COLUMN GENERATION with DYNAMIC CONSTRAINT AGGREGATION in the MASTER PROBLEM

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and team
GERAD

Overview

1. **COLUMN GENERATION**

   To deal with complex constraints

   and reduce number of variables

2. **TASK AGREGATION**

   To reduce number of constraints

3. **RESULTS ON :**

   Integrated pairing-rostering (50-80 tasks per columns)

   Large scale pairing (40 000 flights/month)

AIRLINE CREW SCHEDULING

COMPLEX COLECTIVE AGREMENT CONSTRAINTS

- **NON LINEAR**
- **NON CONVEX**

COMPLEX NON LINEAR COST

- **NON DECRESING FUNCTIONS**

GLOBAL CONSTRAINTS LINKING CREW MEMBERS

- **THOUSANDS OF FLIGHTS TO COVER**
- **BASE CONTRAINTS**

INTEGRALITY

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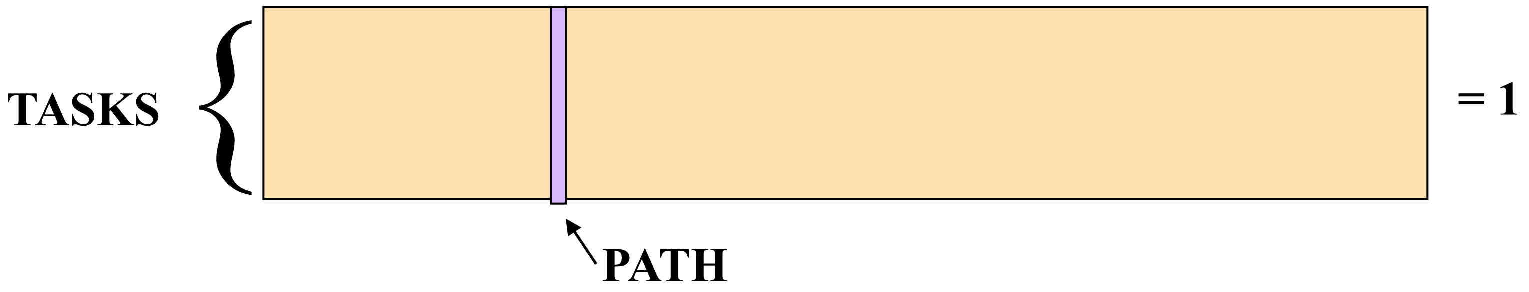
INTEGRALITY

LARGE SCALE AND COMPLEX PROBLEMS

SET PARTITIONING FORMULATION



VARIABLES = FEASIBLE PAIRINGS

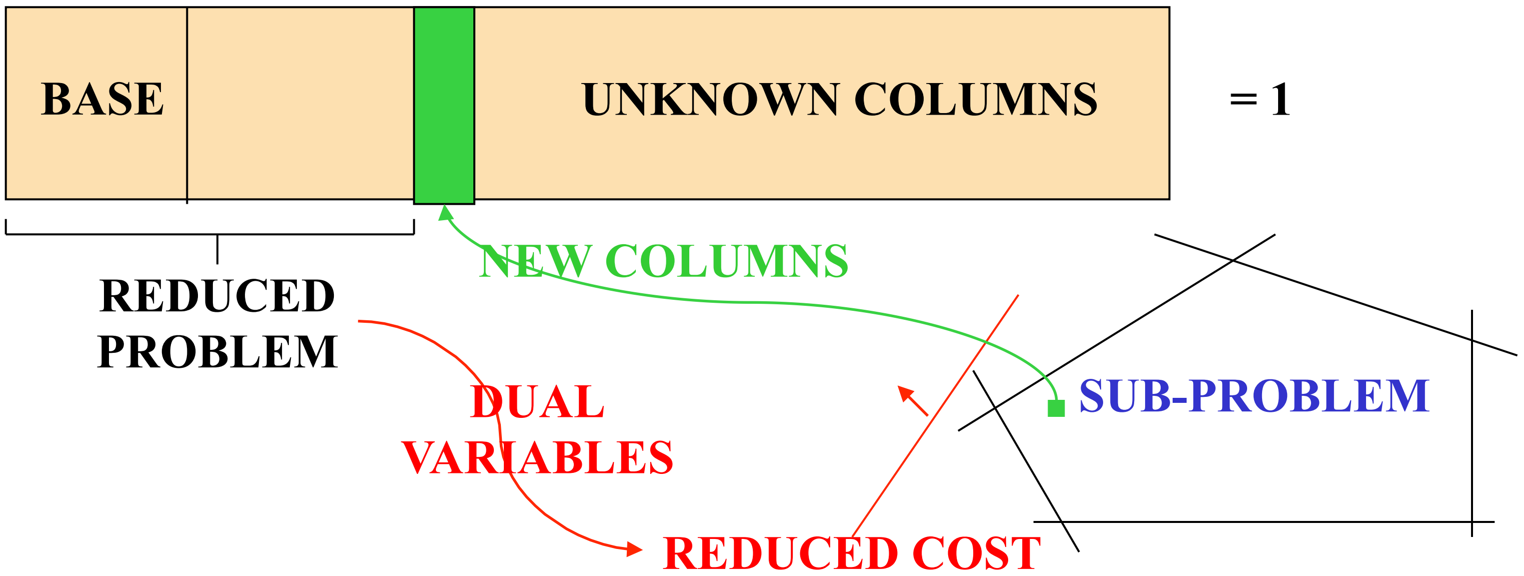
ADVANTAGES

- **SIMPLER CONSTRAINTS**
- **LESS CONSTRAINTS**
- **COMPLEX COSTS CAN BE PRECALCULATED**

DIFFICULTY

- **MILLIONS OF MILLIONS OF VARIABLES**

COLUMN GENERATION



- **SUB-PROBLEM**
  - **MIN COST PATH WITH RESSOURCES CONSTRAINTS**
  - **NON LINEAR, NON CONVEX BUT NON DECREASING FUNCTIONS**
  - **SOLVED AT INTEGRALITY BY DYNAMIC PROGRAMMING**

ADVANTAGES OF COLUMN GENERATION

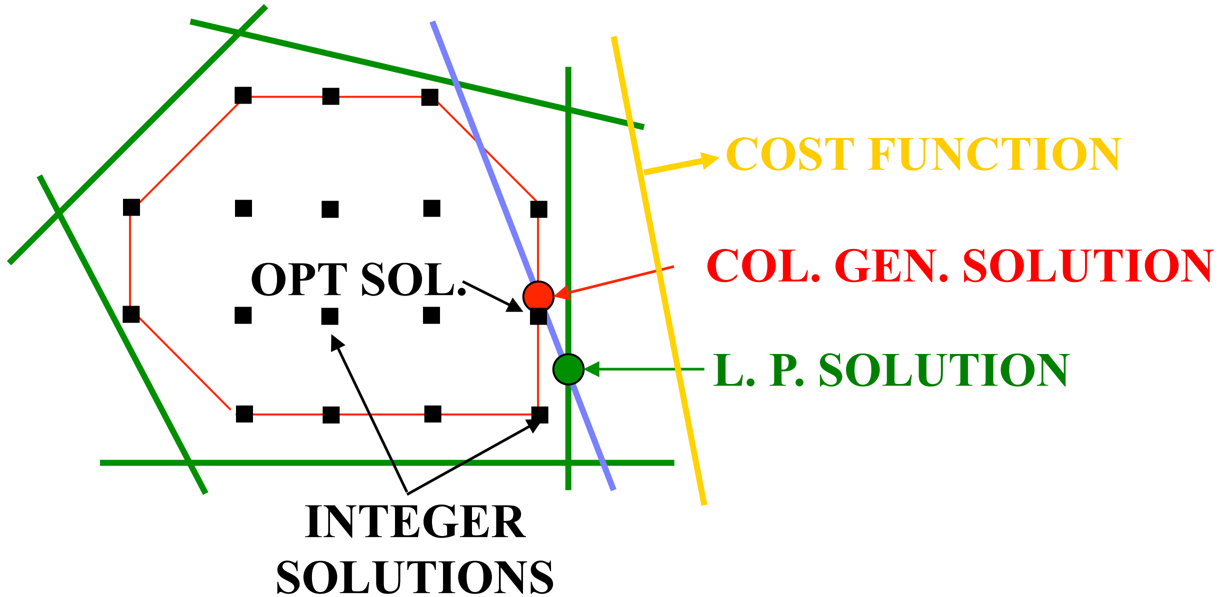
PROBLEM

MIN CX

$AX \leq a$

$BX \leq b$

X INTEGER



ADVANTAGES

- SOLVE SUB-PROBLEM AT INTEGRALITY
- REDUCE INTEGRALITY GAP
- EASIER BRANCH AND BOUND

WEAKNESS of COLUMN GENERATION for LARGE SCALE PROBLEMS

- **M.P. IS SLOW**
  - **SIMPLEX DEGENERATES WHEN SOLUTION IS CLOSE TO INTEGRALITY**
    - **PERTURBATIONS PRODUCE SMALL STEEPS**
    - **INTEGER POINTS METHODS PRODUCE MORE FRACTIONAL SOLUTIONS**

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WORST WHEN THE NUMBER OF TASKS PER COLUMN IS LARGE

TASK AGREGATION to OVERCOME WEAKNESS of COLUMN GENERATION for LARGE SCALE PROBLEMS

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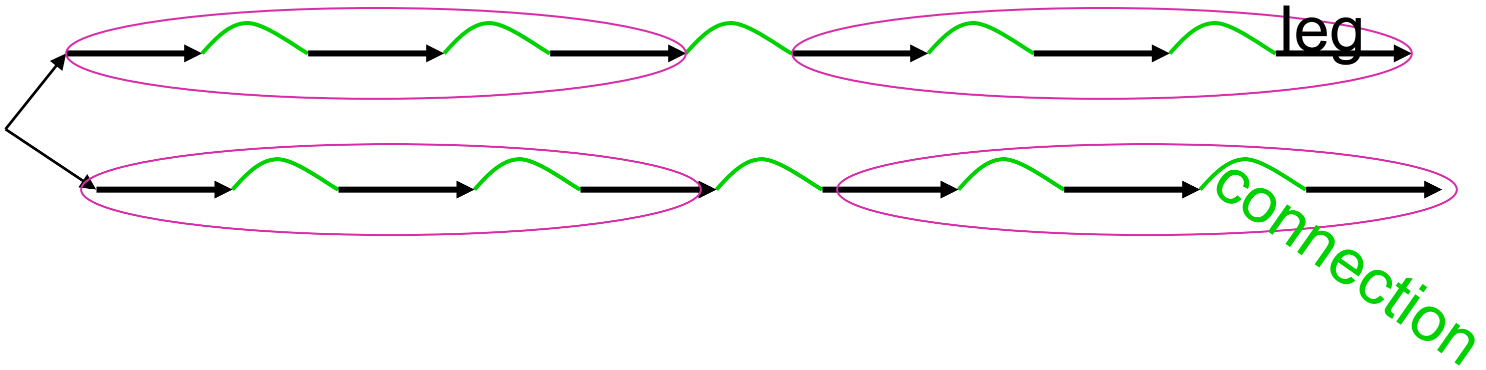
  - **REDUCE NUMBER OF FRACTIONAL VARIABLES**

TASK AGREGATION

- AGGREGATE TASKS IN CLUSTER



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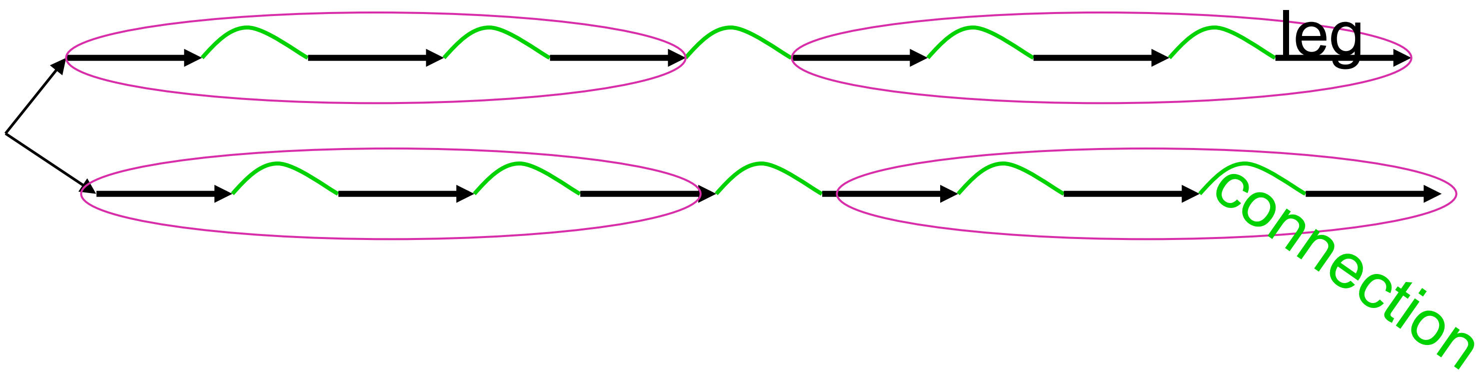
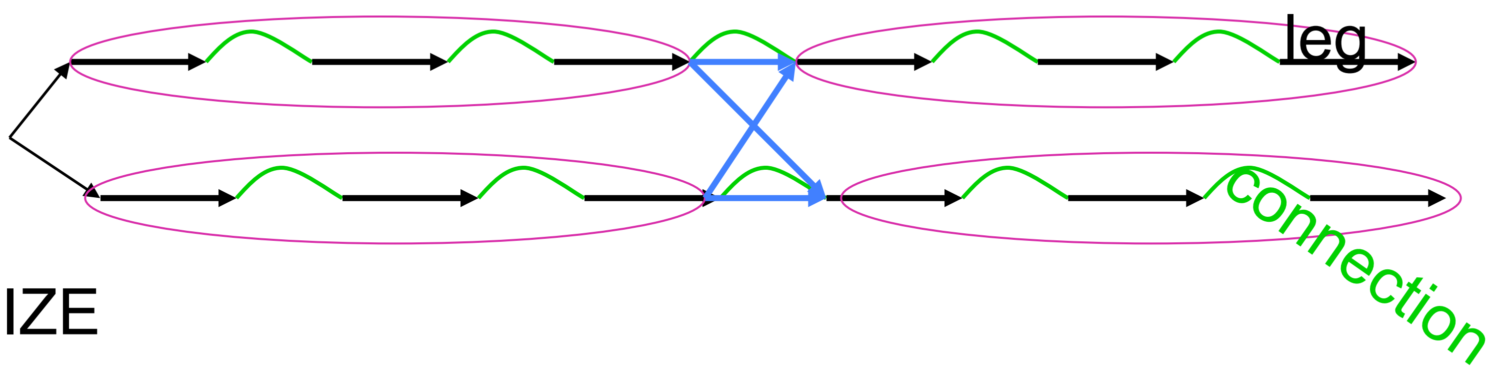


- CLUSTERS CAN COME FROM ANY INITIAL SOLUTION
  - Crew follow aircrafts
  - Any heuristic (windowing, reduced problems, lazy B+B)
  - Solution to reoptimize

TASK AGREGATION

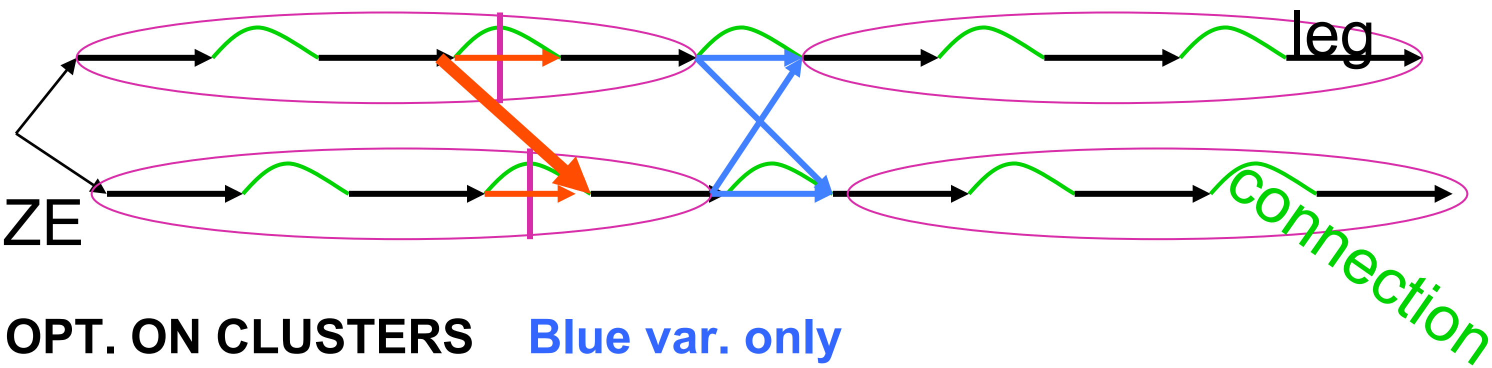
- AGGREGATE TASKS IN CLUSTERS



- OPTIMIZE
  - FAST OPT. ON CLUSTERS Blue var. only
    - Smaller master problem (one constraint per cluster)
    - Smaller sub-problem network (less arcs)

TASK AGREGATION

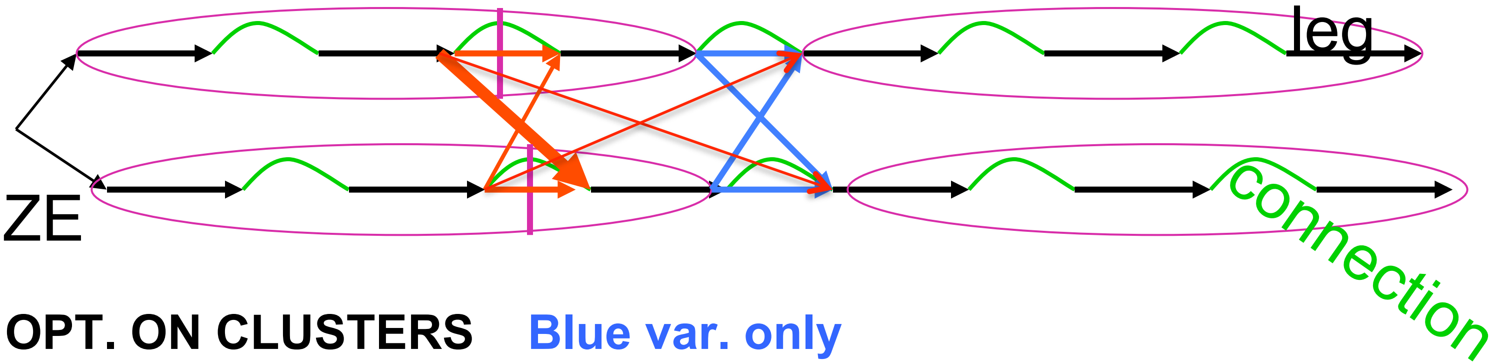
- AGGREGATE LEGS IN CLUSTERS



- OPTIMIZE
  - **FAST OPT. ON CLUSTERS** **Blue var. only**
  - **MODIFY CLUSTERING TO REACH OPTIMALITY**
    - **Add some red var.**
      - Arc with negative reduced cost indentified in the sub-problem
      - Solve the sub-problem with all arcs time to time

TASK AGREGATION

- AGGREGATE LEGS IN CLUSTERS



- OPTIMIZE
  - **FAST OPT. ON CLUSTERS** **Blue var. only**
  - **MODIFY CLUSTERING TO REACH OPTIMALITY**
    - **Add some red var.**
      - Arc with negative reduced cost indentified in the sub-problem
      - Start with partial pricing in the sub-problem (arcs between clusters with large dual variables)

DUAL VARIABLES FOR PRICING IN THE SUB-PROBLEM

(m tasks, p clusters, n variables)

- **p DUAL VARIABLES ARE GIVEN BY THE REDUCED PROBLEM**
  - **REDUCED COSTS OF p COLUMNS (without red arcs) ARE ZERO**
- **FIND m-p DUAL VARIABLES BY COMPLETING THE BASE**
  - **m-p SELECTED COLUMNS (with red arcs) WILL HAVE REDUCED COSTS = 0**
  - **C^{m-p}_{n-p} WAYS TO SELECT m-p VARIABLES**
  - **REDUCED COSTS OF OTHERS VARIABLES (with red arcs) WILL VARY DEEPLY**

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(m tasks, p clusters, n variables)

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  - **C^{m-p}_{n-p} WAYS TO SELECT m-p VARIABLES**
  - **REDUCED COSTS OF OTHERS VARIABLES (with red arcs) WILL VARY DEEPLY**
- **COMPLENENTARY PROBLEM (π_i are variables)**
  - **Z^{MAX} = MAX Z**
  - **REDUCED COSTS OF p COLUMNS (without red arcs) = 0**
  - **REDUCED COSTS OF COLUMNS (with red arcs generated up to date) $\geq$ Z**

QUALITY of the DUAL SOLUTION

- **COMPLENENTARY PROBLEM**
  - **Z^{MAX} = MAX Z**
  - **REDUCED COSTS OF p COLUMNS (without red arcs) = 0**
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- **PROPOSITION 1: THE COMPLENMENTARY PROBLEM PRODUCES CENTRAL REDUCED COSTS**

$\bar{c}_j = c_j - \Sigma \pi_i a_{ij}$ ⟶ decreasing linear relation π_i <-----> $\bar{c}_j$

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↓↑ ↑↓

$\Sigma \pi_i . 1 = \Sigma c_j x_j$ = constant ⟶ **decreasing some dual variables**

increase some other dual variables

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- **COMPLENENTARY PROBLEM**
  - $Z^{MAX} = \text{MAX } Z$
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  PRODUCES CENTRAL REDUCED COSTS

$\bar{c}_j = c_j - \Sigma \pi_i a_{ij}$ ⟶ **decreasing linear relation** π_i <-----> $\bar{c}_j$

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$\Sigma \pi_i . 1 = \Sigma c_j x_j = \text{constant}$ ⟶ **decreasing some dual variables**

increase some other dual variables

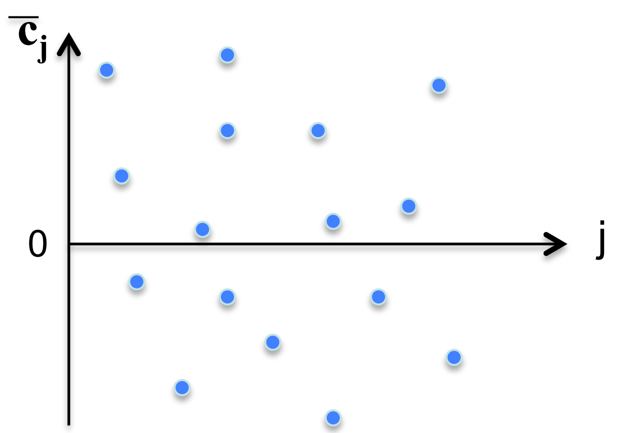
Maximizing the min reduced cost equalize the reduced costs

It stabilizes the column generation

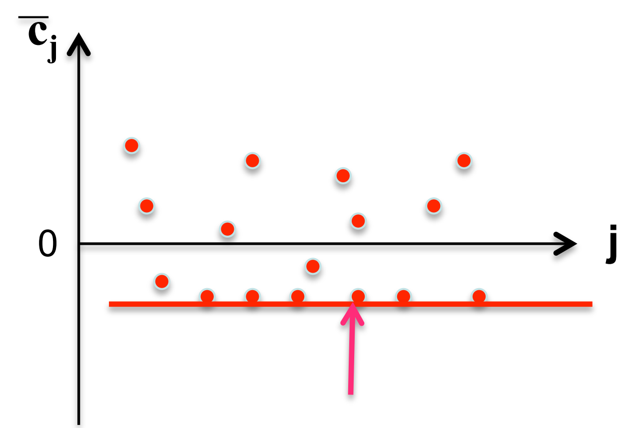
QUALITY of the DUAL SOLUTION

- **COMPLENENTARY PROBLEM**
  - Z^{MAX} = MAX Z
  - **REDUCED COSTS OF p COLUMNS (without red arcs) = 0**
  - **REDUCED COSTS OF COLUMNS (with red arcs generated up to date) ≥ Z**
- **PROPOSITION 2 : AT LESS m REDUCED COSTS = Z in the COMPLENMENTARY PROBLEM SOLUTION**

ANY DUAL SOLUTION



CP SOLUTION



INTERACTION BETWEEN: AGREG. PROB. COMP. PROB. and SUB-PROB.

- **Z^{MAX} SIGNIFICANTLY NEGATIVE**
  - **THE SOLUTION CAN BE SIGNIFICANTLY IMPROVED WITH EXISTING COLUMNS**

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- **Z^{MAX} SIGNIFICANTLY NEGATIVE**
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- **$Z^{MAX} = 0$**
  - **AGREGATED PROBLEM IS OPTIMAL FOR COLUMNS GENERATED UP TO DATE**

INTERACTION BETWEEN: AGREG. PROB. COMP. PROB. and SUB-PROB.

- **Z^{MAX} SIGNIFICANTLY NEGATIVE**
  - **THE SOLUTION CAN BE SIGNIFICANTLY IMPROVED WITH EXISTING COLUMNS**
- **$Z^{MAX} = 0$**
  - **AGREGATED PROBLEM IS OPTIMAL FOR COLUMNS GENERATED UP TO DATE**
- **$Z^{MAX} = 0$ or SMALL NEGATIVE VALUE**
  - **SOLVE THE SUB-PROBLEM**
    - **$Z^{SP} << Z^{MAX}$ ADD THE GENERATED COLUMNS TO IMPROVE THE SOLUTION**
    - **$Z^{SP} \cong Z^{MAX} \cong 0$ STOP. THE SOLUTION IS NEAR OPTIMAL**

EXPERIMENTATION

- **INTEGRATED PAIRING-ROSTERING PROBLEMS**
  - **MONTHLY PROBLEMS**
  - **MEDIUM SIZE: 1000 – 8000 FLIGHTS/MONTH**
- **GLOBAL OPTIMIZATION for PAIRING PROBLEMS**
  - **MEDIUM SIZE: 1000 – 8000 FLIGHTS/MONTH**
  - **LARGE SCALE 40 000 FLIGHTS/MONTH**

INTEGRATED CREW PLANNING

PAIRING	**COVER FLIGHTS WITH PAIRINGS** **(≈10-12 flights/column)**
↓	
ROSTERING	**COVER PAIRINGS WITH ROSTERS** **(≈ 5-7 pairings/column)**
INTEGRATED OPTIMISATION	**COVER FLIGHTS WITH ROSTERS, VERY DENSE COLUMNS** **(50-80 flights/columns)**

INTEGRATED PLANNING WITH CONSTRAINT AGGREGATION

- **SOLVE PAIRING PROBLEM**
- **OPTIMIZE ROSTERS WITH FIXED PAIRINGS**

CLASSICAL SEQUENTIAL APPROACH

- **AGGREGATE FLIGHTS IN THE SAME PAIRING**
- **REOPTIMISE with CONSTRAINTS AGREGATION CHANGING THE PAIRINGS**

CONSTRAINT AGREGATION ALGORITHM

- **(REACH OPTIMAL SOLUTION BY SOLVING SMALL PROBLEMS)**

RESULTS WITH COL. GENERATION AND CONSTRAINT AGREGATION

Problem		Sequential approach *			Integrated approach			
Instance	Flights	CPU (min)	Total cost	Number scheds	CPU (min)	CPU Int/Seq	Cost Svgs %	Scheds Svgs %
I-1	1011	4.0	767 754	33	6.4	1.73	5.74	6.06
I-2	1463	5.8	957 989	34	14.7	2.53	3.60	8.82
I-3	1793	11.4	1 313 391	47	34.7	3.04	3.07	8.51
I-4	5466	522.6	3 502 527	145	996.3	1.84	3.42	5.51
I-5	5639	231.9	4 835 090	247	1401.7	6.04	4.09	2.42
I-6	5755	260.0	5 144 122	223	783.0	3.01	6.75	6.27
I-7	7527	507.6	6 536 094	305	1518.2	2.99	1.50	0.98
Average						**3.02**	**4.02**	**5.51**

(*) NEAR OPTIMAL: L.P. TOLERANCE =10^{-6}, INTEGRALITY GAPS: PAIRING ~0.3%, BLOCS ~0.5%

GLOBAL OPT. for PAIRING PROB.

- **SOLVED FIRST with a COMMERCIAL SOLVER**

  ROLING HORIZON: 3 DAYS WINDOWS, 1 DAY OVERLAP

- **GLOBAL OPTIMIZATION for PAIRING PROBLEMS**

  MEDIUM SIZE: 1000 – 8000 FLIGHTS/MONTH

  LARGE SCALE 10 000 FLIGHTS/WEEK

- **ROLING HORIZON for PAIRING PROBLEMS**

  1 WEEK WINDOWS

  LARGE SCALE 40 000 FLIGHTS/MONTH

MID-SIZE MONTHLY PROBLEMS

Instance	Flights	Stations	CPU (min)	Gap (%)	No. Itrs	Degeneracy (%)	Fat reduction (%)	Deadheads reduction (%)	Reduction in cost (%)
I-1	1011	26	17	0.15	6150	87.33	59.55	77.5	4.52
I-2	1463	35	25	0.29	4667	79.42	32.11	100	1.08
I-3	1793	41	28	0.01	2417	81.03	19.34	100	3.70
I-4	5466	49	278	0.36	1675	80.50	3.2	15.62	0.37
I-5	5639	34	56	0.00	1540	74.35	27.33	18.30	0.38
I-6	5755	52	237	0.13	19279	83.25	72.97	27.69	1.94
I-7	7527	54	141	0.37	1261	79.03	40.37	12.76	1.36
Average							36.41	25.63	1.90

WEEKLY PROBLEMS

- CYCLIC JUNE 2014 > 10 000 FLIGHTS
- INITIAL SOLUTION 3552018, 170 DEAD HEADS

Coût	DH	Temps	Borne N0	Var. Fract. N0	Noeuds	Version
3515062.02 (**1.04%**)	144	6h01m	3477624.6	1370	166	30/06/2015
3429615.70 (**3.45%**)	133	4h38m	3473938.0	1347	39	04/02/2016

- CYCLIC JULY 2014 > 10 000 FLIGHTS
- INITIAL SOLUTION 55156445, 88 DEAD HEADS
- PENALTIES: BASE CONST. , DISTRIBUTION OF DURATION OF PAIRINGS,…

Coût	DH	Temps	Borne N0	Var. Fract. N0	Noeuds	Version
4827421.57 (**12.47%**)	101	4h09m	4966350.3	1093	65	30/06/2015
4572950.60 (**17.09%**)	109	4h39m	4806035.0	2149	255	04/02/2016

- LARGE SAVING ON PENALTIES: 33%, 44%

MONTLY PROBLEM > 40 000 FLIGHTS

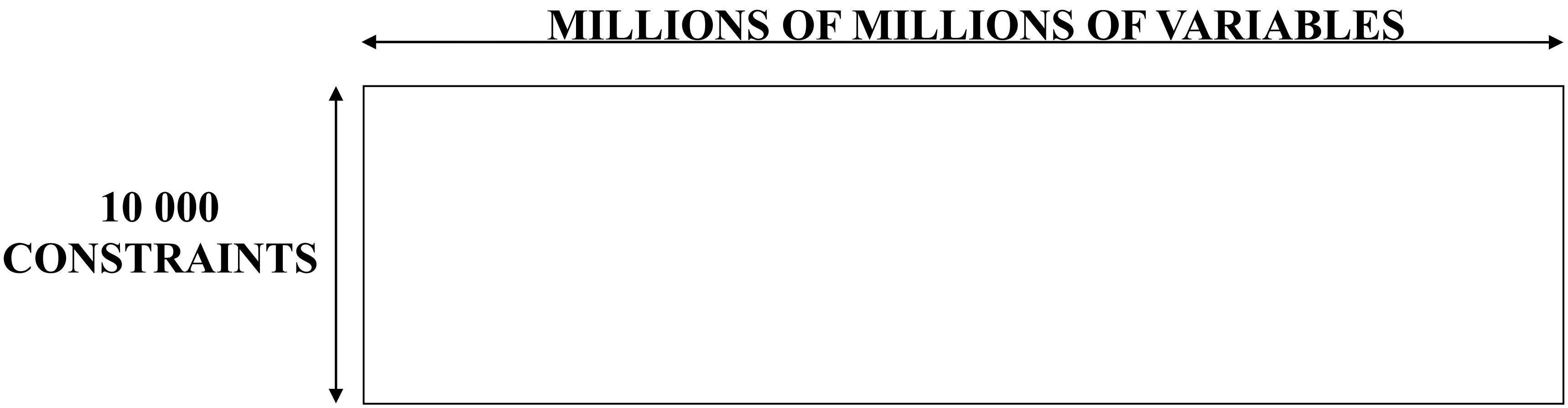
START WITH COPIES OF A WEEKLY SOLUTION
REOPTIMIZE WITH 5 WINDOWS OF 1 WEEK
WITHOUT GLOBAL CONSTRAINTS
WITH GLOBAL CONSTRAINTS

R= $$$$, S= SOFT COSTS, Contr. = PENALTY OF GLOBAL CONTR.

Coût Total	Coût R+S	Coût Contr.	Temps	Cycl. DCA	DCA	Version
15517694.27 (0%)	15517658.27 (0%)	36.0 (0%)	60h12m	*Non*	*Non*	30/06/15
14877947.62 (**4.12%**)	14877947.62 (**4.12%**)	0.0 (**100%**)	46h35m	*Non*	*Oui*	30/06/15
20796331.80 (0%)	16718702.15 (0%)	4077629.65 (0%)	92h03m	*Non*	*Non*	30/06/15
18318902.66 (**11.91%**)	15805630.64 (**5.46%**)	2513272.02 (**38.36%**)	50h48m	*Non*	*Oui*	30/06/15
16756416.41 (**19.43%**)	15194680.80 (**9.12%**)	1561735.61 (**61.70%**)	87h28m	*Oui*	*Non*	30/06/15
16454739.40 (**20.88%**)	15171286.98 (**9.26%**)	1283452.42 (**68.52%**)	65h29m	*Oui*	*Oui*	30/06/15
17786735.91 (**14.47%**)	15542190.98 (**7.04%**)	2244544.93 (**44.95%**)	38h39m	*Oui*	*Oui*	24/01/16
16072276.35 (**22.72%**)	15163078.72 (**9.30%**)	909197.63 (**77.70%**)	62h44m	*Oui*	*Oui*	04/02/16
15837013.10 (**23.85%**)	15147603.21 (**9.40%**)	689409.89 (**83.09%**)	61h20m	*Oui*	*Oui*	04/02/16

CONCLUSIONS ON REDUCING THE NUMBER OF CONSTRAINTS

WE CAN SOLVE HUGE PROBLEMS



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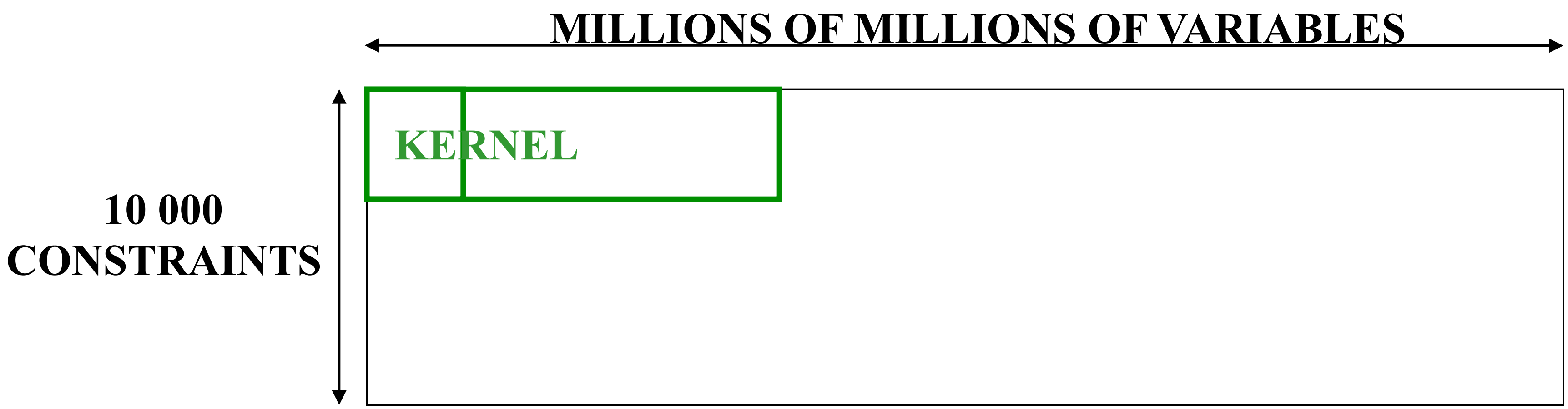

- **SOLVING ONLY A KERNEL PROBLEM MANY TIME**
  - **REDUCE NUMBER OF VARIABLES WITH COLUMN GENERATION**
  - **REDUCE NUMBER OF CONSTRAINTS WITH TASK AGGREGATION**
- **THE KERNEL PROBLEM IS ADJUSTED DYNAMICLY TO REACH OPTIMALITY**

REFERENCE LIST

REDUCING THE NUMBER OF CONSTRAINTS

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