An approach based on Column Generation for solving routing problems with fractional objective function

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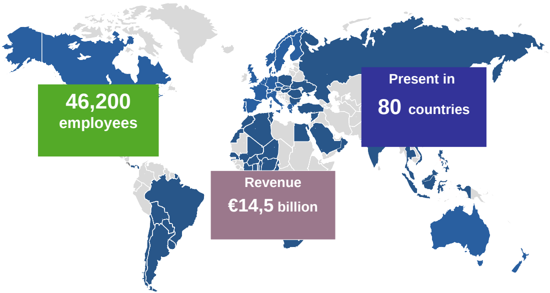
Column Generation 2016, Búzios, Brazil

Outline

Air Liquide

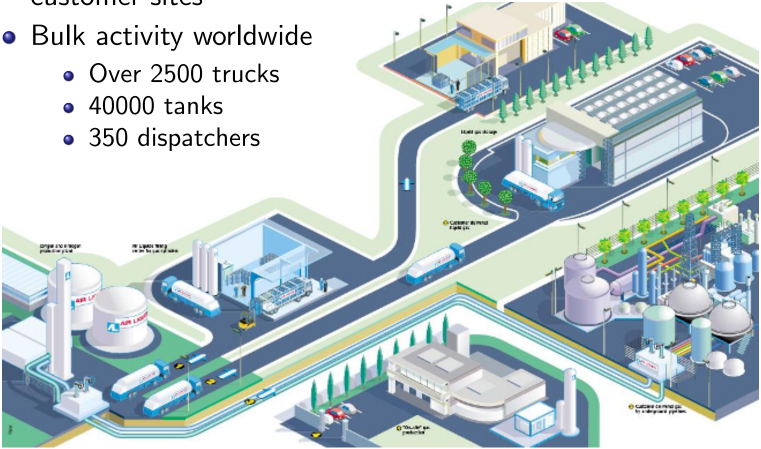
This work arises from the collaboration with Air Liquide.

- Air Liquide is a French multinational company which supplies industrial gases and services to various industries including medical, chemical and electronic manufacturers.
- Founded in 1902, it is world leader in gases for industry, health and the environment and has operations in over 80 countries.



Air Liquide

- The application studied concerns the distribution of a bulk gas from several productions plants to customers.
- Bulk Distribution: Trucks deliver liquefied gases to Tanks at customer sites
- Bulk activity worldwide
  - Over 2500 trucks
  - 40000 tanks
  - 350 dispatchers



Inventory Routing Problem

L.C. Coelho , J.F. Cordeau, G. Laporte [2012]

Inventory Routing Problem (IRP) can be described as the combination of vehicle routing and inventory management problems, in which a supplier has to deliver products to a number of geographically dispersed customers, subject to side constraints. It provides integrated logistics solutions by simultaneously optimizing: inventory management, vehicle routing and delivery scheduling.

Main variants of IRP

Classification of the IRP

Criteria	Possible Options		
Demand	Deterministic	Stochastic	**Dynamic**
Time horizon	**Finite**	Infinite	
Structure	One-to-one	One-to-many	**Many-to-many**
Routing	Direct	Multiple	**Continuous**
Inventory policy	Maxi. level (ML)	Order-up-to-level (OU)	
Inventory decision	Lost sales	Back-order	**Non-negative**
Fleet composition	Homogeneous	**Hererogeneous**	
Fleet size	Single	**Multiple**	

Andersson et al. [2010], Cohelo et. al. [2013]

Brief state-of-the-art for gas-distribution IRP

- **[Ba] Bell, Dalberto, Fisher, Greenfield, Jaikumar, Kedia, Mack, Prutzman 1983**: *First definition of the problem* - **Air Product**
- [Ga] Golden, Assad, Dahl **1984**
- [DB] Dror and Ball **1987**
- [AF] Anily, Federgruen **1990**
- [WL] Webb, Larson **1995**
- [CC] Campbell, Clarke, Kleywegt, and Savelsbergh **1997**
- [C] Christiansen **1999**
- [Qa] Qu, Bookbinder, Iyogun **1999**
- [Ma] Malépart, Boctor, Renaud, Labillois **2002**
- **[Ca] Campbell, Clarke, Savelsbergh 2004**: *Decomposed approach* - **Praxair**
- [GF] Gaur, Fisher **2004**
- [SS] Song and Savelsbergh **2007**
- **[AB] C. Archetti, L. Bertazzi, Gilbert Laporte, M.G. Speranza 2007**
- [Yu] Yugang, Haozun, and Feng **2008**
- **[TB] Thierry Benoist, Frederic Gardi, Antoine Jeanjean 2008** *Integrated approach, local search* - **Air Liquide**
- [An] Anderson et al **(2010)**
- [LG] Leandro C. Coelho, Gilbert Laporte **2012**
- [GR] **Guy Desaulniers, J.G. Rakke, L.C. Coelho 2014**
- [YJ] **Yachao Dong, Jose M. Pinto et al. 2014**: *MIP (Mix Integer Program) model for a real inventory routing* - **Praxair**
- **[AD] C. Archetti, G. Desaulniers, M.G. Speranza 2015**

Prevalent approaches used in the literature

- Heuristic approaches:
  - Local search [TB]
  - Decomposition [Ga,Qa,Ca,Ma] $\Rightarrow$ first inventory, then routing.
- Exact approaches:
  - Compact models [YJ]
  - Branch-and-cut [LG]
  - Branch-and-price [C]
  - Branch-and-cut-and-price [GR]

All the exact approaches are based on mathematical optimization.

State-of-the-art formulations

Archetti et al. [AB]

- Branch-and-cut approach.
- VMI management of the inventory.
- Time horizon divided in days.
- Each customer can be served only once per day (no split).
- No scheduling.
- Single supplier.
- Order up-to-level policy.
- Single vehicle.

State-of-the-art formulations

Desaulniers et al. [GR]

- Branch-and-price approach.
- VMI management of the inventory.
- Time horizon divided in days.
- Each customer can be served only once per day (no split).
- No scheduling.
- Single supplier.
- Max level policy (more flexible).
- Multi-vehicles.
- Homogeneous fleet.

State-of-the-art formulations

Archetti, Desaulniers and Speranza [AD]

- Logistic ratio as objective function
- Dinkelback for dealing with fractional objective functions
- Time horizon divided in days.
- Each customer can be served only once per day (no split).
- No scheduling.
- Single supplier.
- Multi-vehicles.
- Homogeneous fleet.

Overview

State of the art		Our work		
Archetti et al. [AB] 2007 Desaulniers et al. [DR] 2014 Archetti et al. [AD] 2016	⇒ *+subperiods* *+Surrogate log. ratio* *+het. fleet*	Simplified Model (SM) *-Daily shifts* *-No split delivery* *-No layover* *-Restr. start. times* *-No scheduling* *-No q.ty conservation* *-Fixed periods* *-Single-base*	⇒ *+scheduling* *+q.ty conservation* *+flexible periods* *+multi-base*	Full Model (FM) *-single product* *-Only VMI* *-No pref. starting time* *-No hitching costs* *-No missed orders* *-No runouts* *-No preferences*

- We present two generalizations of the models presented in the literature.
- The Full model is the closest to the Air Liquide model.

Aspect not taken into account in SM I

Daily shifts

The time horizon is divided into periods of the lenght of one day.

No split delivery

Each customer can be visited only once per subperiod.

No layover

No layover is allowed.

Restricted starting time

Each shift must start on a given time instant.

No scheduling

No distinction is made between driver, trailers and tractors.

Aspect not taken into account in SM II

No quantity conservation

A vehicle is supposed to return empty to the base.

Fixed periods

Each vehicle can perform only one shift per day and a shift can not cover more than one day.

Single Base

All the vehicles start from the same base.

Aspect not taken into account in SM and FM

Single product

All the products belong to the sale type.

Only VMI

No call-in customers are considered.

No missed orders, No runouts, No preferences

We optimize only the (Surrogate) Logistic Ratio.

No hitching costs

The hitching cost is neglected.

Outline

Simplified Model

Introduction of subperiods

The model presented adds an additional level of granularity in order to take into account the VMI consumption more in detail.

- Each period is subdivided into subperiod.
- Each vehicle can not visit a node more than once in the same subperiod.

Logistic Ratio

The Simplfied Model is capable to minimize both the total costs, the Logistic Ratio or the Surrogate Logistic Ratio.

Heterogeneous fleet

The model allows to use vehicles that are not identical , i.e. with different capacities, time windows, costs, . . .

Outline

Problem Data

- $G = (N_{fc}, E)$ is a complete and undirected graph.
- where $N_{fc} = N_c \cup N_f$ and $\bar{N} = N_{fc} \cup \{0\} \cup \{n+1\}$
- c_{ij} are the routing cost
- T is the set of time periods in the planning horizon.
- SP is the set of subperiods in a period.
- C_i is the inventory capacity of customer/source $i \in N_{fc}$,
- r_i^s units of product in period s produced/consumed by each customer/source $i \in N_{fc}$.
- I_i^0 is the initial Inventory $r_i^0 \leq I_i^0 \leq C_i$.
- A set of M vehicles each one of capacity Q_k are available at the depot, and each vehicle can be used at most once per period to perform a route.

Model description

Variables used

The model proposed uses the following set of variables:

- $\xi \in \{0, 1\}$, one fore every feasible shift.
- $z \in \{0, 1\}$, keeping track of when a shift visit a given node.
- $q \geq 0$, representing the quantity delivered or picked up.
- $I \geq 0$, representing the inventory levels.
- $y \geq 0$, representing the flow of the quantities delivered.

Model description

InventoryConstraints(I, q, z)

Q.ty cons.	$I_i^s = I_i^{s-1} + \sum_{k \in M} q_{ki}^s - r_i^s$	$(\forall i \in N_{fc}, \forall s \in SP),$	(1)
Capacity	$I_i^{s-1} + \sum_{k \in M} q_{ki}^s \leq C_i$	$(\forall i \in N_c, \forall s \in SP),$	(2)
	$I_i^{s-1} + r_i^s \leq C_i$	$(\forall i \in N_f, \forall s \in SP),$	(3)
q-z link	$q_{ki}^s \leq \min\{C_i, Q_k\} z_{ki}^s$	$(\forall i \in N_c, \forall k \in M, \forall s \in SP),$	(4)
	$-q_{ki}^s \leq \min\{C_i, Q_k\} z_{ki}^s$	$(\forall i \in N_f, \forall k \in M, \forall s \in SP),$	(5)
Min q.ty	$I_i^{s-1} + \sum_{k \in M} q_{ki}^s \geq 0$	$(\forall i \in N_f, \forall s \in SP),$	(6)
	$z_{ki}^s \in \{0, 1\}$	$(\forall i \in N_{fc}, \forall k \in M, \forall s \in SP),$	(7)
	$q_{ki}^s \geq 0$	$(\forall i \in N_c, \forall k \in M, \forall s \in SP),$	(8)
	$q_{ki}^s \leq 0$	$(\forall i \in N_f, \forall k \in M, \forall s \in SP),$	(9)
	$I_i^s \geq 0$	$(\forall i \in \overline{N}, \forall s \in SP),$	(10)

Model description

$$\min \quad f(\boldsymbol{\xi}, \mathbf{q}) \tag{11}$$

$$\text{s.t.} \quad \texttt{InventoryConstraints}(I, q, z)$$

$$\texttt{node visit} \quad \sum_{k \in M} \sum_{s \in SP_t} z_{ki}^s \leq 1 \qquad (\forall i \in N_c, \forall t \in T), \tag{12}$$

$$\xi - z \ \texttt{link} \quad \sum_{\ell \in R_{is}} \xi_{k\ell}^t = z_{ki}^s \qquad (\forall i \in N_{fc}, \forall k \in M, \forall t \in T, \forall s \in SP_t), \tag{13}$$

$$\texttt{one shift per veh.} \quad \sum_{\ell \in R} \xi_{k\ell}^t \leq 1 \qquad (\forall k \in M, \forall t \in T), \tag{14}$$

$$\texttt{Capacity} \quad (y_{ij}^{kt} + y_{ji}^{kt})/Q_k = \sum_{\ell \in R} a_\ell \xi_{k\ell}^t \qquad (\forall e \in \overline{E}, \forall k \in M, \forall t \in T), \tag{15}$$

$$\texttt{Flow Cons.} \quad \sum_{j \in N} (y_{ij}^{kt} - y_{ji}^{kt}) = -2 \sum_{s \in SP_t} q_{ki}^s \qquad (\forall i \in N_{fc}, \forall k \in M, \forall t \in T), \tag{16}$$

$$\xi_{k\ell}^t \in \{0, 1\} \qquad (\forall \ell \in R, \forall k \in M, \forall t \in T), \tag{17}$$

$$y_{ij}^{kt} \geq 0 \qquad (\forall i, j \in \overline{N}, i \neq j, \forall k \in M, \forall t \in T). \tag{18}$$

Valid inequalities

The following inequalities are valid and effective:

$$\sum_{\ell \in R_i} \xi_\ell \geq f_i, \quad \forall i \in N_c,$$

where $f_i = \lceil \frac{w_i}{\min\{Q, C_i\}} \rceil$ and $w_i = \sum_{t \in T} \sum_{s \in SP_t} r_i^s - I_i^0$

The objective function I

Total cost

In the literature most of IRP models have as objective function the minimization of the total distribution cost (*Anderson et al 2010, Coelho et al 2012*)

Logistic ratio

The Logistic Ratio(LR) represents the cost per kilogram of the total delivered quantities of products over a given time frame. It allows monitoring of overall efficiency of the logistical distribution process

The objective function II

Surrogate logistic ratio

The Surrogate Logistic Ratio(SLR) can be defined as the global extra cost per kilogram of delivered product, compared to a lower bound of the logistic ratio.

$$SLR = \frac{Total\ cost - Lower\ bound\ cost}{Total\ delivered\ quantity}$$

How to deal with fraction objective function I

$$\min \frac{c^\top x}{d^\top x}$$
$$Ax \geq b$$

- Basic Idea (Charnes and Cooper, 1962): introducing the following variables substitution:

$$\tau = \frac{1}{d^\top x} \qquad \bar{x} = x\tau = \frac{x}{d^\top x}$$

- (valid only if $d^\top x > 0$)

How to deal with fraction objective function II

- Objective function

$$\frac{c^\top x}{d^\top x} \quad \Rightarrow \quad c^\top \bar{x}$$

- constraints:

$$Ax \geq b \quad \Rightarrow \quad \frac{1}{d^\top x} Ax \geq \frac{1}{d^\top x} b \quad \Rightarrow \quad A\bar{x} \geq b\tau$$

- Additional constraint

$$d^\top x = \frac{1}{\tau} \quad \Rightarrow \quad \frac{1}{d^\top x} d^\top x = \frac{1}{\tau d^\top x} \quad \Rightarrow \quad d^\top \bar{x} = 1$$

The use of the variable substitution leads to values of $\bar{x}$ close to zero.

How to deal with fraction objective function III

- the tollerance of the LP solver is not enough to deal with such small values. $\Rightarrow$ A scale factor is used: $\bar{x} = k\frac{x}{d^\top x}$ with $k = 10^4 \div 10^5$.
- Scaling the normalization of the variables allows to obtain the correct solution but it still slows down the averall computational time.
- In practice, a master problem that minimizes the overall costs is $\approx$ 10 times fater than a master that minimizes the Logistic Ratio.

How to deal with fraction objective function IV

Optimizing the logistic ratio corresponds to optimizing the following function:

$$\min \quad \frac{\sum_{l \in \mathcal{R}} c_l \xi_l}{\sum_{s \in S} \sum_{i \in N} q_i^s}$$

If we consider the continuous relaxation of the model proposed we obtain an alternative linear reformulation by imposing the following variables substitution:

$$\bar{\xi}_l = \frac{\xi_l}{\sum_{s \in S} \sum_{i \in N} q_i^s} \quad \bar{I}_i^s = \frac{I_i^s}{\sum_{s \in S} \sum_{i \in N} q_i^s} \quad \bar{q}_i^s = \frac{q_i^s}{\sum_{s \in S} \sum_{i \in N} q_i^s}$$

$$\tau = \frac{1}{\sum_{s \in S} \sum_{i \in N} q_i^s}$$

Model linearized

InventoryConstraints($\bar{I}, \bar{q}, \bar{z}$)

$$\text{Q.ty cons.} \quad \bar{I}_i^s = \bar{I}_i^{s-1} + \sum_{k \in M} \bar{q}_{ki}^s - r_i^s \tau \quad (\forall i \in N_{fc}, \forall s \in SP), \tag{19}$$

$$\text{Capacity} \quad \bar{I}_i^{s-1} + \sum_{k \in M} \bar{q}_{ki}^s \leq C_i \tau \quad (\forall i \in N_c, \forall s \in SP), \tag{20}$$

$$\bar{I}_i^{s-1} + r_i^s \leq C_i \tau \quad (\forall i \in N_f, \forall s \in SP), \tag{21}$$

$$q\text{-}z \text{ link} \quad \bar{q}_{ki}^s \leq \min\{C_i, Q_k\} \bar{z}_{ki}^s \quad (\forall i \in N_c, \forall k \in M, \forall s \in SP), \tag{22}$$

$$-\bar{q}_{ki}^s \leq \min\{C_i, Q_k\} \bar{z}_{ki}^s \quad (\forall i \in N_f, \forall k \in M, \forall s \in SP), \tag{23}$$

$$\text{Min q.ty} \quad \bar{I}_i^{s-1} + \sum_{k \in M} \bar{q}_{ki}^s \geq 0 \quad (\forall i \in N_f, \forall s \in SP), \tag{24}$$

$$\bar{z}_{ki}^s \in \{0, \tau\} \quad (\forall i \in N_{fc}, \forall k \in M, \forall s \in SP), \tag{25}$$

$$\bar{q}_{ki}^s \geq 0 \quad (\forall i \in N_c, \forall k \in M, \forall s \in SP), \tag{26}$$

$$\bar{q}_{ki}^s \leq 0 \quad (\forall i \in N_f, \forall k \in M, \forall s \in SP), \tag{27}$$

$$\bar{I}_i^s \geq 0 \quad (\forall i \in \overline{N}, \forall s \in SP), \tag{28}$$

Model linearized

$$\min \quad f(\bar{\boldsymbol{\xi}}, \bar{\mathbf{q}}) \tag{29}$$

$$\text{s.t.} \quad \texttt{InventoryConstraints}(\bar{I}, \bar{q}, \bar{z})$$

$$\texttt{node visit} \quad \sum_{k \in M} \sum_{s \in SP_t} \bar{z}^s{}_{ki} \leq 1\tau \quad (\forall i \in N_c, \forall t \in T), \tag{30}$$

$$\xi - z \texttt{ link} \quad \sum_{\ell \in R_{is}} \bar{\xi}^t_{k\ell} = \bar{z}^s_{ki} \quad (\forall i \in N_{fc}, \forall k \in M, \forall t \in T, \forall s \in SP_t), \tag{31}$$

$$\texttt{one shift per veh.} \quad \sum_{\ell \in R} \bar{\xi}^t_{k\ell} \leq 1\tau \quad (\forall k \in M, \forall t \in T), \tag{32}$$

$$\texttt{Capacity} \quad (\bar{y}^{kt}_{ij} + \bar{y}^{kt}_{ji})/Q_k = \sum_{\ell \in R} a_\ell \bar{\xi}^t_{k\ell} \quad (\forall e \in \overline{E}, \forall k \in M, \forall t \in T), \tag{33}$$

$$\texttt{Flow Cons.} \quad \sum_{j \in N} (\bar{y}^{kt}_{ij} - \bar{y}^{kt}_{ji}) = -2 \sum_{s \in SP_t} \bar{q}^s_{ki} \quad (\forall i \in N_{fc}, \forall k \in M, \forall t \in T), \tag{34}$$

$$\texttt{New constraint} \quad \sum_{s \in S} \sum_{i \in N} \bar{q}^s_{ki} = 1 \tag{35}$$

$$\bar{\xi}^t_{k\ell} \in \{0, \tau\} \quad (\forall \ell \in R, \forall k \in M, \forall t \in T), \tag{36}$$

$$\bar{y}^{kt}_{ij} \geq 0 \quad (\forall i, j \in \overline{N}, i \neq j, \forall k \in M, \forall t \in T). \tag{37}$$

Binary variables in the Linearized model

$$\xi_l \in \{0, \tau\}$$

- ξ and z variables can be either equal to 0 or to τ
- Continuous relaxation:

$$0 \leq \xi_l \leq \tau$$

- How to impose integrality:

$$\tau - M(1 - \bar{x}_l) \leq \bar{\xi}_l \leq M x_l$$

$$x_l \in \{0, 1\}$$

- An additional set of (exponentially many) variables is added.

Outline

Pricing I

Pricing for the simple model

- In the two proposed version (SM) and (FM), the pricing problem reduces to solve a series of Elementary Shortest Path Problems with Resource Constraints (SPPRC).
- In the simplified model, each node can be visited once per day and each shift spans the time horizon corresponding to exactly one day.
- Therefore: the decision concerning the quantity to delivered in each shift is taken in the master.

Pricing II

Princing for the full model

- In the full model, each node can be visited in several time steps by the same shift and a shift can span the time horizon for more than one time step.
- Therefore: the decision concerning the quantity to delivered in each shift is taken in the pricing.
- The pricing problem does not change if, instead of the total cost, the logistic ratio is minimized.

Outline

Instances and Computational results

- Instances corresponding to up to 75.
- Number of depots: 1.
- Number of sources: 1.
- Number of customers: 5,10,15.
- Number of vehicles: 1,. . . ,5.
- Time horizon: 3, 4, 5.

The instances come from [AB] (Archetti et al., 2007) and we compare with the optimal solutions reported in [AD] (Archetti et al., 2015).

Computational results comparing our lower bound with the optimal solution

nodes		5			10			15		
hor		3	4	5	3	4	5	3	4	5
veh	1	0,17	0,15	0,18	0,16	0,12	0,16	0,17	-	-
	2	0,12	0,10	0,13	0,16	0,13	0,17	0,20	-	-
	3	0,07	0,06	0,05	0,15	0,12	0,15	0,16	-	-
	4	0,04	0,06	0,05	0,12	0,09	0,12	0,14	-	-
	5	0,02	0,03	0,05	0,09	0,07	0,09	0,14	-	-

Table: Instances from the literature, LB vs OPT

Computational results comparing the lower bound with the upper bound obtained with the proposed approach

nodes		5			10			15		
hor		3	4	5	3	4	5	3	4	5
veh	1	0,25	0,22	0,24	0,27	0,22	0,31	0,37	0,28	0,38
	2	0,17	0,12	0,19	0,26	0,23	0,28	0,47	0,48	0,57
	3	0,09	0,13	0,07	0,27	0,24	0,27	0,48	0,52	0,49
	4	0,06	0,08	0,07	0,20	0,19	0,21	0,58	0,53	0,56
	5	0,03	0,03	0,04	0,16	0,13	0,15	0,52	0,71	0,85

Table: Instances from the literature, LB vs HEUR

Computational results comparing the lower bound with the upper bound on the AirLiquide instances

nodes		5			10			15		
hor		3	4	5	3	4	5	3	4	5
veh	1	0,16	0,13	0,13	0,10	0,10	0,18	0,17	0,09	0,18
	2	0,26	0,21	0,16	0,17	0,14	0,13	0,14	0,12	0,10
	3	0,27	0,21	0,16	0,15	0,14	0,12	0,15	0,13	0,10
	4	0,27	0,22	0,18	0,17	0,16	0,13	0,18	0,15	0,10
	5	0,27	0,22	0,17	0,17	0,16	0,12	0,17	0,14	0,10

Table: Air Liquide Instances, LB vs HEUR

Outline

Full Model

The Full Model (FM) is a generalization of the Simplified model.

Scheduling

FM allows to introduce the concept of driver, trailer and tractor.

Quantity conservation

A trailer can return not empty to the base and the quantity in the trailer can be reused in the next shift.

Flexible periods

- The Full Model does not have the concept of periods. A shift can start and finish at any time during the time horizon.
- This generalization changes significantly the difficulties of the pricing.
- Moreover, together with the assumption of quantity conservation makes all the shift strongly interconnected.

Heuristic based on precomputed columns I

The Full Model is extremely complex and the pricing problem is significantly more challenging:

Challenges in the Pricing Problem

- No restricted starting time.
- Quantity conservation is allowed.
- No daily shift periods.

All these aspects increase consistently the search space of the pricing routine and lead to the generation of similar shifts.

Heuristic based on precomputed columns II

An heuristic approach is used to produce feasible shifts

- We decided to use an Heuristic as black box to produce shifts.
- For each instance, several runs of tests are performed with different combination of seeds and optimization parameters.
- The set of columns produced is used heuristically in the Full Model.

- The heuristic is not able to improve the solution over the best solution provided by the heuristic.
- One possible explanation is the poor variability of the columns generated by the heuristic.
- Moreover, it is not easy to generate shift that can mix together, this is due to the strong interconnection among the shifts.

Outline

Conclusions

We presented a real-world application of Inventory Routing Problem

The problem differs from the problems already proposed in the literature

A new column generation approach has been proposed : several interesting methodological features are needed in order to solve the problem

Work in progress: heuristic pricing, exact pricing, computational results routing problems with fractional objective function.