Application of Dantzig-Wolfe Reformulation toBinary Quadratic Problems

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Plan

Context

- In this work we aim at investigating a family of decompositions for Quadratic Problems (QPs).
- A generic QP reads as follows :

Quadratic Problem (QP)

$$(QP) \quad \max\{f(x) = x^\top Q x + L^\top x | Ax \leq b, x \in X\} \ .$$

- $Q \in \mathbb{Q}^{n \times n}$ and not restricted to be convex.
- $L \in \mathbb{Q}^n$.
- $X \subseteq \mathbb{R}^n$ or $X \subseteq \mathbb{N}^n$.

Dantzig-Wolfe decomposition for Binary Quadratic Problems (BQPs)

A generic BQP reads as follows :

Binary Quadratic Problem (BQP)

$$(BQP) \quad \max\{f(x) = x^\top Qx + L^\top x | Ax \leq b, x \in \{0,1\}\} .$$

- Let A', A'' and b', b'' be a generic row partition of the constraint matrix A and of the rhs vector b.
- The continuous relaxation of (BQP) can be strengthened by convexifying the constraints $A''x \leq b''$ (i.e. imposing $x \in conv\{A''x \leq b'', x \in \{0,1\}\}$).

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DWR with quadratic master problem

DWR of constraints A''

(BQP) can be reformulated as follows :

$$
\begin{aligned}
(BQP_{DWR(A'')}) \quad \max \quad & f(x) && && (1)\\
\text{s.t.} \quad & A'x \le b' && [\alpha] && (2)\\
& x_j = \sum_{p \in \mathcal{P}_{DWR(A'')}} x_j^p \lambda^p \qquad j = 1, \ldots, n && [\tau_j] && (3)\\
& \sum_{p \in \mathcal{P}_{DWR(A'')}} \lambda^p = 1 && [\beta] && (4)\\
& x_j \in \{0, 1\} \qquad j = 1, \ldots, n && && (5)\\
& \lambda^p \ge 0 \qquad p \in \mathcal{P}_{DWR(A'')} && && (6)
\end{aligned}
$$

if f is convex and with $\mathcal{P}_{DWR(A'')}$ being the set of extreme points of $conv\{x | A''x \le b'', x \in \{0, 1\}\}$.

DWR with quadratic master problem

Convexification of the objective function

If the objective function is non convex, we need to replace $f(x)$ by an equivalent convex objective function $f'(x)$.

$$
\begin{aligned}
(BQP_{DWR(A'')}) \quad & \max \quad f'(x) && && (7)\\
& \text{s.t.} \quad A'x \leq b' && [\alpha] && (8)\\
& x_j = \sum_{p \in \mathcal{P}_{DWR(A'')}} x_j^p \lambda^p \qquad j = 1, \dots, n && [\tau_j] && (9)\\
& \sum_{p \in \mathcal{P}_{DWR(A'')}} \lambda^p = 1 && [\beta] && (10)\\
& x_j \in \{0, 1\} \qquad j = 1, \dots, n && && (11)\\
& \lambda^p \geq 0 \qquad p \in \mathcal{P}_{DWR(A'')} && && (12)
\end{aligned}
$$

- Variables λ are partially enumerated by solving an additional *pricing* problem.

Let α, τ and β being the dual variables associated to the constraints in the continuous relaxation of $(BQP_{DWR(A'')})$.

Pricing problem

$$
\begin{aligned}
(\Pi_{BQP_{DWR(A'')}}(\tau^*, \beta^*)) \quad \max \quad & \tau^{*\top} x + \beta^* && (13)\\
\text{s.t.} \quad & A'' x \leq b'' && (14)\\
& x_j \in \{0, 1\} \qquad j = 1, \ldots, n && (15)
\end{aligned}
$$

If the optimal value of $(\Pi_{BQP_{DWR(A'')}}(\tau^*, \beta^*))$ is greater than zero, then a column with positive reduced cost is found and added to the master.

- $(BQP_{DWR}) \Rightarrow f(x)$ is quadratic, the pricing problem is (binary) linear.

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DWR with quadratic pricing problem

The objective function can be rewritten directly in terms of the λ variables by introducing $f(\lambda) = \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p$ with

$$c_p = f(x_p) = x_p^\top Q x_p + L^\top x_p \ .$$

$\overline{DWR}$ of constraints A''

$$(BQP_{\overline{DWR}(A'')}) \quad \max \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p \qquad (16)$$

$$\text{s.t.} \quad (8) - (12)$$

DWR with quadratic pricing problem

with $f(x) = \sum_{p \in \mathcal{P}_{DWR(A'')}} c_p \lambda_p$, the pricing problem reduces to the following quadratic problem :

Pricing problem

$$
\begin{aligned}
(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \quad & x^\top Q x + L^\top x + {\tau^*}^\top x + \beta^* && (17)\\
\text{s.t.} \quad & A'' x \le b'' && (18)\\
& x_j \in \{0, 1\} \qquad j = 1, \ldots, n && (19)
\end{aligned}
$$

where Q is not required to be convex.

- $(BQP_{\overline{DWR}}) \Rightarrow f(\lambda)$ is linear, the pricing problem is (binary) quadratic.

DWR with quadratic pricing problem

The objective function can still be modified using the convexified objective function $f'(x)$.
The pricing reduces to the following quadratic problem :

Pricing problem

$$
\begin{aligned}
(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad & \max \quad f'(x) + {\tau^*}^\top x + \beta^* & & (20)\\
& \text{s.t.} \quad A''x \le b'' & & (21)\\
& \quad x_j \in \{0, 1\} \qquad j = 1, \ldots, n & & (22)
\end{aligned}
$$

Plan

(kQKP) Formulation

Notations

n : number of items
a_j : weight of item j ($j = 1, \ldots, n$)
b : capacity of the knapsack
c_{ij} : profit associated with the selection of items i and j ($i, j = 1, \ldots, n$)
k : number of items to be filled in the knapsack

Assumptions

$c_{ij} \in \mathbb{N}\ i, j = 1, \ldots, n,\ a_j \in \mathbb{N}\ j = 1, \ldots, n,\ b \in \mathbb{N}$
$\max_{j=1,\ldots,n} a_j \leq b < \sum_{j=1}^{n} a_j$
$k \in \{1, \ldots, k_{max}\}$

(kQKP) Formulation

Mathematical formulation

$$
\text{(kQKP)} \begin{cases} \max\ f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j \\ \text{s.t.} \quad \sum_{j=1}^{n} a_j x_j \leq b \qquad (1) \\ \quad \sum_{j=1}^{n} x_j = k \qquad (2) \\ \quad x_j \in \{0,1\} \qquad j = 1,\ldots,n \end{cases}
$$

- without constraint (2) : the 0-1 quadratic knapsack problem (QKP)
- without constraint (1) : the k-cluster problem

Reformulations for (kQKP)

Reformulation of the objective function $f(x)$

QCR/MQCR Method

($f(x)$ can be reformulated by exploiting the property $x^2 = x$)

$\Downarrow$

convex problem

Reformulation of the feasible region $\{x | Ax \leq b, x \in \{0, 1\}\}$

Danzig-Wolfe Reformulation

(a subset of constraints is substituted by its convex hull)

$\Downarrow$

tighter formulation

Quadratic Convex Reformulation (QCR) : a two-phase method

[Billionnet, Elloumi and M-C. Plateau, Discrete Applied Mathematics, 2009]

- **Phase 1** : Reformulate the objective function $f(x)$ into an equivalent 0-1 program with a **concave** quadratic objective function $f_{u,\alpha}(x)$.
  $\rightarrow$ an equivalent **convex** 0-1 program
- **Phase 2** : Apply a standard 0-1 convex quadratic solver to this new problem.

QCR - Phase 1 : Addition of two functions null on the feasible set

- $q_u(x) = \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right)$
- $q_\alpha(x) = \sum_{i=1}^{n} \alpha_i x_i \left(\sum_{j=1}^{n} x_j - k\right)$ or $q_v(x) = v \left(\sum_{j=1}^{n} x_j - k\right)^2$

q_α and q_v are implied by the cardinality constraint.

QCR : A two-phase method

QCR - Phase 1 : Reformulation 1

New objective function (concave iff quadratic terms matrix is PSD) :

$$f_{u,\alpha}(x) = f(x) - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - \sum_{i=1}^{n} \alpha_i x_i \left(\sum_{j=1}^{n} x_j - k\right)$$

Reformulation 2 [Faye, Roupin, 4OR, 2007 ; Billionnet, Elloumi and Lambert, Math. Prog., 2012]

$$f_{u,v}(x) = f(x) - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - v \left(\sum_{j=1}^{n} x_j - k\right)^2$$

- identical bounds.
- consumes the least amount of computation times.

QCR : A two-phase method

Best values for the parameters $u \in \mathbb{R}^n$ and $v \in \mathbb{R}$

Solve the semi-definite relaxation of the problem $(E - kQKP)$:

$$E - kQKP_{(SDP1)} \begin{cases} \max & \sum_{i=1}^n \sum_{j=1}^n c_{ij} X_{ij} \\ \text{s.t.} & X_{ii} = x_i \qquad\qquad i = 1, \ldots, n \quad (u^*) \\ & \sum_{i=1}^n \sum_{j=1}^n X_{ij} - 2k \sum_{j=1}^n x_j = -k^2 \qquad (v^*) \\ & \sum_{j=1}^n a_j x_j \leq b \\ & \sum_{j=1}^n x_j = k \\ & \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 \\ & x \in \mathbb{R}^n,\ X \in \mathbb{R}^{n \times n} \end{cases}$$

An improved convex 0-1 quadratic reformulation for $(E-kQKP)$ (1/4) [Billionnet, Elloumi and Lambert, Math. Prog., 2012 and Ji, Zheng and Sun, Pacific Journal of Optimization, 2012]

From

$$f(x) = x^t Cx$$

Consider the decomposition of $f(x)$

$$f(x) = x^t(C-M)x + x^t Mx,$$

where

$$C - M \preceq 0$$

and

$$M = Diag(u) + P + N$$

with $u \in \mathbb{R}^n$, $P \in \mathbb{R}_+^{n\times n}$ and $N \in \mathbb{R}_-^{n\times n}$

An improved convex 0-1 quadratic reformulation for $(E-kQKP)$ (2/4)

$$f(x) = x^t(C - Diag(u) - P - N)x + x^t(Diag(u) + P + N)x$$

$$= x^t(C - Diag(u) - P - N)x + u^t x + \sum_{i,j=1}^n [P_{ij}x_i x_j + N_{ij}x_i x_j]$$

$$= x^t(C - Diag(u) - P - N)x + u^t x + \sum_{i,j=1}^n [P_{ij}s_{ij} - N_{ij}t_{ij}]$$

where
$s_{ij} = min(x_i, x_j)$,
$t_{ij} = max(0, x_i + x_j - 1)$
$(x_i x_j = min(x_i, x_j) = max(0, x_i + x_j - 1)$ for any $x_i, x_j \in \{0, 1\})$

An improved convex 0-1 quadratic reformulation for $(E-kQKP)$ (3/4)

Relax
$s_{ij}=min(x_i,x_j)$ to two linear inequalities $s_{ij}\leq x_i$ and $s_{ij}\leq x_j$
and
$t_{ij}=-max(0,x_i+x_j-1)$ to $t_{ij}\leq 0$ and $t_{ij}\leq 1-x_i-x_j$.
without affecting the optimal solution of the problem.

The reformulation is equivalent to the convex 0-1 quadratic program :

$$\begin{cases} \max & f(x)=x^t(C-Diag(u)-P-N)x+u^tx+\sum_{i,j=1}^n[P_{ij}s_{ij}+N_{ij}t_{ij}] \\ \text{s.t.} & \sum_{j=1}^n a_jx_j\leq b \\ & \sum_{j=1}^n x_j=k \\ & s_{ij}\leq x_i,\ s_{ij}\leq x_j \quad i,j=1,\dots,n \\ & t_{ij}\leq 0,\ t_{ij}\leq 1-x_i-x_j \quad i,j=1,\dots,n \\ & x_j\in\{0,1\} \quad j=1,\dots,n \end{cases}$$

An improved convex 0-1 quadratic reformulation for $(E - kQKP)$ (4/4)

The optimal parameters (u^*, v^*, P^*, N^*) can be found by solving a SDP problem :

$$
(E - kQKP_{SDP2}) \left\{
\begin{array}{llll}
\max & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} X_{ij} & & \\
\text{s.t.} & X_{ii} = x_i & i = 1, \dots, n & (u^*) \\
& \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} - 2k \sum_{j=1}^{n} x_j = -k^2 & i = 1, \dots, n & (v^*) \\
& \sum_{j=1}^{n} a_j x_j \leq b & & \\
& \sum_{j=1}^{n} x_j = k & & \\
& X_{ij} \leq x_i, \; X_{ij} \leq x_j & i,j = 1, \dots, n & (P^*) \\
& X_{ij} \geq x_i + x_j - 1, \; X_{ij} \geq 0 & i,j = 1, \dots, n & (N^*) \\
& \begin{pmatrix} 1 & x^t \\ x & X \end{pmatrix} \succeq 0 & & \\
& x \in \mathbb{R}^n, \; X \in S_n & &
\end{array}
\right.
$$

The application of the QCR method leads to the following reformulation of $(kQKP)$:

$$(kQKP^{u^*,v^*}) \quad \max \ f_{u^*,v^*}(x) = f(x) - \sum_{i=1}^{n} u_i^* \left(x_i^2 - x_i\right) - v^* \left(\sum_{j=1}^{n} x_j - k\right)^2 \tag{23}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_j x_j \leq b \tag{24}$$

$$\sum_{j=1}^{n} x_j = k \tag{25}$$

$$x \in \{0, 1\} \tag{26}$$

The application of the MQCR method leads to the following reformulation of $(kQKP)$:

$$
\begin{aligned}
(kQKP_M^{u^*,v^*,P^*,N^*}) \quad \max \quad & f_{u^*,v^*,P^*,N^*}(x,s,t) \\
& = x^T(C - Diag(u) - P - N)x + u^T x + \sum_{i,j=1}^{n} [P_{ij}s_{ij} + N_{ij}t_{ij}] \\
& - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - v \left(\sum_{j=1}^{n} x_j - k\right)^2 \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_j x_j \leq b \\
& \sum_{j=1}^{n} x_j = k \\
& s_{ij} \leq x_i, \; s_{ij} \leq x_j \qquad i,j = 1,\ldots,n \\
& t_{ij} \leq 0, \; t_{ij} \leq 1 - x_i - x_j \qquad i,j = 1,\ldots,n \\
& x_j \in \{0,1\} \qquad j = 1,\ldots,n
\end{aligned}
$$

DWR with a quadratic master problem

Decompositions with a master objective function of the shape $f(x) = x^\top Q x + L^\top x$.

- PRO : linear pricing problem.
- CON : the objective function must be convex. $\Rightarrow$ DWR must be applied to $(kQKP^{u^*,v^*})$ or to $(kQKP_M^{u^*,v^*,P^*,N^*})$

DWR with a quadratic master problem

When applied to $(kQKP^{v^*,u^*})$, DWR gives the following model :

$$
\begin{aligned}
(kQKP^{u^*,v^*}_{DWR(1)}) \quad \max \quad & f_{u^*,v^*}(x) && && (27)\\
\text{s.t.} \quad & \sum_{j=1}^{n} x_j = k && && (28)\\
& x_j = \sum_{p \in \mathcal{P}_{(1)}} x_j^p \lambda^p && j = 1, \dots, n & [\tau_j] & \quad (29)\\
& \sum_{p \in \mathcal{P}_{(1)}} \lambda^p = 1 && & [\beta] & \quad (30)\\
& x_j \in \{0, 1\} && j = 1, \dots, n & & \quad (31)\\
& \lambda^p \geq 0 && p \in \mathcal{P}_{(1)} & & \quad (32)
\end{aligned}
$$

$\mathcal{P}_{(1)}$ is the set of extreme points of $conv\{x | \sum_{j=1}^{n} a_j x_j \leq b, x \in \{0, 1\}\}$

DWR with a quadratic master problem

The following reformulations can be obtained :

- $(kQKP^{u^*,v^*}_{DWR(1)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1)})$: knapsack constraint is convexified
- $(kQKP^{u^*,v^*}_{DWR(2)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(2)})$: cardinality constraint is convexified
- $(kQKP^{u^*,v^*}_{DWR(1-2)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1-2)})$: both constraints are convexified

DWR with a quadratic pricing problem

Decompositions with a master objective function of the shape $f(x) = c^\top \lambda$.

- The pricing problem is quadratic.
- No restriction is imposed on the convexity of the objective function of the starting compact model.

DWR with a quadratic pricing problem

When applied to $(kQKP)$, DWR gives the following model :

$$
\begin{aligned}
(kQKP_{\overline{DWR}(1)}) \quad \max \quad & \sum_{p \in \mathcal{P}_{(1)}} c^p \lambda^p && && (33)\\
\text{s.t.} \quad & x_j = \sum_{p \in \mathcal{P}_{(1)}} x_j^p \lambda^p && j = 1, \ldots, n && [\phi_j] \quad (34)\\
& \sum_{j=1}^{n} x_j = k && && [\gamma] \quad (35)\\
& \sum_{p \in \mathcal{P}_{(1)}} \lambda^p = 1 && && [\theta] \quad (36)\\
& x_j \in \{0, 1\} && j = 1, \ldots, n && (37)\\
& \lambda^p \geq 0 && p \in \mathcal{P}_{(1)} && (38)
\end{aligned}
$$

with $\mathcal{P}_{(1)}$ being the set of extreme points of $conv\{x | \sum_{j=1}^{n} a_j x_j \leq b, x \in \{0, 1\}\}$ and with $c^p = \sum_{j=1}^{n} c_{ij} x_i^p x_j^p$.

DWR with a quadratic pricing problem

The following reformulations can be obtained :

- $(kQKP_{\overline{DWR}(1)})$, $(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,\overline{DWR}(1)})$: knapsack constraint is convexified
- $(kQKP_{\overline{DWR}(2)})$, $(kQKP^{u^*,v^*}_{\overline{DWR}(2)})$, $(kQKP^{u^*,v^*,P^*,N^*}_{M,\overline{DWR}(2)})$: cardinality constraint is convexified

Plan

Experimental environment

- Carried out on an Intel i7-2600 quad core 3.4 GHz with 8 GB of RAM, using only one core
- CSDP integrated into COIN-OR for solving SDP programs
- CPLEX 12.6.2 with default settings
- Average values over 10 instances
- $n \in \{50, 60, \dots, 100\}$
- $k \in [1, n/4]$, $b \in [50, 30k]$, a_j, $c_{ij} \in [1, 100]$

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP_M^{u^*,v^*,P^*,N^*})$ Gap	$(kQKP_M^{u^*,v^*,P^*,N^*})$ Time	$(kQKP_{DWR(1)}^{u^*,v^*})$ Gap	$(kQKP_{DWR(1)}^{u^*,v^*})$ Time	$(kQKP_{M,DWR(1-2)}^{u^*,v^*,P^*,N^*})$ Gap	$(kQKP_{M,DWR(1-2)}^{u^*,v^*,P^*,N^*})$ Time
50	*25*	102,65	0,02	30,89	1,05	38,36	1,97		
	50	150,56	0,06	25,25	0,94	31,05	3,04		
	75	230,29	0,12	105,16	1,09	114,26	1,55		
	100	356,48	0,24	63,81	0,99	69,45	1,56		
60	*25*	60,76	0,04	130,92	0,04	149,25	1,55		
	50	93,73	0,11	15,08	2,61	19,48	3,86		
	75	212,67	0,25	141,08	2,09	151,22	1,67		
	100	284,47	0,44	62,30	2,08	69,06	2,39		
70	*25*	130,23	0,06	38,03	5,11	46,84	4,82		
	50	177,07	0,19	72,81	4,27	80,44	6,83		
	75	382,36	0,44	56,26	3,45	63,77	3,25		
	100	252,21	0,77	60,23	3,97	66,08	3,91		
80	*25*	111,24	0,08	34,05	7,98	41,90	5,64		
	50	271,64	0,26	55,44	9,59	64,09	4,67		
	75	313,33	0,66	83,58	7,42	92,31	4,64		
	100	424,40	1,22	43,25	8,88	49,18	5,32		
90	*25*	118,45	0,13	112,80	13,75	129,31	4,74		
	50	248,57	0,48	83,15	12,38	92,19	4,52		
	75	388,68	1,06	37,90	5,63	42,13	6,95		
	100	390,02	1,85	26,54	11,27	29,94	5,58		
100	*25*	169,43	0,16	73,90	23,49	82,78	6,80		
	50	145,72	0,49	17,38	28,06	21,83	8,58		
	75	260,26	1,25	21,67	18,37	27,22	6,23		
	100	472,97	2,44	98,60	23,14	106,58	6,37		
Avg		**239,51**	**0,53**	**62,09**	**8,24**	**69,95**	**4,43**		

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP_M^{u^*,v^*,P^*,N^*})$ Gap	$(kQKP_M^{u^*,v^*,P^*,N^*})$ Time	$(kQKP_{DWR(1)}^{u^*,v^*})$ Gap	$(kQKP_{DWR(1)}^{u^*,v^*})$ Time	$(kQKP_{M,DWR(1-2)}^{u^*,v^*,P^*,N^*})$ Gap	$(kQKP_{M,DWR(1-2)}^{u^*,v^*,P^*,N^*})$ Time
50	*25*	102,65	0,02	30,89	1,05	38,36	1,97	29,15	9,30
	50	150,56	0,06	25,25	0,94	31,05	3,04	23,66	9,71
	75	230,29	0,12	105,16	1,09	114,26	1,55	100,88	8,06
	100	356,48	0,24	63,81	0,99	69,45	1,56	61,58	6,19
60	*25*	60,76	0,04	130,92	0,04	149,25	1,55	126,07	10,89
	50	93,73	0,11	15,08	2,61	19,48	3,86	14,19	19,05
	75	212,67	0,25	141,08	2,09	151,22	1,67	136,22	8,99
	100	284,47	0,44	62,30	2,08	69,06	2,39	60,72	14,82
70	*25*	130,23	0,06	38,03	5,11	46,84	4,82	36,52	33,25
	50	177,07	0,19	72,81	4,27	80,44	6,83	70,77	54,37
	75	382,36	0,44	56,26	3,45	63,77	3,25	54,57	22,19
	100	252,21	0,77	60,23	3,97	66,08	3,91	58,94	26,50
80	*25*	111,24	0,08	34,05	7,98	41,90	5,64	32,87	71,19
	50	271,64	0,26	55,44	9,59	64,09	4,67	53,65	43,98
	75	313,33	0,66	83,58	7,42	92,31	4,64	81,47	43,42
	100	424,40	1,22	43,25	8,88	49,18	5,32	42,06	34,95
90	*25*	118,45	0,13	112,80	13,75	129,31	4,74	109,63	44,66
	50	248,57	0,48	83,15	12,38	92,19	4,52	81,65	66,75
	75	388,68	1,06	37,90	5,63	42,13	6,95	37,12	102,54
	100	390,02	1,85	26,54	11,27	29,94	5,58	26,02	113,84
100	*25*	169,43	0,16	73,90	23,49	82,78	6,80	72,72	99,90
	50	145,72	0,49	17,38	28,06	21,83	8,58	17,19	219,77
	75	260,26	1,25	21,67	18,37	27,22	6,23	21,50	158,30
	100	472,97	2,44	98,60	23,14	106,58	6,37	96,82	127,40
Avg		**239,51**	**0,53**	**62,09**	**8,24**	**69,95**	**4,43**	**60,25**	**56,25**

Even optimizing over the convex hull of the feasible region does not improve significantly the dual bound provided by DWR with a quadratic master.
The reasons for this behaviour are :

- The optimal solution of the continuous relaxation is attained in the interior of the convex hull of the feasible region.

n	50				60				70				80				90				100			
%	25	50	75	100	25	50	75	100	25	50	75	100	25	50	75	100	25	50	75	100	25	50	75	100
	0	2	0	0	0	2	0	0	0	2	0	0	0	1	0	0	0	1	0	0	0	1	0	0

TABLE: $(kQKP_{DWR})$, root node, quadratic master, Number of optimal solution in the interior of the convex hull of the feasible region

- The objective function over the feasible region is “flat” (due to the convexification via u^* and v^*).

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1-2)})$ Gap	$(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1-2)})$ Time	$(kQKP_{\overline{DWR}(1)})$ Gap	$(kQKP_{\overline{DWR}(1)})$ Time	$(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$ Gap	$(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$ Time
50	*25*	102,65	0,02	29,15	9,30	19,43	0,30		
	50	150,56	0,06	23,66	9,71	28,97	0,08		
	75	230,29	0,12	100,88	8,06	119,44	0,05		
	100	356,48	0,24	61,58	6,19	96,34	0,04		
60	*25*	60,76	0,04	126,07	10,89	79,77	0,14		
	50	93,73	0,11	14,19	19,05	13,13	0,25		
	75	212,67	0,25	136,22	8,99	157,01	0,07		
	100	284,47	0,44	60,72	14,82	77,53	0,07		
70	*25*	130,23	0,06	36,52	33,25	16,80	1,20		
	50	177,07	0,19	70,77	54,37	47,85	4,30		
	75	382,36	0,44	54,57	22,19	65,16	0,15		
	100	252,21	0,77	58,94	26,50	111,61	0,09		
80	*25*	111,24	0,08	32,87	71,19	11,67	210,59		
	50	271,64	0,26	53,65	43,98	42,69	0,77		
	75	313,33	0,66	81,47	43,42	95,92	14,00		
	100	424,40	1,22	42,06	34,95	74,78	0,13		
90	*25*	118,45	0,13	109,63	44,66	57,50	1692,40		
	50	248,57	0,48	81,65	66,75	63,79	190,64		
	75	388,68	1,06	37,12	102,54	59,35	15,16		
	100	390,02	1,85	26,02	113,84	61,68	0,20		
100	*25*	169,43	0,16	72,72	99,90	41,07	926,10		
	50	145,72	0,49	17,19	219,77	14,75	577,86		
	75	260,26	1,25	21,50	158,30	47,82	3,77		
	100	472,97	2,44	96,82	127,40	154,35	0,36		
Avg		**198,20**	**0,32**	**60,25**	**57,02**	**54,56**	**202,10**		

n	δ	$(kQKP)$ Gap	$(kQKP)$ Time	$(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1-2)})$ Gap	$(kQKP^{u^*,v^*,P^*,N^*}_{M,DWR(1-2)})$ Time	$(kQKP_{\overline{DWR}(1)})$ Gap	$(kQKP_{\overline{DWR}(1)})$ Time	$(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$ Gap	$(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$ Time
50	*25*	102,65	0,02	29,15	9,30	19,43	0,30	0,00	
	50	150,56	0,06	23,66	9,71	28,97	0,08	0,00	
	75	230,29	0,12	100,88	8,06	119,44	0,05	0,00	
	100	356,48	0,24	61,58	6,19	96,34	0,04	0,00	
60	*25*	60,76	0,04	126,07	10,89	79,77	0,14	0,00	
	50	93,73	0,11	14,19	19,05	13,13	0,25	0,00	
	75	212,67	0,25	136,22	8,99	157,01	0,07	0,00	
	100	284,47	0,44	60,72	14,82	77,53	0,07	0,00	
70	*25*	130,23	0,06	36,52	33,25	16,80	1,20	0,00	
	50	177,07	0,19	70,77	54,37	47,85	4,30	0,00	
	75	382,36	0,44	54,57	22,19	65,16	0,15	0,00	
	100	252,21	0,77	58,94	26,50	111,61	0,09	0,00	
80	*25*	111,24	0,08	32,87	71,19	11,67	210,59	0,00	
	50	271,64	0,26	53,65	43,98	42,69	0,77	0,00	
	75	313,33	0,66	81,47	43,42	95,92	14,00	0,00	
	100	424,40	1,22	42,06	34,95	74,78	0,13	0,00	
90	*25*	118,45	0,13	109,63	44,66	57,50	1692,40	0,00	
	50	248,57	0,48	81,65	66,75	63,79	190,64	0,00	
	75	388,68	1,06	37,12	102,54	59,35	15,16	0,00	
	100	390,02	1,85	26,02	113,84	61,68	0,20	0,00	
100	*25*	169,43	0,16	72,72	99,90	41,07	926,10	0,00	
	50	145,72	0,49	17,19	219,77	14,75	577,86	0,00	
	75	260,26	1,25	21,50	158,30	47,82	3,77	0,00	
	100	472,97	2,44	96,82	127,40	154,35	0,36	0,00	
Avg		**198,20**	**0,32**	**60,25**	**57,02**	**54,56**	**202,10**	**0,00**	

An alternative look at $(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$

Pricing problem

$$
\begin{aligned}
(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*, \beta^*)) \quad \max \;\; & f_{u,v}(x) + {\tau^*}^\top x + \beta^* && (39)\\
\text{s.t.} \;\; & Ax \le b && (40)\\
& x_j \in \{0,1\} \qquad j = 1, \dots, n && (41)
\end{aligned}
$$

$$
f_{u,v}(x) = f(x) - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - v \left(\sum_{j=1}^{n} x_j - k\right)^2
$$

$$
f_{u,v}(x) = x^\top \tilde{Q} x + \tilde{L}^\top x - v \left(\sum_{j=1}^{n} x_j - k\right)^2
$$

An alternative look at $(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$

Pricing problem

$$(\Pi_{BQP_{\overline{DWR}(A'')}}(\tau^*,\beta^*)) \quad \max \quad f_{u,v}(x) + {\tau^*}^\top x + \beta^* \tag{42}$$

$$\text{s.t.} \quad Ax \leq b \tag{43}$$

$$x_j \in \{0,1\} \qquad j = 1,\ldots,n \tag{44}$$

$$f_{u,v}(x) = f(x) - \sum_{i=1}^{n} u_i \left(x_i^2 - x_i\right) - v\left(\sum_{j=1}^{n} x_j - k\right)^2$$

$$f_{u,v}(x) = x^\top \tilde{Q} x + \tilde{L}^\top x - v\left(\sum_{j=1}^{n} x_j - k\right)^2$$

The coefficient v can be viewed as penalty term for the cardinality constraint.

An alternative look at $(kQKP^{u^*,v^*}_{DWR(1)})$

- Solving directly

$$
\begin{aligned}
\max \quad & x^\top \tilde{Q} x + \tilde{L}^\top x - v \left(\sum_{j=1}^{n} x_j - k \right)^2 \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_j x_j \leq b \\
& x_j \in \{0,1\} \qquad\qquad j = 1, \ldots, n
\end{aligned}
$$

is computationally non convenient.

An alternative look at $(kQKP^{u^*,v^*}_{\overline{DWR}(1)})$

$$\begin{aligned} \max \quad & x^\top \tilde{Q} x + \tilde{L}^\top x - v\left(\sum_{j=1}^{n} x_j - k\right)^2 + \tau^{*\top} x + \beta^* \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_j x_j \leq b \\ & x_j \in \{0,1\} \qquad\qquad j = 1,\ldots,n \end{aligned}$$

- The dual variables “lead” the pricing problem and make it significantly easier to solve.

An alternative look at $(kQKP^{u^*,v^*}_{DWR(1)})$

$$\max \quad x^\top \tilde{Q} x + \tilde{L}^\top x - v \left(\sum_{j=1}^{n} x_j - k \right)^2 + \tau^{*\top} x + \beta^*$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_j x_j \leq b$$

$$x_j \in \{0, 1\} \qquad j = 1, \ldots, n$$

- The dual variables “lead” the pricing problem and make it significantly easier to solve.
- IDEA : warm-start with columns heuristically generated ⇒ good dual variables.

n	δ	Time opt $(kQKP)$	Time opt $(kQKP_M^{u^*,v^*,P^*,N^*})$	Time root $(kQKP_{DWR(1)}^{u^*,v^*})$
50	*25*	3,7	1,0	40.1
	50	150,8	1.0	34.7
	75	213,1	0.7	14.7
	100	(8) 53,1	1,5	33,2
60	*25*	3,0	0,9	27.9
	50	282,4	1,4	32.5
	75	(9) 50,9	3,3	16.9
	100	(8) 188,6	3,2	39,3
70	*25*	23,7	3,4	64.4
	50	(6) 213,8	8,4	72.1
	75	(8) 873,0	16,2	55.1
	100	(4) 60,2	14,6	95,9
80	*25*	226,6	7,8	78.0
	50	(8) 872,9	26,8	66.7
	75	(5) 278,7	47,8	85.5
	100	(6) 1469,5	96,5	129,5
90	*25*	(9) 585,9	23,3	89.4
	50	(6) 3708,5	67,8	102.3
	75	(2) 2850,5	735,1	491.4
	100	(3) 146,2	180,4	195,7
100	*25*	2308,1	65,4	145.8
	50	(6) 1724,3	308,0	289.1
	75	(2) 4243,47	856,8	656.7
	100	(5) 2658,8	649,7	552,3

TABLE: $(kQKP_{DWR(1)}^{u^*,v^*})$ root node vs CPLEX

Plan

Conclusion and perspectives

Conclusion

- Dantzig-Wolfe Reformulation and convexification approaches can be used together.
- DWR is crucial to obtain strong dual bounds.

Perspectives

- Apply Dantzig-Wolfe Reformulation approach to other 0-1 quadratic problems.
- Derive an exact method (Branch & Price) combining Dantzig-Wolfe Reformulation and convexification.