\mathcal{I}_\mathcal{P}} \frac{\ln(v_j)}{|I_P|}$.
2. *If* $g(s) = -\sum_{j\in\mathcal{I}_P} \frac{1}{s_j}$ *then* $f(v) = 2\sum_{j\in\mathcal{I}_P} \frac{\sqrt{v_j}}{|I_P|} - 1$.

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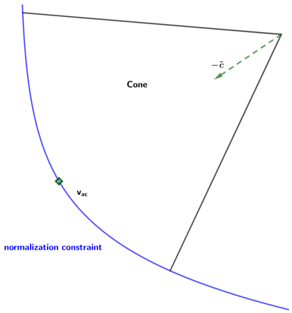

local analytic center
$\sum_{j \in I_P} \frac{-\ln(v_j)}{|I_P|} - \frac{w}{|I_P|} \leq 1$

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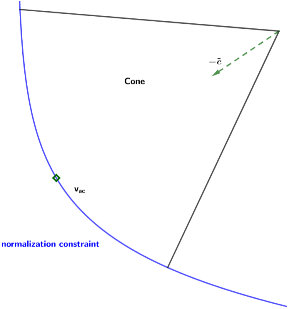

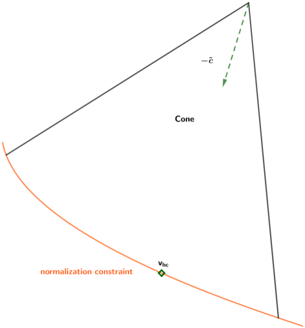

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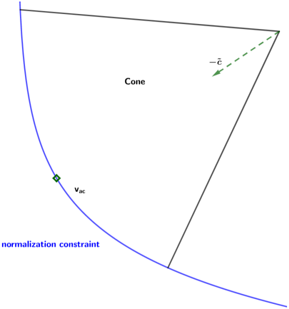

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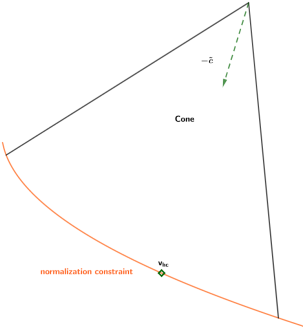

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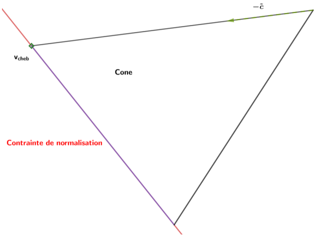
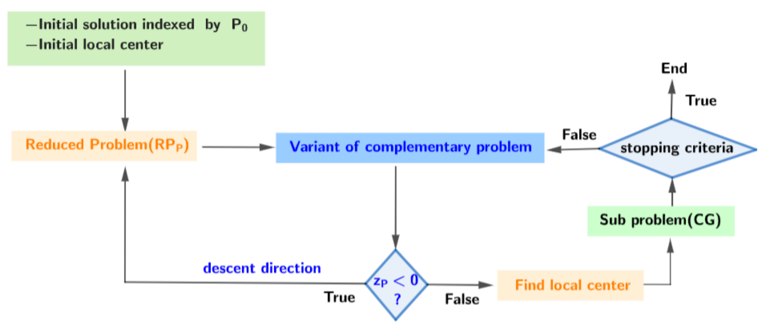


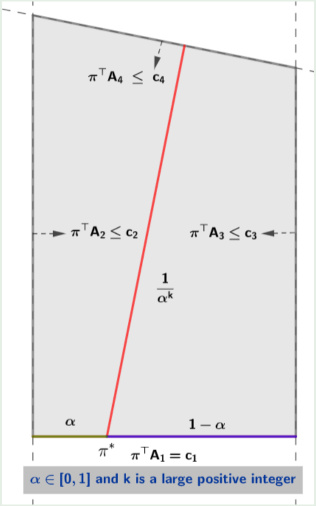
local Chebyshev center

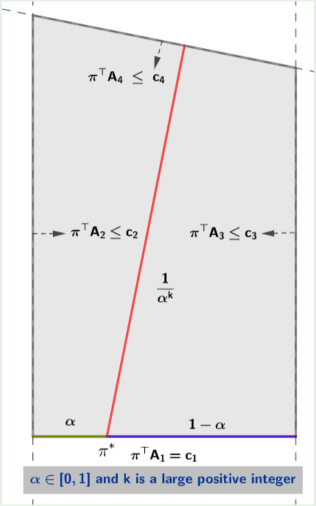
$$\sum_{j\in I_P} \| A_j \|_* v_j \leq 1$$

Generic local center cutting plane method

- Local center cutting plane method (LCCPM) = a kind of IPS with the appropriate complementary problem

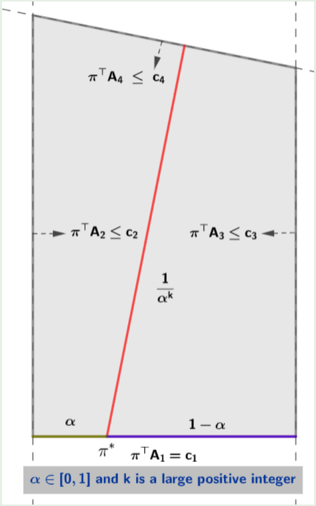



$\pi^\top A_4 \le c_4$
$\pi^\top A_2 \le c_2$
$\pi^\top A_3 \le c_3$
$\frac{1}{\alpha^k}$
α
$1-\alpha$
π^*
$\pi^\top A_1 = c_1$
$\alpha \in [0,1]$ and k is a large positive integer



→ Local analytic center α_{ac}:

$$\max_{\alpha} \quad \ln(\alpha) + \ln(1-\alpha) + \ln\left(\frac{1}{\alpha^k}\right)$$
$$\text{s.t.:} \quad 1 > \alpha > 0,$$



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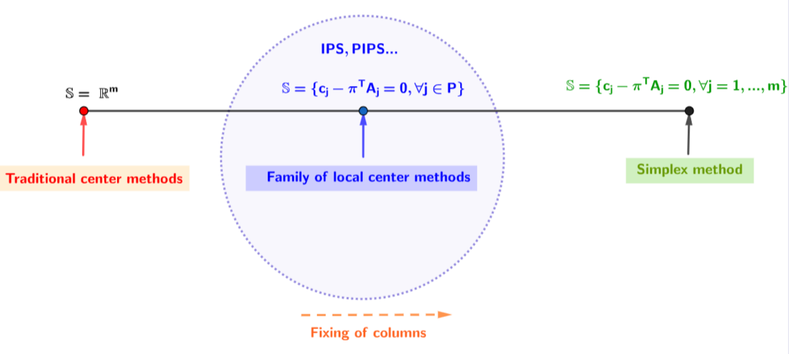
→ Local harmonic center α_{hc}:

$$\min_{\alpha} \quad \frac{1}{\alpha} + \frac{1}{1-\alpha} + \alpha^k$$
$$\text{s.t.:} \quad 1 > \alpha > 0,$$

→ Local Chebyshev center α_{cheb}:

$$\max r$$
$$\text{s.t.:} \quad \alpha \geq r, 1-\alpha \geq r, \frac{1}{\alpha^k} \geq r,$$
$$1 > \alpha > 0, r \geq 0,$$

→ $\frac{\frac{1}{2}-\alpha_{ac}}{\frac{1}{2}-\alpha_{hc}} = \frac{(1-2\alpha_{ac})}{\alpha_{hc}^{k-1}\alpha_{hc}^2(1-\alpha_{hc})^2} \gg 1$


IPS, PIPS...
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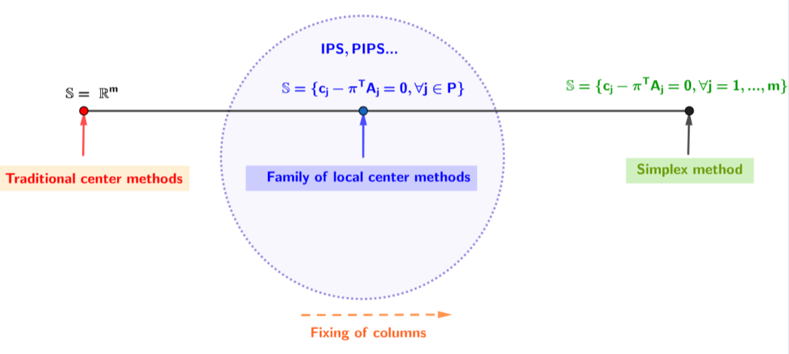
Traditional center methods
Family of local center methods
Simplex method
Fixing of columns



Classical center methods	Local center methods
localization set	local dual polyhedron approximation
the center is calculated at each iteration	when there is a need
involves all columns	some incompatible columns
good complexities	good enough complexities!

Research trends

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  - $\rightarrow$ **Idea : A variant of LCCPM based on PIPS decomposition instead of IPS.**

Keep calm and optimize!

References I

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