Column generation for vector packing with bounds from multidimensional dual-feasible functions

Cláudio Alves[‡], José Valério de Carvalho[‡], François Clautiaux[⋆], and Jürgen Rietz[‡]

‡ Dept. Produção e Sistemas, Universidade do Minho, Braga, Portugal
⋆ Institut de Mathématiques de Bordeaux, Université de Bordeaux, France

vc@dps.uminho.pt

2016 International Workshop on Column Generation - GERAD
Búzios, 22-25 May 2016

Motivation

- Dual Feasible Functions (DFF) tailored to a given problem provide feasible solutions of the dual of its column generation model,
- and bounds (by the Weak Duality Theorem).
- Theoretical study of properties of DFF provide tools to enhance DFF to maximal DFF (and enhanced bounds).
- Using a set of different DFF improves the chance of getting a very good bound, and is computationally "inexpensive".

Previous results for 1-dimensional DFF (Clautiaux, Alves, VC, AOR 2010)

- DFF provides very good lower bounds for the 1-dim. BPP, when
- using several families of DFF ($f_0^k, f_{FS,1}^k, f_{VDB,2}^k, f_{CCM,1}^k, f_{BJ,1}^k, f_{LL,2}^k$).
- Lower bound is $LB_{DFF} = \max\{LB_{f_0^k}, \ldots, LB_{f_{LL,2}^k}\}$.
- LB_{DFF} is typically much better than bound from LP relaxation.
- $LB_{DFF} \leq z_{CG}$, but can obtained in a small fraction of time.

Contents

Two optimization problems

Problem 1: Bin-Packing Problem (BPP)

- A company has to deliver 10 items with weight 0.4 and 40 items with weight 0.3.
- Each vehicles can carry a weight of 1.
- How many vehicles are needed to carry all the items?

Problem 2: Vector Packing Problem (VPP)

- A company has to deliver 4 items with weight and volume:

item	1	2	3	4	vehicle
weight	3	2	4	1	5
volume	2	3	1	2	4

- Each vehicles can carry a weight of 5 and a volume of 4.
- How many vehicles are needed to carry all the items?

(Trivial) lower bounds

Bin-Packing Problem (BPP) [Martello and Toth (1990)]:

$$LB_T = \lceil \sum_{i=1}^{m} b_i w_i / W \rceil.$$

- $LB_T = \lceil (10 \times 0.4 + 40 \times 0.3)/1 \rceil = 16$: at least, 16 vehicles are needed.

m-dimensional Vector Packing Problem (VPP) [Spieksma (1994)]:

$$L_C = \max_{d=1,\ldots,m} \left\{ \left\lceil \sum_{i=1}^{n} l_{id}/1 \right\rceil \right\}.$$

After scaling:

item	1	2	3	4	vehicle
weight	0.6	0.4	0.8	0.2	1
volume	0.5	0.75	0.25	0.5	1

- $L_C = \max\{\lceil 0.6 + 0.4 + 0.8 + 0.2 \rceil, \lceil 0.5 + 0.75 + 0.25 + 0.5 \rceil\} = 2$: at least, 2 vehicles are needed.

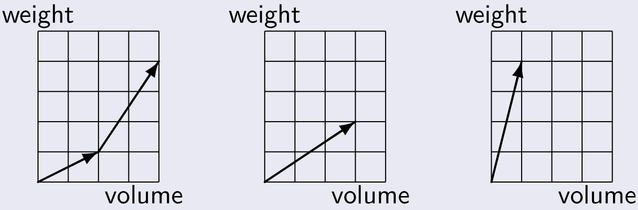
Examples: lower bounds and optimal solutions

BPP: lower bound $LB_T = 16$ vehicles, (one) optimal solution=17 vehicles:

- Ten vehicles carrying: $0.4 + 0.3 + 0.3 = 1$
- Six vehicle carrying: $0.3 + 0.3 + 0.3 = 0.9$
- One vehicle carrying: $0.3 + 0.3 = 0.6$

VPP: lower bound $L_C = 2$ vehicles, (one) optimal solution $=$ 3 vehicles:

item	1	2	3	4	bin
weight	3	2	4	1	5
volume	2	3	1	2	4



Can we get better lower bounds?

Yes: column generation (CG),

- but solving plain CG models may be very time consuming.

Yes, typically: Dual Feasible Functions (DFF)

- DFF provide feasible solutions of the dual polytope of the CG model of the Bin Packing Problem (BPP) / Cutting Stock Problem (CSP),
- which has **all** the (exponentially many) constraints that correspond to **all** feasible cutting patterns.
- We aim at getting strong lower bounds,
- by finding dual feasible solutions, without column generation,
- (*i.e.*, without enumerating any dual constraint).
- DFF are derived from the structure of the columns.

DFF were first used by D. Johnson (PhD thesis, MIT, 1973),
coined by G. Lueker (1983), and
used by Fekete and Schepers (2001) in combinatorial algorithms.

BPP/CSP: dual perspective of CG model

- a column in the primal is equivalent to a constraint in the dual.

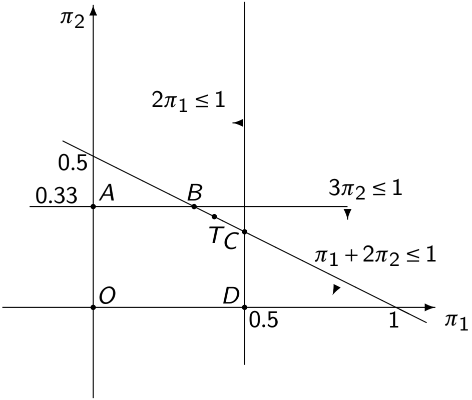
$$(Primal)\quad \begin{array}{ll} \min & cx \\ s.t. & Ax \geq b \\ & x \geq 0 \end{array} \qquad\qquad (Dual)\quad \begin{array}{ll} \max & \pi b \\ s.t. & \pi A \leq c \\ & \pi \geq 0 \end{array}$$

- Same example: bins of size 1, and items of size 0.4 and 0.3.
- Set K of **all** maximal feasible cutting patterns:

  $K = \{(y_1, y_2) : 0.4y_1 + 0.3y_2 \leq 1, y_1, y_2 \in \mathbb{N}\} = \{(2,0),(1,2),(0,3)\}$

$$(Primal\ )\quad \begin{array}{lllll} \min & 1x_1 & +1x_2 & +1x_3 & \\ s.t. & 2x_1 & +1x_2 & & \geq b_1 \\ & & +2x_2 & +3x_3 & \geq b_2 \\ & & x_1, x_2, x_3 \geq 0 & & \end{array} \qquad (Dual)\quad \begin{array}{llll} \max & b_1\pi_1 & +b_2\pi_2 & \\ s.t. & 2\pi_1 & & \leq 1 \\ & 1\pi_1 & +2\pi_2 & \leq 1 \\ & & 3\pi_2 & \leq 1 \\ & \pi_1, \pi_2 \geq 0 & & \end{array}$$

Dual polytope of CG model and the trivial lower bound



Dual polytope of CG model:

$$\begin{cases} 2\pi_1 & & \le 1 \\ 1\pi_1 & +2\pi_2 & \le 1 \\ & 3\pi_2 & \le 1 \\ \pi_1,\ \pi_2 \ge 0 \end{cases}$$

Dual solution $\hat{\pi}_i = w_i / W, i = 1, \ldots, m$, is **always** feasible.

Example: demand of 10 items of size 0.4 and 40 items of size 0.3.

- Dual solution $(\hat{\pi}_1, \hat{\pi}_2)^\top = (0.4, 0.3)^\top$ is solution T.
- $(b_1, b_2)^\top = (10, 40)^\top$
- $LB_T = \lceil 10 \times 0.4 + 40 \times 0.3 \rceil = 16$ bins.

Dual feasible functions (DFF)

Definition: A function $f:[0,1]\to[0,1]$ is said to be dual feasible if, for any finite set of real numbers $S\subseteq[0,1]$,

$$\sum_{x\in S} x\le 1 \Rightarrow \sum_{x\in S} f(x)\le 1$$

Fekete,Schepers'01:

- Let $I:=(x_1,\ldots,x_n)$ be a BPP instance and let f be a DFF.
- Any lower bound for the transformed BPP instance $f(I):=(f(x_1),\ldots,f(x_n))$ is also a lower bound for I.

Sketch of proof:

- Definition: If a packing of a set of items with (original) sizes is feasible, then it is **also** feasible with the transformed sizes.

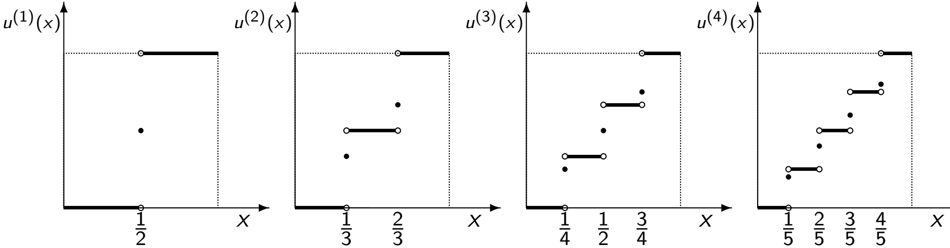
$\therefore$ Value of optimal solution of CG model of transformed BPP instance $\le$ value of optimal solution of CG model of original BPP instance.

$\therefore$ Trivial lower bound of transformed BPP instance: $\lceil \sum_{i\in I} f(x_i)/1\rceil$ is a valid bound for original problem.

Example: DFF $u^{(k)}(x)$, $k \in \mathbb{N}\setminus\{0\}$ [Fekete,Schepers'2001]

DFF $u^{(k)}(x)$, $k \in \mathbb{N}\setminus\{0\}$, slightly improves a function by Lueker'83.

$$u^{(k)}: \quad [0,1] \to [0,1]$$
$$x \mapsto \begin{cases} x, & \text{for } (k+1)x \in \mathbb{Z}, \\ \frac{\lfloor (k+1)x \rfloor}{k}, & \text{otherwise.} \end{cases}$$



Example: getting LB from DFF $u^{(k)}(x), k = 1, 2, 3, \ldots$

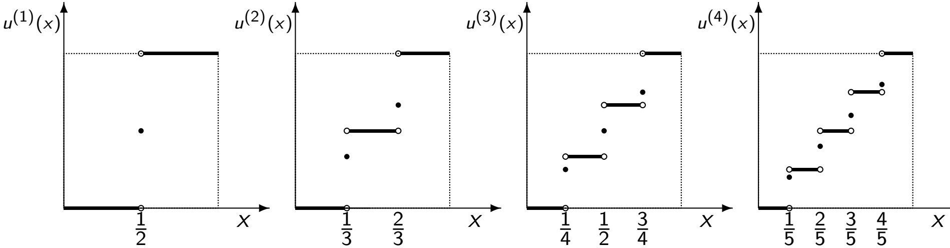
- $I = (\overbrace{0.4, \ldots, 0.4}^{\text{10 items}}, \overbrace{0.3, \ldots, 0.3}^{\text{40 items}})$.

  $u^{(1)}(0.4) = 0;$
  $u^{(1)}(0.3) = 0.$ $\rightarrow LB_{u^{(1)}} = \lceil (10 \times 0 + 40 \times 0)/1 \rceil = 0$ bins.

  $u^{(2)}(0.4) = 0.5;$
  $u^{(2)}(0.3) = 0.$ $\rightarrow LB_{u^{(2)}} = \lceil (10 \times 0.5 + 40 \times 0)/1 \rceil = 5$ bins.

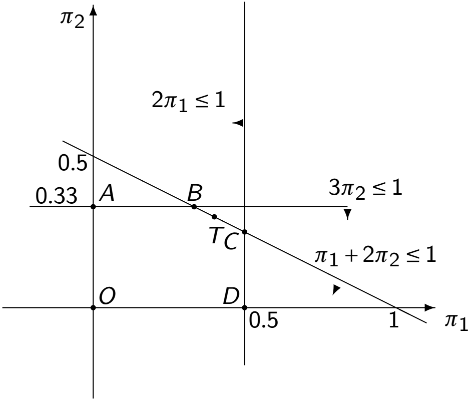
  $u^{(3)}(0.4) = 0.33;$
  $u^{(3)}(0.3) = 0.33.$ $\rightarrow LB_{u^{(3)}} = \lceil (10 \times 0.33 + 40 \times 0.33)/1 \rceil = 17$ bins.

  $\ldots$



Example: getting dual feasible dual solutions from DFF

- The 3 first DFF in the previous page map to the dual solutions $O = (0,0)^\top$, $D = (0.5,0)^\top$ and $B = (0.33, 0.33)^\top$, respectively.
- Other DFF might map to solutions A and C.



Dual problem of CG model:

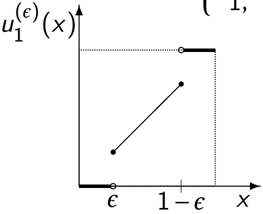
$$
\begin{array}{llll}
\max & b_1\pi_1 & +b_2\pi_2 & \\
s.t. & 2\pi_1 & & \leq 1 \\
 & 1\pi_1 & +2\pi_2 & \leq 1 \\
 & & 3\pi_2 & \leq 1 \\
 & \pi_1, \pi_2 \geq 0 & &
\end{array}
$$

Just one other example of DFF: $u_1^{(\epsilon)}$ [FeketeSchepers'01]

Let $\epsilon \in \left[0, \frac{1}{2}\right]$

$$u_1^{(\epsilon)}: \quad [0,1] \to [0,1]$$

$$x \mapsto \begin{cases} 0, & \text{for } x < \epsilon, \\ x, & \text{for } \epsilon \leq x \leq 1-\epsilon, \\ 1, & \text{for } x > 1-\epsilon, \end{cases}$$



Formalizes Martello and Toth's lower bound L2 for bin-packing.

The m-dimensional vector packing problem (mD-VPP)

Input: instance $E := (n; \mathbf{L}; \mathbf{b})$ with a set $\{1, 2, \ldots, n\}$ of n items

- sizes are normalized: bins are m-dimensional unit cubes.
- item sizes: matrix $\mathbf{L} = (l_{11}, l_{12}, \ldots, l_{1m}; \ldots; l_{n1}, l_{n2}, \ldots, l_{nm}) \in [0,1]^{n \times m}$ (with $\mathbf{l}_i$ being the i^{th} row-vector of $\mathbf{L}$)
- order demands $\mathbf{b} = (b_1, \ldots, b_n)^\top \in (\mathbb{N} \setminus \{0\})^n$.

Output

- Find a partition of the set of items into a minimum number of subsets such that the items in each subset fit into a bin, *i.e.* the sum of the sizes in each dimension does not exceed 1 for any subset.

A pattern $\mathbf{a} \in \mathbb{N}^n$ is feasible, if the capacity constraints hold on all the m dimensions:

$$\sum_{i=1}^{n} a_i \times \ell_{id} \leq 1, d = 1, \ldots, m.$$

A column generation (CG) model for the mD-VPP

Caprara and Toth (2001):

$$
\begin{aligned}
z_{CG} = \min \quad & \sum_{p \in P} \lambda_p \\
\text{s.t.} \quad & \sum_{p \in P} a_{ip} \lambda_p \geq b_i, \;\; i = 1, \dots, n, \\
& \lambda_p \geq 0, \text{ and integer}, \;\; p \in P.
\end{aligned}
$$

- λ_p : number of times a pattern p is used.
- P : set of feasible patterns satisfying the capacity constraints.
- a_{ip} : number of times an item i is used in pattern p.
- Pricing subproblem: multidimensional knapsack problem that does not have the integrality property: $L_C \leq z_{CG}$.
- z_{CG} : bound is quite strong, but time consuming for large instances.

Vector packing dual-feasible functions

Definition

A function $f : [0,1]^m \to [0,1]$ is a *vector packing dual-feasible function* (VP-DFF), if for all instances of the *mD*-VPP and all feasible patterns $\mathbf{a} \in \mathbb{N}^n$,

$$\sum_{i=1}^{n} a_i \times \ell_{id} \le 1, d = 1, \ldots, m \quad \Rightarrow \quad \sum_{i=1}^{n} a_i \times f(\mathbf{l}_i^\top) \le 1.$$

Definition

A VP-DFF f is *maximal* (VP-MDFF), if there is no other VP-DFF g with $g(\mathbf{x}) \ge f(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^m$.

General properties of VP-MDFF

Theorem

Any VP-MDFF $f : [0,1]^m \to [0,1]$ *has necessarily the following properties:*

1. f *is superadditive, i.e. for all* $\mathbf{x}, \mathbf{y} \in [0,1]^m$ *with* $\mathbf{x} + \mathbf{y} \le \mathbf{w}$, *it holds that*
$$f(\mathbf{x}) + f(\mathbf{y}) \le f(\mathbf{x} + \mathbf{y});$$
2. f *is non-decreasing:*
$$f(\mathbf{x}) \le f(\mathbf{y}), \text{ if } \mathbf{o} \le \mathbf{x} \le \mathbf{y} \le \mathbf{w};$$
3. f *is symmetric, i.e. for all* $\mathbf{x} \in [0,1]^m$, *it holds that*
$$f(\mathbf{x}) + f(\mathbf{w} - \mathbf{x}) = 1,$$
and particularly $f(\mathbf{w}) = 1$ *and* $f(\frac{1}{2}\mathbf{w}) = 1/2$.

where: $\mathbf{o} := (0, 0, \ldots, 0)^\top \in \mathbb{R}^m$, $\mathbf{w} := (1, 1, \ldots, 1)^\top \in \mathbb{R}^m$ and, for given vectors $\mathbf{s}, \mathbf{t} \in \mathbb{R}^m$, $\mathbf{s} \le \mathbf{t}$ will stand for $s_i \le t_i$, $i = 1, \ldots, m$.

Class I: functions based on projections into 1-dim. domains

Proposition

Let $g : [0,1] \to [0,1]$ *be a MDFF and* $\mathbf{u} \in \mathbb{R}_+^m$ *with* $\mathbf{u}^\top \mathbf{w} = 1$.
The function $f : [0,1]^m \to [0,1]$ *with*

$$f(\mathbf{x}) := g(\mathbf{u}^\top \mathbf{x})$$

is a VP-MDFF.

- We used several 1-dimensional MDFF and generated $\mathbf{u}$ vectors randomly.

Example: 2-dim. VP-MDFF f from MDFF g

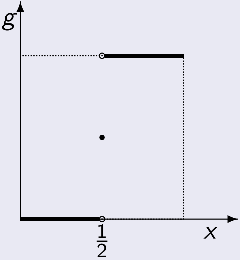
$g : [0,1] \to [0,1]$ is the MDFF in the Figure.

Function $f : [0,1]^2 \to [0,1]$ with

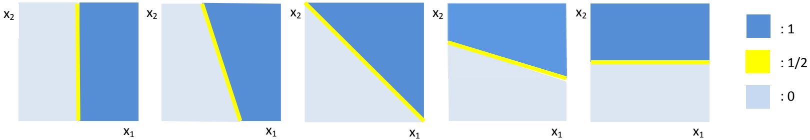
$f(\mathbf{x}) := g(u_1 x_1 + u_2 x_2)$,

for $\mathbf{u} \in \mathbb{R}^2_+$ with $u_1 + u_2 = 1$,

is a 2-dim. VP-MDFF.



Colors indicate value of VP-MDFF $f(\mathbf{x}) := g(\mathbf{u}^\top \mathbf{x})$, for $\mathbf{x} \in [0,1]^2$, for each $\mathbf{u}$:



$\mathbf{u}^\top = (1,0)$ $\mathbf{u}^\top = (\frac{3}{4}, \frac{1}{4})$ $\mathbf{u}^\top = (\frac{1}{2}, \frac{1}{2})$ $\mathbf{u}^\top = (\frac{1}{4}, \frac{3}{4})$ $\mathbf{u}^\top = (0,1)$

Example: ..., and its use to compute Lower Bounds

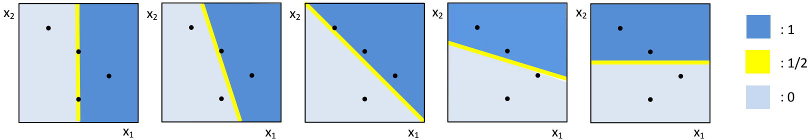
After scaling:

item	1	2	3	4	bin
weight (x_2)	0.6	0.4	0.8	0.2	1
volume (x_1)	0.5	0.75	0.25	0.5	1

- Each dot represents an item vector.
- Each item vector has demand one.
- Sum of f values is a dual objective function value (lower bound).

Lower bound for each value of $\mathbf{u}$:

1/2+1+0+1/2=**2** 1+1+0+0=2 1+1+1+0=3 1+0+1+0=2 1+0+1+0=**2**



- Best lower bound (=3) from VP-MDFF f for $\mathbf{u}^\top = (\frac{1}{2}, \frac{1}{2})$.

Example: ... and the same in the CG model

- CG model with all the maximal packings for the 2d-VPP instance:

(v,w)	item	λ_1	λ_2	λ_3		dual feasible solutions for $\mathbf{u}^\top =$ $(1,0)$	$(\frac{3}{4},\frac{1}{4})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{4},\frac{3}{4})$	$(0,1)$
$(0.5,0.6)$	1	1			≥ 1	$1/2$	1	1	1	1
$(0.75,0.4)$	2		1		≥ 1	1	1	1	0	0
$(0.25,0.8)$	3			1	≥ 1	0	0	1	1	1
$(0.5,0.2)$	4	1		1	≥ 1	$1/2$	0	0	0	0
$(1,1)$	min	1	1	1						

- Note that dual solutions $\pi^u = (\pi_1^u, \ldots, \pi_n^u), \forall \mathbf{u}$, obey dual constraints, *i.e.*, $\sum_{i=1}^n a_{ip}\pi_i^u \leq 1, \forall p \in P$.
- Best lower bound (=3) from dual feasible solution for $\mathbf{u}^\top = (\frac{1}{2}, \frac{1}{2})$.

- tailored to the instance (data dependent).
- Let u_1, u_2 be a feasible (not necessarily optimal) dual solution for an instance of the *mD*-VPP with 2 items of sizes $\mathbf{s}$ and $\mathbf{t}$ ($\mathbf{s}, \mathbf{t} \in (0,1]^m$) demanded at least once.

Proposition

The function $f : [0,1]^m \to [0,1]$ *is a superadditive VP-DFF:*

$$f(\mathbf{x}) := \max\{a_1 u_1 + a_2 u_2 | a_1, a_2 \in \mathbb{N}, a_1 \mathbf{s} + a_2 \mathbf{t} \leq \mathbf{x}\}.$$

- Deriving the function requires solving an integer optimization problem for every argument $\mathbf{x}$, but the complexity remains low, if the possible values $a_1, a_2 \in \mathbb{N}$ are bounded by a small constant.
- The complexity to calculate a_1 and a_2 is pseudo-polynomial.

We generated Class V superadditive *mD*-VPP-DFF for every pair of item vectors.

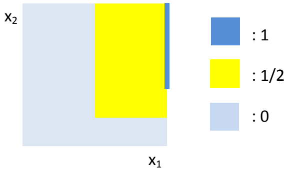
Example

- Let $\mathbf{s} = (0.5, 0.2)^\top, \mathbf{t} = (0.5, 0.6)^\top$.
- Pick $u_1 = 0.5, u_2 = 0.5$ as the dual feasible solution.

Optimization problem is:

$$\begin{aligned} f(\mathbf{x}) = \max \quad & 0.5a_1 + 0.5a_2 \\ \text{s.t.} \quad & 0.5a_1 + 0.5a_2 \leq x_1 \\ & 0.2a_1 + 0.6a_2 \leq x_2 \\ & a_1, a_2 \in \mathbb{N} \end{aligned}$$

Resulting $f(\mathbf{x})$:



Function is not maximal, but can be enhanced into a VP-MDFF by enforcing symmetry

Creating VP-MDFFs by forcing symmetry

An m-dimensional superadditive vector function can be enhanced into a VP-MDFF by forcing symmetry.

Proposition

Let $f : [0,1]^m \to [0,1]$ be a superadditive function, and M be any subset of $[0,1]^m \setminus \{\frac{1}{2}\mathbf{w}\}$ such that:

1. *for all $\mathbf{x} \in [0,1]^m \setminus \{\frac{1}{2}\mathbf{w}\}$, the following equivalence holds:*

$$\mathbf{x} \in M \iff \mathbf{w} - \mathbf{x} \notin M;$$

2. *for any $\mathbf{x}, \mathbf{y} \in M$, it holds that*

$$\mathbf{x} + \mathbf{y} \not\leq \mathbf{w}.$$

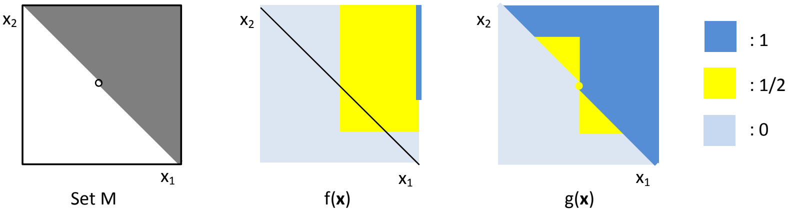
The following function $g : [0,1]^m \to [0,1]$ which is built from f is a VP-MDFF:

$$g(\mathbf{x}) := \begin{cases} 1/2, & \text{if } 2\mathbf{x} = \mathbf{w}, \\ 1 - f(\mathbf{w} - \mathbf{x}), & \text{if } \mathbf{x} \in M, \\ f(\mathbf{x}), & \text{otherwise.} \end{cases}$$

Example: building a 2-dim. VP-MDFF by forcing symmetry

Figure shows:

- A valid choice for set M (in grey).
- $f(\mathbf{x})$: a 2-dimensional superadditive vector function.
- $g(\mathbf{x})$: an enhanced 2-dimensional VP-MDFF.



- Note that g dominates f, but it is not the only function to dominate f.

Test instances for the 2-dimensional VPP

set I: proposed by Caprara and Toth (2001) and Spieksma (1994)

- Ten sets, each set divided into 4 groups of 10 instances with a number of items per instance $n \in \{25, 50, 100, 200\}$.
- Total of 400 instances.

set II: adapted from Berkey and Wang (1987) and Martello and Vigo (1998)

- Ten sets, each set divided into 5 groups of 10 instances, with a number of items per instance $n \in \{20, 40, 60, 80, 100\}$.
- Total of 350 instances: only used groups where the bound of Spieksma is strictly smaller than the column generation bound (groups 1,3,5,7,8,9,10).

PC with an Intel Core i3 CPU with 2.27 GHz and 4GB of RAM, with 600 seconds time limit.

Overview of computational results: lower bounds

set I

- For 309 of the 400 instances (77.3%), at least, one VP-DFF gave a lower bound equal to z_{CG}.
- For the remaining instances, the difference between z_{CG} and the best VP-DFF bounds is 1.87, on average.

set II

- For 255 of the 350 instances (72.9%), at least, one VP-DFF gave a lower bound equal to z_{CG}.
- For the remaining instances, the difference between z_{CG} and the best VP-DFF bounds is 1.2, on average.

- VP-DFF of Class V (enhanced by enforcing symmetry) were one of the most successful.

Overview of computational results: column generation

- VP-MDFF improve significantly our plain branch-and-price (B&P):

set I

- in some sets, total computing time is reduced up to nearly 99%;
- in one set, plain B&P solved none, but B&P with VP-MDFF solves instances in less than one second.

set II

- instances are solved efficiently with plain B&P, but B&P with VP-MDFF reduces total computing time by up to nearly 63%, on average.

We used a branching scheme that allows to:

- deep in the tree, in the branches enforcing "item vector i goes with item vector j", to add the two vectors i and j, and update the item vector list to derive a stronger (than in the root node) bound, before calling the pricing subproblem of the node.

Conclusions

- The concept of DFF was extended to the multidimensional case yielding m-dimensional vector packing dual-feasible functions,
- which generate strong lower bounds in a small fraction of a second.
- Strong lower bounds along the search tree improve branch-and-price performance.
- We also addressed extensions of DFF to Orthogonal Packing, Bin-Packing with Conflicts and other problems.

Thank you for your attention.

Bibliography

- Alves, C., Clautiaux, F., VC, Rietz, J., Dual-Feasible Functions for Integer Programming and Combinatorial Optimization: Basics, Extensions and Applications, Springer, 2016.
  `http://www.springer.com/br/book/9783319276021`
- C. Alves, VC, F. Clautiaux J. Rietz, Multidimensional dual-feasible functions and fast lower bounds for the vector packing problem, European Journal of Operational Research, 233(1), 43-63 (2014).
- J. Rietz, C. Alves, VC, On the extremality of maximal dual feasible functions, Operations Research Letters, 40(1), 25-30 (2012).
- J. Rietz, C. Alves, VC, Worst-case analysis of maximal dual feasible functions, Optimization Letters 6(8): 1687-1705 (2012).
- J. Rietz, VC, C. Alves, Theoretical investigations on maximal dual feasible functions, Operations Research Letters, 38, 174-178 (2010).
- F. Clautiaux, C. Alves, VC, A survey of dual-feasible and superadditive functions, Annals of Operations Research, 179(1), 317-342 (2010).
- J. Carlier, E. Néron, Computing redundant resources for the resource constrained project scheduling problem. European Journal of Operational Research, 176(3), 1452-1463 (2007).
- J. Carlier, F. Clautiaux, A. Moukrim, New reduction procedures and lower bounds for the two-dimensional bin packing problem with fixed orientation. Computers and Operations Research, 34(8), 2223-2250 (2007).
- S. Fekete, J. Schepers, New classes of fast lower bounds for bin packing problems. Mathematical Programming, 91, 11-31 (2001).

End

Acknowledgement

- FCT - Fundação para a Ciência e Tecnologia within the Project Scope: PEst-OE/EEI/UI0319/2014.