Dantzig-Wolfe Reformulations for the Stable Set Problem

(and Possible Extensions to Related Problems)

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Integer Programming and Polyhedra

- original problem

$$\begin{array}{rrcll} \max & c^T x & & & \\ \text{s. t.} & a_i^T x & \leq & b_i & \forall i \in I \\ & x & \in & \mathbb{Z}^n \cap [L, U] & \end{array}$$

- $L, U \in \mathbb{Z}^n$ finite bounds on variables

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- $L, U \in \mathbb{Z}^n$ finite bounds on variables

- integer hull

$$P_{IP} := \operatorname{conv}\{x \in \mathbb{Z}^n \cap [L, U] : a_i^T x \leq b_i \ \forall i \in I\}$$

- set of feasible solutions to the LP relaxation

$$P_{LP} := \{x \in \mathbb{Q}^n \cap [L, U] : a_i^T x \leq b_i \ \forall i \in I\}$$

Dantzig-Wolfe Reformulation for IPs

- choose subset $I' \subseteq I$
- let $X(I') := \text{conv}\{x \in \mathbb{Z}^n \cap [L, U] : a_i^T x \leq b_i \quad \forall i \in I'\}$
- remark: $X(I) = P_{IP}$

- reformulate every $x \in X(I')$ as convex combination of extreme points of $X(I')$
- introduce one variable per extreme point

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- remark: $X(I) = P_{IP}$

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- introduce one variable per extreme point

- corresponds to convexification of constraints with index in I'

$\rightarrow$ Dantzig-Wolfe polytope:

$$P_{DW}(I') := \{x \in \mathbb{Q}^n \cap [L, U] : a_i^T x \leq b_i \; \forall i \in I \setminus I', \\ x \in X(I')\}$$

Strength of Reformulations

- inclusion relation

$$P_{IP} \subseteq P_{DW}(I') \subseteq P_{LP}$$

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$$\begin{array}{ccccc} P_{IP} & \subseteq & P_{DW}(I') & \subseteq & P_{LP} \\ \| & & & & \| \\ P_{DW}(I) & & & & P_{DW}(\emptyset) \end{array}$$

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- we want to investigate the strength of such reformulations
  - when is the reformulation weakest possible?
  - when is the reformulation strongest possible?
  - ...

Strength of Reformulations

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  - when is the reformulation weakest possible?
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- we focus on the stable set problem in this talk!

Stable Set Problem

- let $G = (V, E)$ be a graph with $n := |V|$ and weights $w \in \mathbb{Z}^n_{\geq 0}$
- $S \subseteq V$ is called stable set if no nodes of S are adjacent
- find stable set S^* with maximum weight
- maximum weight is called weighted stability number $\alpha_w(G)$

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- find stable set S^* with maximum weight
- maximum weight is called weighted stability number $\alpha_w(G)$
- IP formulation

$$
\begin{array}{llcll}
\max & \sum_{v \in V} w_v \cdot x_v & & & \\
\text{s. t.} & x_u + x_v & \leq & 1 & \forall \{u, v\} \in E \\
 & x & \in & \{0, 1\}^n &
\end{array}
$$

D-W Reformulation for the Stable Set Problem

- inclusion relation

$$P_{IP} \quad \subseteq \quad P_{DW}(I') \quad \subseteq \quad P_{LP}$$

D-W Reformulation for the Stable Set Problem

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$$\downarrow$$

$$\mathrm{STAB}(G)$$

- stable set polytope $\mathrm{STAB}(G)$

D-W Reformulation for the Stable Set Problem

- inclusion relation

$$\begin{array}{ccccc} P_{IP} & \subseteq & P_{DW}(I') & \subseteq & P_{LP} \\ \downarrow & & & & \downarrow \\ \mathrm{STAB}(G) & & & & \mathrm{FRAC}(G) \end{array}$$

- stable set polytope $\mathrm{STAB}(G)$
- fractional stable set polytope $\mathrm{FRAC}(G)$

D-W Reformulation for the Stable Set Problem

- inclusion relation

$$P_{IP} \subseteq P_{DW}(I') \subseteq P_{LP}$$

$P_{IP} \downarrow \text{STAB}(G)$

$P_{DW}(I') \downarrow \text{DW}(G, G')$

$P_{LP} \downarrow \text{FRAC}(G)$

- stable set polytope $\text{STAB}(G)$
- fractional stable set polytope $\text{FRAC}(G)$

- choose $E' \subseteq E$ and define $G' := (V, E')$

$$\text{DW}(G, G') := \{x \in [0,1]^n : x_u + x_v \leq 1 \ \forall \{u, v\} \in E \setminus E', \\ x \in \text{STAB}(G')\}$$

Bipartite Graphs

Theorem (Nemhauser and Trotter 1974)

$\mathrm{STAB}(G) = \mathrm{FRAC}(G)$ *iff* G *is bipartite.*

Corollary

If G *is bipartite, then for all* $E' \subseteq E$ *and* $G' = (V, E')$ *holds*

$$\mathrm{STAB}(G) = \mathrm{DW}(G, G') = \mathrm{FRAC}(G).$$

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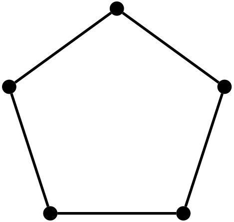
$$\mathrm{STAB}(G) = \mathrm{DW}(G, G') = \mathrm{FRAC}(G).$$

- G is bipartite iff G contains no odd cycle

Odd Cycles/Holes

- odd hole is odd cycle without chords (induced odd cycle)
- 3-cycles/3-cliques/triangles are considered holes in this talk!



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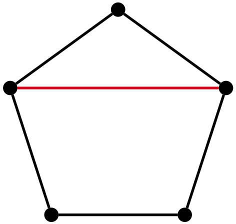

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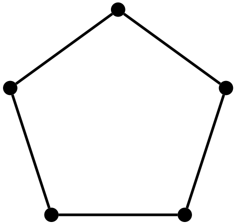

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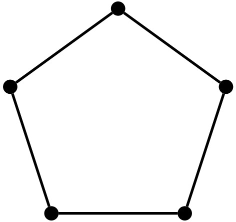

- odd cycle inequality for odd cycle C is valid for $\mathrm{STAB}(G)$

$$\sum_{v \in V(C)} x_v \leq \frac{|V(C)| - 1}{2}$$

Weakest Possible Reformulation

Theorem

$\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$ *iff* $\ldots$?

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Proof sketch "$\Leftarrow$":

$$\begin{aligned}\mathrm{DW}(G, G') = \{x \in [0,1]^n : \; & x_u + x_v \leq 1 \; \forall \{u, v\} \in E \backslash E', \\ & x \in \mathrm{STAB}(G')\} \\ \mathrm{FRAC}(G) = \{x \in [0,1]^n : \; & x_u + x_v \leq 1 \; \forall \{u, v\} \in E \backslash E', \\ & x_u + x_v \leq 1 \; \forall \{u, v\} \in E'\}\end{aligned}$$

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$$\begin{aligned} G' \text{ is bipartite} \quad & \Leftrightarrow \quad \mathrm{STAB}(G') = \mathrm{FRAC}(G') \\ & \Rightarrow \quad \mathrm{DW}(G, G') = \mathrm{FRAC}(G)\end{aligned}$$

Weakest Possible Reformulation

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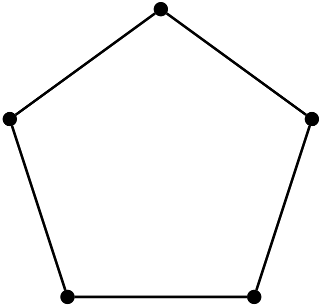
Proof sketch "$\Rightarrow$": Assume G' is not bipartite
$\Rightarrow G'$ contains an odd cycle C



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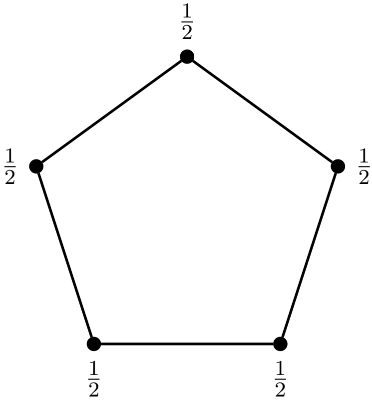

- let $\bar{x}_v = \begin{cases} \frac{1}{2} & v \in V(C) \\ 0 & \text{else} \end{cases}$

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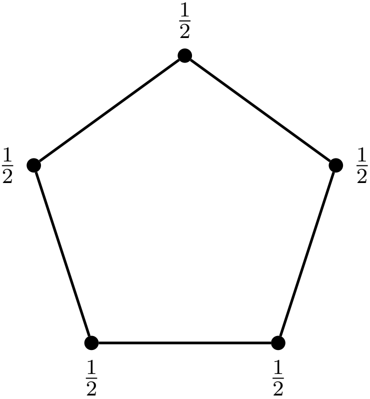

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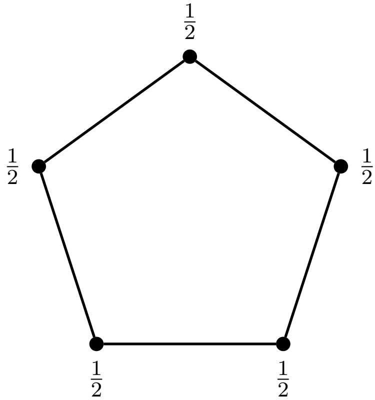

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- odd cycle ineq. not satisfied

$\Rightarrow \bar{x} \notin \mathrm{STAB}(G') \supseteq \mathrm{DW}(G, G')$

$\Rightarrow \bar{x} \notin \mathrm{DW}(G, G')$ ↯

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Strongest Possible Reformulation

Theorem

$\mathrm{DW}(G, G') = \mathrm{STAB}(G)$ *iff* ... ?

Strongest Possible Reformulation

Theorem

$\mathrm{DW}(G, G') = \mathrm{STAB}(G)$ *iff* G' *contains all odd holes of* G.

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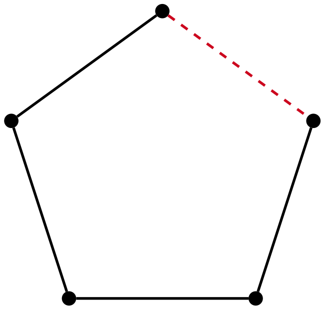
Proof sketch "$\Rightarrow$": Assume $\exists$ odd hole H of G with $H \not\subseteq G'$



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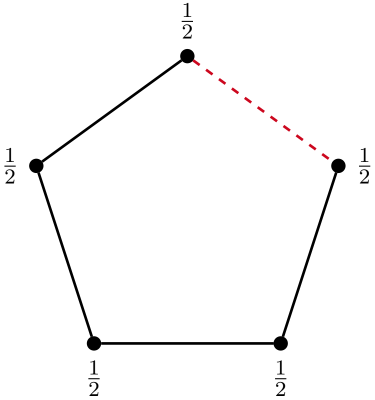

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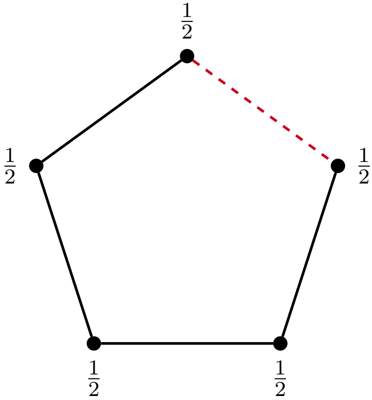

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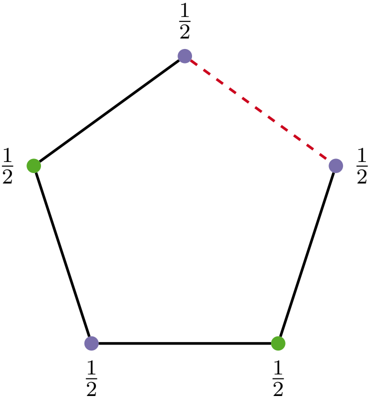

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- $\bar{x} \in \mathrm{STAB}(G')$ (conv. comb. of • and •)
- edge ineq.s satisfied

$\Rightarrow \bar{x} \in \mathrm{DW}(G, G')$ ↯

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Proof sketch "$\Leftarrow$":

- let $\sum_{v \in V} \pi_v x_v \leq \pi_0$ be a facet of $\mathrm{STAB}(G)$
- (neither $x_v \geq 0$ nor $x_u + x_v \leq 1$)
- idea: prove $\sum_{v \in V} \pi_v x_v \leq \pi_0 \quad \forall x \in \mathrm{DW}(G, G')$

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- $\pi \geq 0$, $\pi_0 > 0$, and $\pi_0 = \alpha_\pi(G)$ holds

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- idea: prove $\sum_{v \in V} \pi_v x_v \leq \pi_0 \quad \forall x \in \mathrm{DW}(G, G')$
- $\pi \geq 0$, $\pi_0 > 0$, and $\pi_0 = \alpha_\pi(G)$ holds
- $V_0 := \{v \in V : \pi_v > 0\}$, $G_0 := G[V_0] = (V_0, E_0)$
- G_0 with weighting π is called facet-graph

Strongest Possible Reformulation

Proof sketch "$\Leftarrow$" cont'd:

- $e \in E_0$ critical in G_0 if $\alpha_\pi(G_0 - e) > \alpha_\pi(G_0)$
- G_0 is α_π-critical if every edge is critical

- $\exists$ spanning α_π-critical subgraph $T_0 \subseteq G_0$ with $\alpha_\pi(T_0) = \alpha_\pi(G_0)$ (Sewell, 1990)
- (T_0 with weighting π is still a facet-graph)

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- (T_0 with weighting π is still a facet-graph)
- idea: if T_0 is covered by G', i.e., $T_0 \subseteq G'$, then

$$\sum_{v \in V_0} \pi_v x_v \leq \pi_0 \quad \forall x \in \mathrm{STAB}(T_0) \supseteq \mathrm{STAB}(G')_{|V_0} \supseteq \mathrm{DW}(G, G')_{|V_0}$$

$\rightarrow$ prove that every $e \in E(T_0)$ is part of some odd hole in G_0

Strongest Possible Reformulation

Proof sketch "$\Leftarrow$" cont'd:

Lemma

$\exists$ spanning α_π-critical subgraph $T_0 \subseteq G_0$ s.t. every edge $e \in E(T_0)$ is part of an odd hole H_e of G_0, i.e., $e \in E(H_e)$.

(Proof sketch later)

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(Proof sketch later)

$\Rightarrow$ every $e \in E(T_0)$ is part of some odd hole of G_0

$\Rightarrow$ every $e \in E(T_0)$ is part of some odd hole of G

$\Rightarrow$ T_0 covered by G', i.e., $T_0 \subseteq G'$

$\Rightarrow$ $\sum_{v \in V} \pi_v x_v \leq \pi_0 \quad \forall x \in \mathrm{DW}(G, G')$

$\Rightarrow$ $\mathrm{STAB}(G) = \mathrm{DW}(G, G')$

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Proof sketch for $e = \{u, v\} \in E(T_0)$ critical in G_0

- proof idea due to Andrásfai (1966)

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- let S be MWSS in G_0 with $u, v \notin S$ ($\exists x : \pi^T x = \pi_0$ and $x_u + x_v \neq 1$) (Sewell, 1990)
- let S^+ be MWSS in $G_0 - e$
  - $\pi(S) < \pi(S^+)$
  - $u, v \in S^+$ holds (otherwise S^+ stable in G_0)

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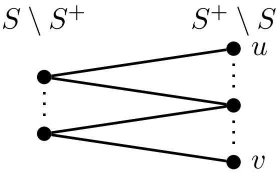
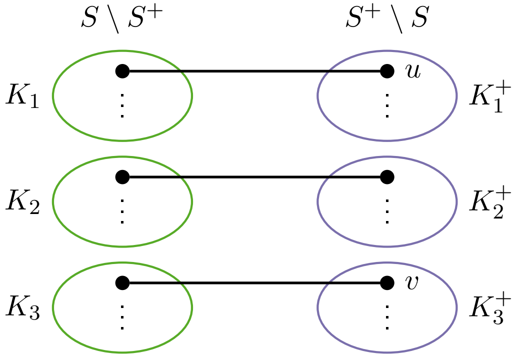


Strongest Possible Reformulation

- assume u and v are in different connected components



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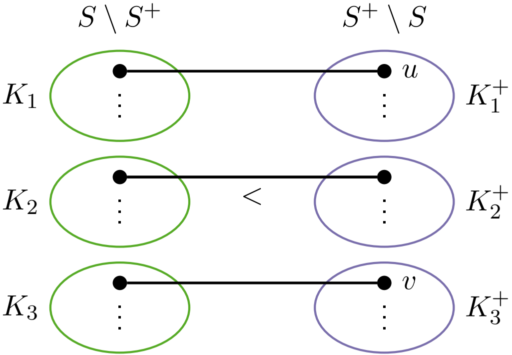

$\Rightarrow$ $\exists\ i$ with $\pi(K_i) < \pi(K_i^+)$

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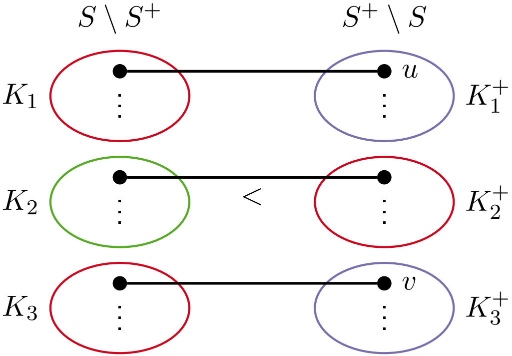
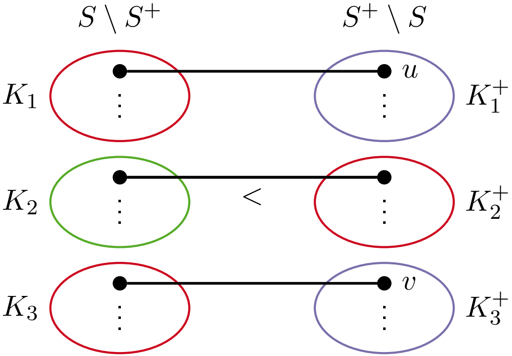


$\Rightarrow$ $\exists\, i$ with $\pi(K_i) < \pi(K_i^+)$

$\Rightarrow$ $S \setminus K_i \cup K_i^+$ is stable in G_0 with $\pi(S \setminus K_i \cup K_i^+) > \pi(S)$ ↯

Strongest Possible Reformulation

- assume u and v are in different connected components



$\Rightarrow$ $\exists\, i$ with $\pi(K_i) < \pi(K_i^+)$

$\Rightarrow$ $S \setminus K_i \cup K_i^+$ is stable in G_0 with $\pi(S \setminus K_i \cup K_i^+) > \pi(S)$ ↯

$\Rightarrow$ $\exists$ u-v-path P in $G[S \setminus S^+ \cup S^+ \setminus S]$ of even length

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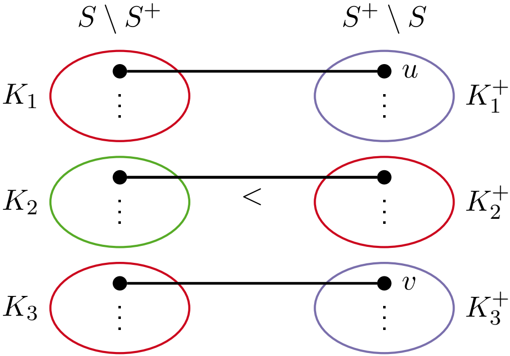

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$\Rightarrow$ shortest P plus $\{u, v\}$ is odd hole in G_0

What We Learned

Theorem

$\mathrm{DW}(G, G') = \mathrm{FRAC}(G)$ *iff* G' *is bipartite.*

Theorem

$\mathrm{DW}(G, G') = \mathrm{STAB}(G)$ *iff* G' *contains all odd holes of* G.

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Related Problems

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- node covering problem
- clique problem
- (matching problem)
- ...

- set packing problem
- set covering problem
- set partitioning problem
- independence system problem

Related Problems

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Related Problems

- can we prove similar results for other (related) problems?

- node covering problem ✓
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- set packing problem (✓)
- set covering problem ✗
- set partitioning problem ✗
- independence system problem ✗

Related Problems

- can we prove similar results for other (related) problems?

- node covering problem ✓
- clique problem ✓
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- ...

- **set packing problem** (✓)
- set covering problem ✗
- set partitioning problem ✗
- independence system problem ✗

Set Packing Problem

- let $A \in \{0,1\}^{m \times n}$ be a matrix and let $w \in \mathbb{Z}^n$ be a vector
- the vector $\mathbf{1} := (1, \ldots, 1)^T$ is of suitable dimension

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- set packing problem:

$$
\begin{array}{lrcl}
\max & w^T x & & \\
\text{s.t.} & Ax & \leq & \mathbf{1} \\
& x & \in & \{0,1\}^n
\end{array}
$$

- columns of A represent sets; rows represent conflicts

D-W Reformulation for the Set Packing Problem

- inclusion relation

$$P_{IP} \subseteq P_{DW}(I') \subseteq P_{LP}$$

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D-W Reformulation for the Set Packing Problem

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$$\begin{array}{ccccc} P_{IP} & \subseteq & P_{DW}(I') & \subseteq & P_{LP} \\ \downarrow & & & & \\ \mathrm{SP}(A) & & & & \end{array}$$

- set packing polytope $\mathrm{SP}(A)$

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- fractional set packing polytope $\mathrm{FSP}(A)$

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\downarrow & & \downarrow & & \downarrow \\
\mathrm{SP}(A) & & & & \mathrm{FSP}(A) \\
& & \mathrm{DW}(A, A') & &
\end{array}
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- set packing polytope $\mathrm{SP}(A)$
- fractional set packing polytope $\mathrm{FSP}(A)$

- choose subset $I' \subseteq \{1, \ldots, m\}$ of rows and define $A' := A_{I'}$

$$
\mathrm{DW}(A, A') := \{x \in [0,1]^n : A_{I \setminus I'} \leq \mathbf{1}, \\ x \in \mathrm{SP}(A')\}
$$

(Fractional) Set Packing vs. Stable Set Polytope

- let $G(A) = (V(A), E(A))$ be the conflict graph of A, i.e.,

$$V(A) = \{1, \ldots, n\}$$
$$E(A) = \{ij : \exists r \text{ s.t. } a_{ri} \neq 0, a_{rj} \neq 0\}$$

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$$\mathrm{FSP}(A) \subseteq \mathrm{FRAC}(G(A))$$

(added "some" clique inequalities)

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- for the integer hulls holds

$$\mathrm{SP}(A) = \mathrm{STAB}(G(A))$$

(different description of the same conflicts)

Weakest Possible Reformulation

Theorem (Sachs 1970)

$\mathrm{SP}(A) = \mathrm{FSP}(A)$ *iff* A *is perfect.*

Corollary

If A' *is perfect, then* $\mathrm{DW}(A, A') = \mathrm{FSP}(A)$.

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Theorem (Sachs 1970)

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If A' *is perfect, then* $\mathrm{DW}(A, A') = \mathrm{FSP}(A)$.

Theorem (Chvátal 1975)

A *is perfect iff its non-dominated rows form the clique-node matrix of a perfect graph.*

Lemma

If $\exists \tilde{G}$ *perfect with* $G(A') \subseteq \tilde{G} \subseteq G(A)$ *and clique inequalities of* $\tilde{G}$ *are dominated by* $Ax \leq \mathbf{1}$*, then* $\mathrm{DW}(A, A') = \mathrm{FSP}(A)$.

Strongest Possible Reformulation

Corollary

If for every $e \in E(A)$ *contained in an odd hole of* $G(A)$ *holds* $e \in E(A')$, *then* $\mathrm{DW}(A, A') = \mathrm{SP}(A)$.

Lemma

If $\mathrm{DW}(A, A') = \mathrm{SP}(A)$, *then for every* $e \in E(G(A))$ *contained in an odd hole/antihole of* $G(A)$ *of size* ≥ 5 *holds* $e \in E(G(A'))$

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If $\mathrm{DW}(A, A') = \mathrm{SP}(A)$*, then for every* $e \in E(G(A))$ *contained in an odd hole/antihole of* $G(A)$ *of size* ≥ 5 *holds* $e \in E(G(A'))$

- what to do with edges/conflicts only contained in odd holes of size 3?

Future Work

- investigate dual bound instead of polytope
- ideas for detector
- further extend ideas to other problems

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