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p-step formulations for vehicle routing problems: a generalized class of models based on column generation

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Outline

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- Conclusions and future work.

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$$\lambda_r = \begin{cases} 1, & \text{if route } r \text{ is used in the solution;} \\ 0, & \text{o.w.} \end{cases}$$

Two-index vehicle flow formulation (CVRP)

$$\min \quad \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{\substack{j=1 \\ j \neq i}}^{n+1} x_{ij} = 1, \qquad i = 1, \ldots, n,$$

$$\sum_{\substack{i=0 \\ i \neq h}}^{n} x_{ih} - \sum_{\substack{j=1 \\ j \neq h}}^{n+1} x_{hj} = 0, \qquad h = 1, \ldots, n,$$

$$\sum_{j=1}^{n} x_{0j} \leq K,$$

$$y_j \geq y_i + q_j x_{ij} - Q(1 - x_{ij}), \qquad i, j = 0, \ldots, n+1,$$

$$d_i \leq y_i \leq Q, \qquad i = 0, \ldots, n+1,$$

$$x_{ij} \in \{0, 1\}, \qquad i, j = 0, \ldots, n+1.$$

Set partitioning formulation (CVRP)

$$
\begin{aligned}
\min \quad & \sum_{r \in \mathcal{R}} c_r \lambda_r \\
\text{s.t.} \quad & \sum_{r \in \mathcal{R}} a_{ri} \lambda_r = 1, \quad i = 1, \ldots, n, \\
& \sum_{r \in \mathcal{R}} \lambda_r \leq K, \\
& \lambda_r \in \{0, 1\}, \qquad r \in \mathcal{R}.
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- R: set of feasible (complete) routes;
- Routes generated by solving Resource Constrained Elementary Shortest Path Problems (or relaxations).

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- They are particular cases of a generalized formulation.

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- For each $s \in \mathcal{S}$ we have the decision variable:

$$\lambda_s = \begin{cases} 1, & \text{if partial path } s \text{ is used in the solution;} \\ 0, & \text{o.w.} \end{cases}$$

p-step formulation (CVRP)

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\begin{aligned}
\text{min} \quad & \sum_{s \in \mathcal{S}} c_s \lambda_s \\
\text{s.t.} \quad & \sum_{s \in \mathcal{S}} e_i^s \lambda_s = 1, && i = 1, \dots, n, \\
& \sum_{s \in \mathcal{S}} a_i^s \lambda_s = 0, && i = 1, \dots, n, \\
& \sum_{s \in \mathcal{S}} a_0^s \lambda_s \le K, && \\
& \varphi_j \ge \varphi_i + q_j \sum_{s \in \mathcal{S}_{ij}} \lambda_s - Q(1 - \sum_{s \in \mathcal{S}_{ij}} \lambda_s), && i = 0, \dots, n, \; j = 1, \dots, n+1, \\
& q_i \le \varphi_i \le Q, && i = 1, \dots, n, \\
& \lambda_s \in \{0, 1\}, && s \in \mathcal{S}.
\end{aligned}
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p-step formulation (CVRP)

- c_s is the total cost of traversing arcs in path s;
- e^s and a^s are $n+1$-vectors defined as

$$e_i^s = \begin{cases} +1, & \text{if } i \text{ is visited by path } s\text{, but it is not the last node of } s, \\ 0, & \text{otherwise,} \end{cases}$$

$$a_i^s = \begin{cases} +1, & \text{if } i \text{ is the first node visited by path } s, \\ -1, & \text{if } i \text{ is the last node visited by path } s, \\ 0, & \text{otherwise,} \end{cases}$$

  for all $i = 0, 1, \ldots, n$ and $s \in \mathcal{S}$.
- $\mathcal{S}_{ij} \subset \mathcal{S}$ contains only the paths that traverse arc (i,j) for a pair $i,j \in \mathcal{N}$.

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p-step formulation (VRPTW)

$$
\begin{array}{lll}
\text{min} & \sum_{s \in \mathcal{S}} c_s \lambda_s & \\
\text{s.t.} & \sum_{s \in \mathcal{S}} e_i^s \lambda_s = 1, & i = 1, \ldots, n, \\
& \sum_{s \in \mathcal{S}} a_i^s \lambda_s = 0, & i = 1, \ldots, n, \\
& \sum_{s \in \mathcal{S}} a_0^s \lambda_s \leq K, & \\
& \varphi_j \geq \varphi_i + q_j \sum_{s \in \mathcal{S}_{ij}} \lambda_s - Q(1 - \sum_{s \in \mathcal{S}_{ij}} \lambda_s), & i = 0, \ldots, n, \; j = 1, \ldots, n+1, \\
& \omega_j \geq \omega_i + (s_i + t_{ij}) \sum_{s \in \mathcal{S}_{ij}} \lambda_s - M(1 - \sum_{s \in \mathcal{S}_{ij}} \lambda_s), & i = 0, \ldots, n, \; j = 1, \ldots, n+1, \\
& q_i \leq \varphi_i \leq Q, & i = 1, \ldots, n, \\
& w_i^a \leq \omega_i \leq w_i^b, & i = 0, \ldots, n+1, \\
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- In the $(n + 1)$-step formulation all arcs are already attached, so the model has less freedom when assigning value to them (and the feasible set of the linear relaxation becomes more restricted);
- The number of paths is huge (exponential) and requires column generation, label-setting and branch-and-price.

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Proposition 2. Let $\tilde{z}_p$ be the optimal value of the linear relaxation of the p-step formulation, for $p = 1, \ldots, n+1$. For any $p \in \{1, \ldots, n\}$, we have that $\tilde{z}_{p+1} \geq \tilde{z}_p$.

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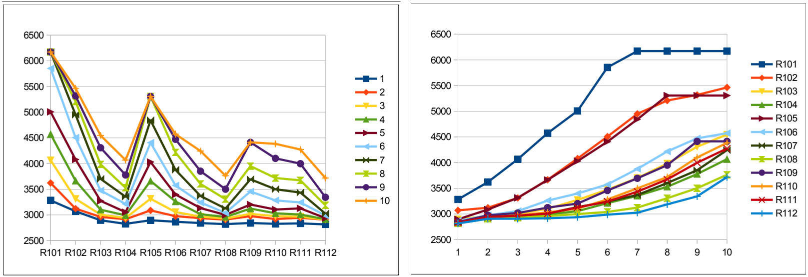

Figure 1. Objective values of the linear relaxations for class R1.

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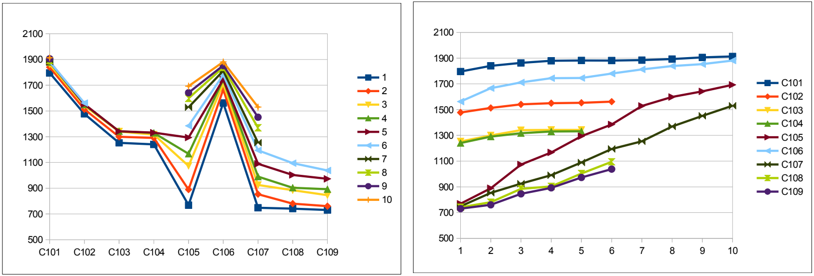

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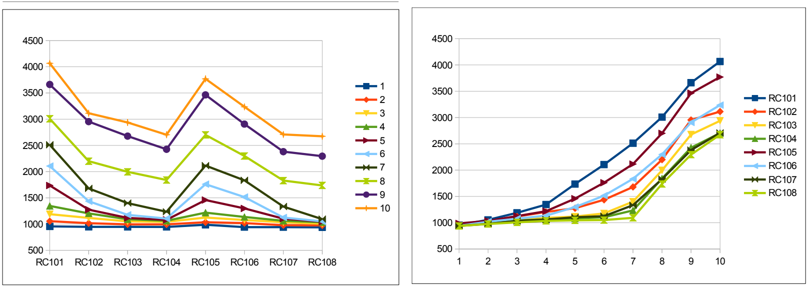

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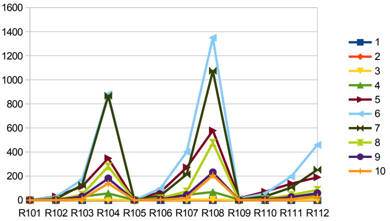

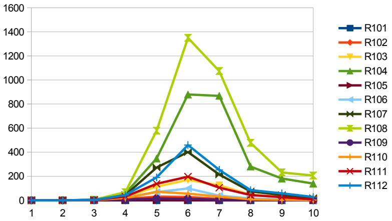

Figure 4. Computational times to solve the linear relaxations for class R1.

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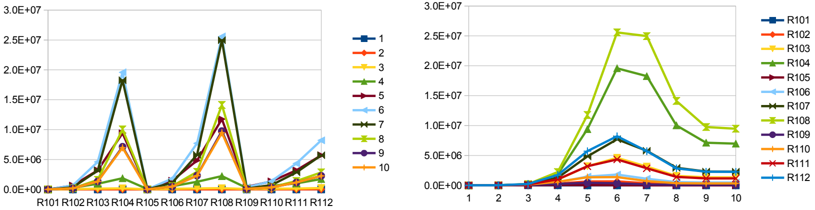

Figure 5. Number of paths generated for class R1.

Column generation for p-step formulations

Restricted master problem, with $\overline{\mathcal{S}} \subset \mathcal{S}$:

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- Theoretical results and computational evidence relating the bounds provided by p-step formulations;
- Column generation for p-step formulations;
- Future developments: branch-price-and-cut for p-step formulations; extending the idea of p-step formulations to other types of optimization problems.

Thank you!

Acknowledgments

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My webpage

`http://www.dep.ufscar.br/docentes/munari`

PDCGM webpage

`http://www.maths.ed.ac.uk/~gondzio/software/pdcgm.html`