The Fixed Charge Transportation Problem: A Strong Formulation Based On Lagrangian Decomposition and Column Generation

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Column generation 2016

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What is this talk about?

Strong lower bounding for the fixed charge transportation problem by

- Lagrangian decomposition:
  supply and demand side copies of the shipping variables
- Dual cutting plane method (column generation)

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Why?

- Lagrangian decomposition can give strong formulations
- Strong formulations of interest in column generation
- Combination not utilised in many papers
- Preliminary work to study the strength of the formulation:
  theoretically and empirically

Outline

Lagrangian decomposition / Lagrangian relaxation

Consider the problem

$$\text{(P)} \quad \min\{cx \text{ s.t. } Ax \le b,\ Cx \le d,\ x \in X\} =$$
$$\min\{cx \text{ s.t. } Ay \le b,\ Cx \le d,\ x = y,\ x \in X,\ y \in Y\},$$
where Y is such that $X \subseteq Y$

Lagrangian decomposition / Lagrangian relaxation

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Lagrangian decomposition

$$\text{(LD)} \quad \min\{cx + u(x-y) \text{ s.t. } Ay \leq b,\ Cx \leq d,\ x \in X,\ y \in Y\} =$$
$$\min\{(c+u)x \text{ s.t. } Cx \leq d,\ x \in X\} + \min\{-uy \text{ s.t. } Ay \leq b,\ y \in Y\}$$

Lagrangian decomposition / Lagrangian relaxation

Consider the problem

$$
\begin{aligned}
\text{(P)} \quad & \min\{cx \text{ s.t. } Ax \leq b,\ Cx \leq d,\ x \in X\} = \\
& \min\{cx \text{ s.t. } Ay \leq b,\ Cx \leq d,\ x = y,\ x \in X,\ y \in Y\}, \\
& \text{where } Y \text{ is such that } X \subseteq Y
\end{aligned}
$$

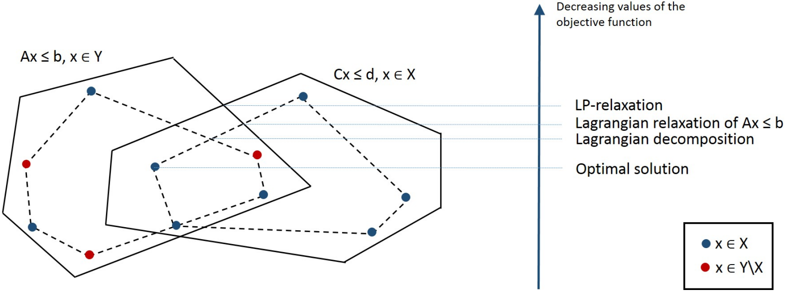
Lagrangian decomposition

$$
\begin{aligned}
\text{(LD)} \quad & \min\{cx + u(x - y) \text{ s.t. } Ay \leq b,\ Cx \leq d,\ x \in X,\ y \in Y\} = \\
& \min\{(c + u)x \text{ s.t. } Cx \leq d,\ x \in X\} + \min\{-uy \text{ s.t. } Ay \leq b,\ y \in Y\}
\end{aligned}
$$

Lagrangian relaxation w.r.t. one of the constraint groups, for example $Ax \leq b$

$$
\begin{aligned}
\text{(LR)} \quad & \min\{cx + v(Ax - b) \text{ s.t. } Cx \leq d,\ x \in X\} = \\
& \min\{(c + vA)x \text{ s.t. } Cx \leq d,\ x \in X\} + vb, \\
& \text{where } v \geq 0
\end{aligned}
$$

Strength of bounds



[*Guignard and Kim(1987)*]: The Lagrangian decomposition bound is as least as strong as the strongest of

- the Lagrangian relaxation bound when $Ax \leq b$ is relaxed
- the Lagrangian relaxation bound when $Cx \leq d$ is relaxed

and there is a chance that it is stronger!

Related work

Very few papers on Lagrangian decomposition and column generation:

- [*Pimentel et al.(2010)*]:
  The multi-item capacitated lot sizing problem
  - Branch-and-price implementations for two types of Lagrangian relaxation and for Lagrangian decomposition
  - Lagrangian decomposition: No gain in bound compared to Lagrangian relaxation when capacity is relaxed
- [*Letocart et al.(2012)*]:
  The 0-1 bi-dimensional knapsack problem and the generalised assignment problem
  - Illustrates the concept
  - No full comparison of bounds, conclusions not possible

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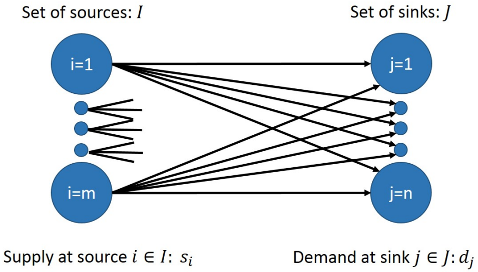
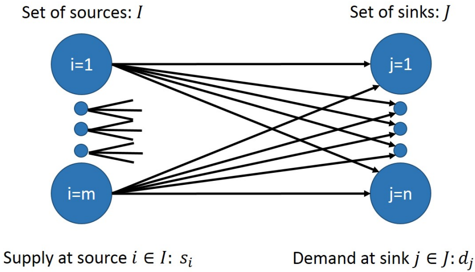
Our work this far:
Find an application where we gain in strength compared to the strongest obtainable from Lagrangian relaxation and investigate further ...

Fixed charge transportation problem (FCTP)



For each arc (i, j), $i \in I$, $j \in J$:
u_{ij} = $\min(s_i, d_j)$ = upper bound
c_{ij} = unit cost for shipping
f_{ij} = fixed cost for shipping

Fixed charge transportation problem (FCTP)



For each arc (i, j), $i \in I$, $j \in J$:
$u_{ij} = \min(s_i, d_j)$ = upper bound
c_{ij} = unit cost for shipping
f_{ij} = fixed cost for shipping

Variables:
x_{ij} = amount shipped from source i to sink j, $i \in I$, $j \in J$

Concave cost function:

$$g_{ij}(x_{ij}) = \begin{cases} f_{ij} + c_{ij}x_{ij} & \text{if } x_{ij} > 0 \\ 0 & \text{if } x_{ij} = 0 \end{cases} \qquad i \in I,\ j \in J$$

Fixed charge transportation problem (FCTP)

$$
\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{j \in J} g_{ij}(x_{ij}) \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} = s_i \qquad i \in I \\
& \sum_{i \in I} x_{ij} = d_j \qquad j \in J \\
& x_{ij} \geq 0 \qquad i \in I, \ j \in J
\end{aligned}
$$

- Polytope of feasible solutions, minimisation of concave objective ⇒ Optimal solution at an extreme point (can be non-global local optima at extreme points)
- MIP-formulation: A binary variable to indicate if there is flow on an arc or not

MIP-formulation of FCTP

$$
\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{j\in J}(c_{ij}x_{ij} + f_{ij}y_{ij}) \\
\text{s.t.} \quad & \sum_{j\in J} x_{ij} = s_i \qquad i \in I \\
& \sum_{i\in I} x_{ij} = d_j \qquad j \in J \\
& x_{ij} \le u_{ij}y_{ij} \qquad i \in I,\ j \in J \\
& x_{ij} \ge 0 \qquad i \in I,\ j \in J \\
& y_{ij} \in \{0,1\} \qquad i \in I,\ j \in J
\end{aligned}
$$

The reformulation: variable splitting

Supply and demand side duplicates of the shipping variables: x_{ij}^s and x_{ij}^d
Introduce a parameter ν: $0 \leq \nu \leq 1$

$$
\begin{array}{lll}
\min & \nu \sum_{i \in I} \sum_{j \in J} g_{ij}(x_{ij}^s) + (1-\nu) \sum_{j \in J} \sum_{i \in I} g_{ij}(x_{ij}^d) & \\
\text{s.t.} & x_{ij}^s = x_{ij}^d & i \in I,\ j \in J \\
& \sum_{j \in J} x_{ij}^s = s_i & i \in I \\
& \sum_{i \in I} x_{ij}^d = d_j & j \in J \\
& x_{ij}^s, x_{ij}^d \geq 0 & i \in I,\ j \in J,
\end{array}
$$

The reformulation: inner representation

Let each column correspond to an extreme point of a set

$$X_i^s = \{x_{ij}^s,\ j \in J \mid \sum_{j \in J} x_{ij}^s = s_i,\ 0 \leq x_{ij}^s \leq u_{ij},\ j \in J\}, \quad i \in I,$$

or of a set

$$X_j^d = \{x_{ij}^d,\ i \in I \mid \sum_{i \in I} x_{ij}^d = d_j,\ 0 \leq x_{ij}^d \leq u_{ij},\ i \in I\}, \quad j \in J$$

The flow from one source / to one sink is a convex combination of extreme point flows, introduce:

$\lambda_{ip}^s =$ convexity weight for extreme point $p \in \tilde{P}_i^s$ of set $X_i^s,\ i \in I$

and

$\lambda_{jp}^d =$ convexity weight for extreme point $p \in \tilde{P}_j^d$ of set $X_j^d,\ j \in J$

The reformulation: column oriented formulation

$$\min \quad \nu \sum_{i\in I}\sum_{j\in J} g_{ij}\left(\sum_{p\in\tilde{P}_i^s}\lambda_{ip}^s x_{ijp}^s\right) + (1-\nu)\sum_{j\in J}\sum_{i\in I} g_{ij}\left(\sum_{p\in\tilde{P}_j^d}\lambda_{jp}^d x_{ijp}^d\right)$$

$$\text{s.t.} \quad \sum_{p\in\tilde{P}_i^s} x_{ijp}^s\lambda_{ip}^s = \sum_{p\in\tilde{P}_j^d} x_{ijp}^d\lambda_{jp}^d \qquad i\in I,\ j\in J$$

$$\sum_{p\in\tilde{P}_i^s}\lambda_{ip}^s = 1 \qquad i\in I$$

$$\sum_{p\in\tilde{P}_j^d}\lambda_{jp}^d = 1 \qquad j\in J$$

$$\lambda_{ip}^s \geq 0 \qquad p\in\tilde{P}_i^s,\ i\in I$$

$$\lambda_{jp}^d \geq 0 \qquad p\in\tilde{P}_j^d,\ j\in J$$

The reformulation: approximating the objective

The objective is bound below by its linearisation:

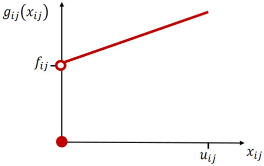
$$\nu \sum_{i\in I} \sum_{j\in J} g_{ij} \left(\sum_{p\in\tilde{P}_i^s} \lambda_{ip}^s x_{ijp}^s \right) + (1-\nu) \sum_{j\in J} \sum_{i\in I} g_{ij} \left(\sum_{p\in\tilde{P}_j^d} \lambda_{jp}^d x_{ijp}^d \right)$$

$$\geq \nu \sum_{i\in I} \sum_{p\in\tilde{P}_i^s} \left(\sum_{j\in J} g_{ij}(x_{ijp}^s) \right) \lambda_{ip}^s + (1-\nu) \sum_{j\in J} \sum_{p\in\tilde{P}_j^d} \left(\sum_{i\in I} g_{ij}(x_{ijp}^d) \right) \lambda_{jp}^d$$

What is lost by the linearisation?

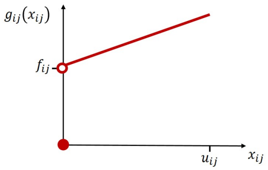
The reformulation: arc cost

True cost

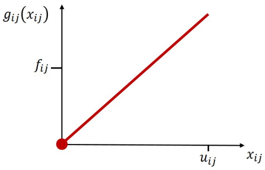


The reformulation: arc cost

True cost



In LP-relaxation of MIP-formulation



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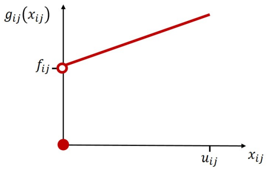

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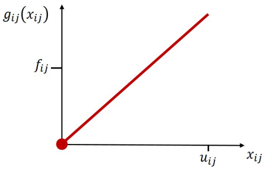
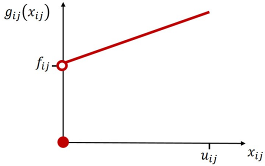


After our reformulation (here for supply side, similarly for sink side):

- Extreme points: true cost
- Non-extreme points:
  $x_{ij}^s = \sum_{p \in \tilde{P}_i^s:\ \lambda_{ip}^s > 0} x_{ijp}^s \lambda_{ip}^s$
  - if $x_{ij}^s > 0$ and there is $p \in \tilde{P}_i^s$: $\lambda_{ip}^s > 0$ and $x_{ip}^s = 0$, then f_{ij} is decreased by $\Delta = \sum_{p \in \tilde{P}_i^s:\ \lambda_{ip}^s > 0,\ x_{ip}^s = 0} f_{ij} \lambda_{ip}^s$
  - otherwise: true cost

The reformulation: arc cost

True cost



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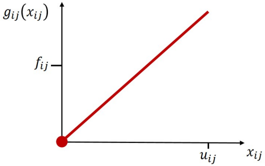

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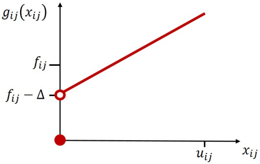

Example: For $x_{ij}^s = 4 = 6\frac{2}{3} + 0\frac{1}{3}$ we have $x_{ij1}^s = 6,\ \lambda_{i1}^s = \frac{2}{3};\ x_{ij2}^s = 0,\ \lambda_{i2}^s = \frac{1}{3};\ \Delta = \frac{1}{3} f_{ij}$

The reformulation: column generation subproblem

For a single source $i \in I$ (and similarly for the sinks):

$$\begin{aligned}
\min \quad & \sum_{j \in J} \left(\nu g_{ij}(x_{ij}^s) - \alpha_{ij} x_{ij}^s \right) - \beta_i \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij}^s = s_i \\
& x_{ij}^s \leq u_{ij} \qquad j \in J \\
& x_{ij}^s \geq 0 \qquad j \in J
\end{aligned}$$

- Concave minimization problem with an optimal solution at an extreme point of X_i^s
- MIP-formulation:
  Binary variable to indicate if there is flow on an arc or not

Theoretical strength

[*Guignard and Kim(1987)*]: The Lagrangian decomposition bound
(= convexification over source and sink side)
as least as strong as the strongest of

- the Lagrangian relaxation bound when $Ax \leq b$ is relaxed
  (= convexification over source side)
- the Lagrangian relaxation bound when $Cx \leq d$ is relaxed
  (= convexification over sink side)

and there is a chance that it is stronger!

Theoretical strength: type of strength?

The constraints in the MIP-formulation of the column generation subproblem (source side, similarly for the sink side):

$$\sum_{j \in J} x_{ij}^{s} = s_i$$

$$x_{ij}^{s} \leq u_{ij} y_{ij}^{s}, \; j \in J$$

$$y_{ij}^{s} \in \{0,1\}, \; j \in J$$

$$0 \leq x_{ij}^{s}, \; j \in J$$

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$$0 \leq x_{ij}^s, \; j \in J$$

$+$

Implied inequality:

$$\sum_{j \in J} u_{ij} y_{ij}^s \geq s_i$$

Theoretical strength: type of strength?

The constraints in the MIP-formulation of the column generation subproblem (source side, similarly for the sink side):

$$\begin{aligned} &\sum_{j \in J} x_{ij}^s = s_i \\ &x_{ij}^s \leq u_{ij} y_{ij}^s, \ j \in J \\ &y_{ij}^s \in \{0,1\}, \ j \in J \\ &0 \leq x_{ij}^s, \ j \in J \end{aligned} \quad + \quad \text{Implied inequality:} \ \sum_{j \in J} u_{ij} y_{ij}^s \geq s_i \quad = \quad \begin{aligned} &\sum_{j \in J} x_{ij}^s = s_i \\ &x_{ij}^s \leq u_{ij} y_{ij}^s, \ j \in J \\ &\sum_{j \in J} u_{ij} y_{ij}^s \geq s_i \\ &y_{ij}^s \in \{0,1\}, \ j \in J \\ &0 \leq x_{ij}^s, \ j \in J \end{aligned}$$

- Knapsack constraints over the binary variables
- At least (exactly?) that type of strength

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Empirical strength

Comparison:

- Convexification over both sides = Lagrangian decomposition
- Convexifiation over one side, see bounds obtained by [*Roberti et al.(2015)*] (A new formulation based on extreme flow patterns from each source; derive valid inequalities; exact branch and price; column generation to compute lower bounds)

Empirical strength

Table: Instance characteristics

instance			total	r. v. c.		r. f. c.	
type	ID	size	supply	inf	sup	inf	sup
Roberti-set3	*Table5-(1-10)*	70×70	682-819	0	0	200	800
Roberti-set3	*Table6-(1-10)*	70×70	705-760	7	32	200	800
Roberti-set3	*Table7-(1-10)*	70×70	654-808	18	83	200	800

r.v.c. = range of variable costs; r.f.c. = range of fixed costs

- Compare LBDs by comparing gaps (relative deviations) calculated as (UBD-LBD)/LBD in percent
- The UBDs used are the best known, most of them are verified to be optimal

Empirical strength

instance ID	cost ratio (%)	gap (%) FCTP-LP	Roberti et al.	our
Table5-1	100	19.4	9.7	6.0
Table5-2	100	16.7	10.6	5.2
Table5-3	100	16.3	10.0	5.7
Table5-4	100	18.9	8.8	4.5
Table5-5	100	17.3	9.1	4.5
Table5-6	100	18.5	9.1	4.2
Table5-7	100	18.8	10.4	5.8
Table5-8	100	17.2	8.0	4.4
Table5-9	100	16.2	8.1	4.4
Table5-10	100	16.8	9.7	5.4
AVG		**17.6**	**9.3**	**5.0**

instance ID	cost ratio (%)	gap (%) FCTP-LP	Roberti et al.	our
Table6-1	79	16.1	10.3	5.0
Table6-2	78	14.0	8.2	4.7
Table6-3	78	16.6	8.3	4.8
Table6-4	79	12.9	8.6	4.0
Table6-5	79	16.4	8.4	4.7
Table6-6	78	14.5	7.9	4.6
Table6-7	78	15.6	6.8	4.8
Table6-8	79	16.0	9.1	5.6
Table6-9	79	15.5	9.7	4.4
Table6-10	79	14.5	8.2	4.7
AVG		**15.2**	**8.6**	**4.7**

instance ID	cost ratio (%)	gap (%) FCTP-LP	Roberti et al.	our
Table7-1	58	10.0	5.4	3.5
Table7-2	58	12.9	8.1	6.2
Table7-3	59	11.2	5.6	3.5
Table7-4	60	13.3	7.9	4.8
Table7-5	60	13.7	6.4	5.3
Table7-6	58	10.1	6.4	3.5
Table7-7	59	11.2	7.1	3.9
Table7-8	59	11.5	6.6	3.9
Table7-9	59	11.4	6.9	4.4
Table7-10	59	10.8	6.9	3.1
AVG		**11.6**	**6.7**	**4.2**

Average improvement in gap thanks to convexification,
"from first side" + "from second side"

- *Table5*: 47% + 46%
- *Table6*: 43% + 45%
- *Table7*: 42% + 37%

Both convexifications contribute!

Other observations this far

- Property of the dual function: Constant along the direction **e**
- Stabilsation: No improvement – the opposite!
- Because of the property of the dual function: Tried regularisation instead to favour solutions with a small l_1-norm

Conclusions ...

... this far:

- Strong formulation for the fixed charge transportation problem
- Empirically we gain significantly in strength from the convexification over both sides

Further studies:

- Properties of the dual function, cf. experiences from subgradient methods?
- Understand the effects of stabilization. Customized techniques?
- How strength depends on instance characteristics

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Letocart L, Nagih A, Touati-Moungla N (2012) Dantzig-Wolfe and Lagrangian decompositions in integer linear programming. International Journal of Mathematics in Operational Research 4(3):247–262.

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Thanks for listening!