Social Network Analysis and Community Detection by Decomposing a Graph into Relaxed Cliques

Column Generation 2016, Búzios, Brazil

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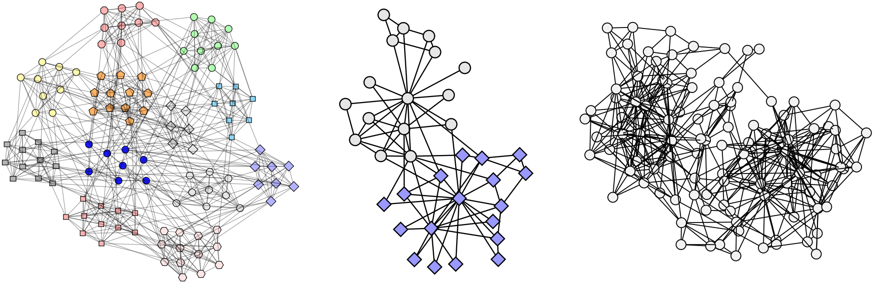
Network Analysis:

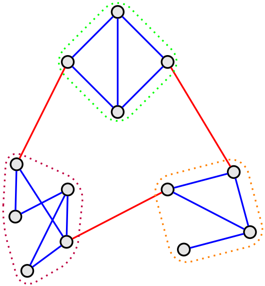
Graphs representing real systems are no random graphs
→ community structure or clustering



Important applications in many networked systems from biology, sociology, computer science, engineering, economics, politics, linguistics, etc.

Network Analysis:

1. Community Detection (Fortunato, 2010)
   - Divide the graph into groups
   - Few edges between groups, relatively many internal edges
   - Maximize modularity (Newman and Girvan, 2004)
   - Groups have no specific structure



2. Relaxed Cliques (Pattillo *et al.*, 2013a)
   - Subgraphs with a specific structure
   - Find maximum cardinality/weight relaxed clique

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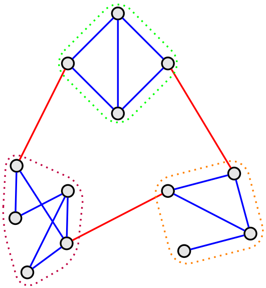

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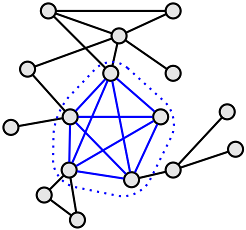

Outline

1. Introduction and problem description
2. Branch-and-Price algorithm
   - Master program and pricing problem
   - Branching
3. Computational results
4. Conclusions

Basic Notation

We assume that a simple graph $G = (V, E)$ with finite node set V and edge set E is given.

For any subset $S \subseteq V$, the subgraph $G[S] := (S, E \cap (S \times S))$ is called the induced subgraph of S.

A set S is a clique if $G[S]$ is complete.
A set S is an independent (stable) set if $G[S]$ is edgeless.
A clique/independent set is maximum if there is no clique/independent set with larger cardinality.

Examples:

Maximum Clique

Maximum Independent Set

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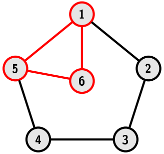
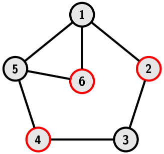


Maximum Clique



Maximum Independent Set

Cliques and Clique Relaxations

A clique S forms an extreme subset in the following senses:

Degree	Every node $i \in S$ has maximum degree $(=|S|-1)$
Distance	The distance $d(i,j)$ between any two nodes $i,j \in S$ is minimal $(=1)$
Density	$G[S]$ has maximum density $(=1)$; Note: density of (V', E') is $\frac{|E'|}{\binom{|V'|}{2}}$
Connectivity	The vertex connectivity $\kappa(G[S])$ is maximum $(=|S|-1)$

Network analysis:

- Cliques can model cohesive substructures/communities (Luce and Perry, 1949).
- However, requirements of a clique were found too restrictive!

Cliques and Clique Relaxations

Different clique relaxations have been considered in the literature (see Pattillo *et al.*, 2013a, for an overview):

Type of relaxation	Definition	Based on	Hered-itary	Connected
k-core	$\delta(G[S]) \geq k$	Degree	no	$\lvert S\rvert \leq 2k+1$
s-plex	$\delta(G[S]) \geq \lvert S\rvert - s$	Degree	yes	$\lvert S\rvert \geq 2s-1$
s-clique	$d_G(i,j) \leq s \;\forall\; i,j \in S$	Dist.	yes	$s=1$
s-club	$d_{G[S]}(i,j) \leq s \;\forall\; i,j \in S$	Dist.	no	always
γ-quasi-clique	$\rho(G[S]) \geq \gamma$	Density	no	$\left\lceil \gamma\binom{\lvert S\rvert}{2} - \binom{\lvert S\rvert-1}{2} \right\rceil \geq 1$
s-defective cl.	$\lvert E(G[S])\rvert \geq \binom{\lvert S\rvert}{2} - s$	Density	yes	$\lvert S\rvert \geq s+2$
k-block	$\kappa(G[S]) \geq k$	Connect.	no	always
s-bundle	$\kappa(G[S]) \geq \lvert S\rvert - s$	Connect.	yes	$\lvert S\rvert \geq s+1$

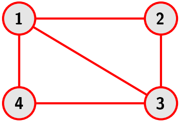
Note: $\delta(G)$ minimum degree, $d_G(i,j)$ distance, $\rho(G)$ edge density, $\kappa(G)$ vertex connectivity

Denote by Π the graph property of being a specific type of relaxed clique.

Clique Relaxations: Examples

1-defective clique

'one missing edge allowed in induced subgraph'



$\Rightarrow$ **2-plex**

'one missing edge allowed at each node'

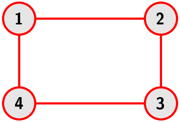
2-club

'distance between nodes in S at most 2 in induced subgraph'

$\Rightarrow$ **2-clique**

'distance between nodes of S at most 2 in graph'

2-plex



$\not\Rightarrow$ 1-defective clique

2-clique

$\not\Rightarrow$ 2-club

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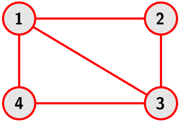

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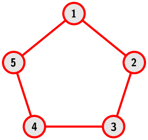

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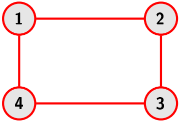

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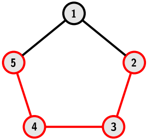


$\not\Rightarrow$ 2-club

Weak-Heredity

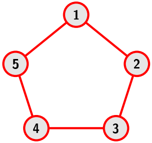
S with satisfies Π
$\Rightarrow$ any $S' \subset S$ satisfies Π.

Weak-Hereditary relaxed cliques:

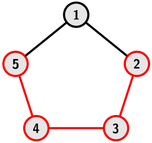
s-plex, s-clique, s-defective clique, s-bundle

Non-hereditary relaxed cliques:

k-core, s-club, γ-quasi-clique, k-block



2-club



no 2-club

Connectivity

For all $i, j \in S$ there exists a path between i and j in $G[S]$.

Connected relaxed cliques:

s-club, k-block

Non-connected relaxed cliques:

k-core, s-plex, s-clique, s-bundle, γ-quasi-clique, s-defective clique

2-defective clique

Note:

- Non-connected communities may not be reasonable
  $\rightarrow$ **Connected relaxed cliques**
- Connectivity is non-hereditary

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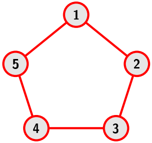
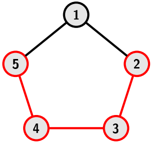


2-club



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Large Relaxed Cliques

The optimization-related literature related to relaxed cliques is (as far as we know) exclusively on finding maximum/inclusion maximal relaxed cliques.

Type of relaxation	Exact Approach and Reference
k-core	polynom. solvable, see (Kosub, 2004)
s-plex	B&C: (Balasundaram *et al.*, 2011), B&B: (Trukhanov *et al.*, 2013; Gschwind *et al.*, 2015)
s-clique	clique in the sth power graph
s-club	B&C: (Almeida and Carvalho, 2012, 2013), B&B: (Bourjolly *et al.*, 2002; Mahdavi Pajouh and Balasundaram, 2012), MIP: (Bourjolly *et al.*, 2000; Veremyev and Boginski, 2012), SAT: (Wotzlaw, 2014)
γ-quasi-clique	MIP: (Pattillo *et al.*, 2013b), B&B: (Pajouh *et al.*, 2014)
s-defective clique	B&C: (Sherali and Smith, 2006), B&B: (Trukhanov *et al.*, 2013; Gschwind *et al.*, 2015)
k-block	polynom. solvable, see (Kammer and Täubig, 2005)
s-bundle	B&B: (Gschwind *et al.*, 2015)

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Generic MIP-formulation:

- $x_i \in \{0, 1\}$ indicates if vertex $i \in V$ is in the relaxed clique S
- $y_e \in \{0, 1\}$ indicates if $G[S]$ contains edge $e \in E$

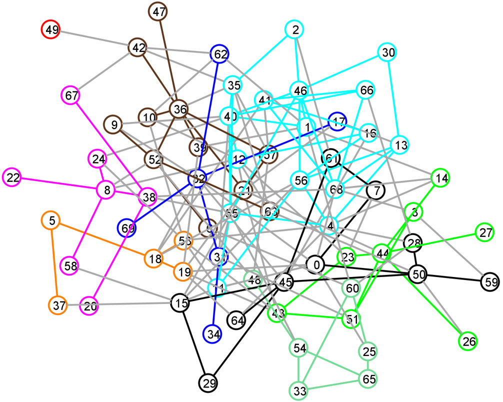
$$\max \quad \sum_{i \in V} x_i \tag{1}$$

$$\text{s.t.} \quad (x, y) = (x_i, y_{ij}) \in \mathscr{F}(G) \tag{2}$$

Note: $\mathscr{F}(G)$ is a polytope such that $(x, y) \in \mathscr{F}(G)$ if and only if $G[S]$ with $S = \{i \in V : x : i = 1\}$ satisfies Π.

The problems we consider in the following are partitioning and covering a graph with a minimum number of relaxed cliques:

Example: Partitioning with 3-Clubs



Special case:

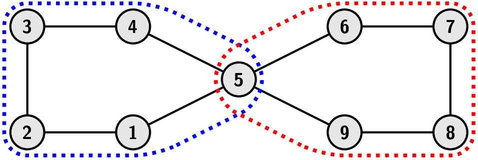
1-club = $(|S|-1)$-core = 1-plex = $\cdots$ = clique

- Clique Cover
- Vertex Coloring on Complement Graph

The problems we consider in the following are partitioning and covering a graph with a minimum number of relaxed cliques:

Partitioning $\neq$ **covering** for non-hereditary Π

Covering Solution:
Two 2-Clubs



Partitioning Solution:
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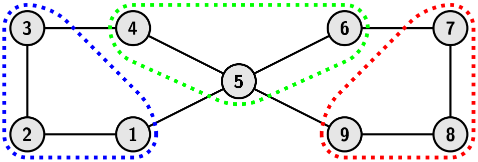

Partitioning and Covering a Graph with Relaxed Cliques

The problems we consider in the following are partitioning and covering a graph with a minimum number of relaxed cliques:

Generic **compact formulation**:

- Let UB_{RC} be an upper bound for the problem
- Index set $H = \{1, ..., UB_{RC}\}$ to refer to the individual relaxed cliques in the partitioning/covering
- Introduce indicator variables $z^h \in \{0, 1\}$, $h \in H$
- Duplicate x_i and y_e variables and constraints for each $h \in H$

$$
\begin{aligned}
\min \quad & \sum_{h \in H} z^h \\
\text{s.t.} \quad & \sum_{h \in H} x_i^h = (\geq) 1 && i \in V \\
& z^h \geq x_i^h && i \in V, h \in H \\
& (x_i^h, y_{ij}^h) \in \mathscr{F}(G) && h \in H
\end{aligned}
$$

Interesting problem variants:

	General		With connectivity required	
Type of relaxation	Partitioning	Covering	Partitioning	Covering
k-core	✗	✗	✗	✗
s-plex	✔	✗	✔	✔
s-clique	✗	✗	✔	✔
s-club	✔	✔	✗	✗
γ-quasi-clique	✔	✔	✔	✔
s-defective clique	✔	✗	✔	✔
k-block	✗	✗	✗	✗
s-bundle	✔	✗	✔	✔

Note:

- ✗ (blue) Some vertices may have a degree smaller than k
- ✗ (green) Determination of k-connected components (Kammer and Täubig, 2005)
- ✗ (orange) Clique cover in sth power graph
- ✗ (red) Partitioning = covering for hereditary Π
- ✗ (purple) s-Club is always connected

Master problem derived from Dantzig-Wolfe decomposition of the generic compact formulation and subsequent aggregation:

- Ω the set of all S satisfying Π
- Indicators a_{iS} for $i \in V, S \in \Omega$ with $a_{iS} = 1$ if $i \in S$, and 0 otherwise

$$
\begin{aligned}
\min \quad & \sum_{S \in \Omega} \mathbb{1}^\top \lambda_S \\
\text{s.t.} \quad & \sum_{S \in \Omega} a_{iS} \lambda_S = 1 \quad (\text{or} \geq 1) \quad \text{for all } i \in V \qquad (\text{dual } \pi_i) \\
& \lambda \geq \mathbf{0} \quad (\in \mathbb{R}^{|\Omega|})
\end{aligned}
$$

Pricing Problem

Reduced Costs (rdc) of a relaxed clique S:

$$\tilde{c}_S = 1 - \sum_{i \in S} \pi_i$$

Pricing Problem:

$$1 - \tilde{c}_S = \max \quad \sum_{i \in V} \pi_i x_i \qquad \text{s.t.} \quad (x_i, y_{ij}) \in \mathscr{F}(G)$$

This is a maximum weight relaxed clique problem (MW-RC)

- Generalization of the maximum (cardinality) relaxed clique problem
- For most types of relaxed cliques this has not been studied in the literature so far
- MIP-solver (+ cutting plane algorithm) or combinatorial B&B
- Weights $\pi_i \in \mathbb{R}$ can be negative in case of partitioning

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Branching for Relaxed Clique Partitioning

Branching schemes: (Pricing problem preserving)

- **Generic Branching Rule (GBR)**
  Branching on infeasible components of the support graph
  - Pros
    1. Applicable to all relaxed cliques
    2. Basic idea: easy to understand
    3. Simple to implement (remove edges from given graph)
  - Cons
    1. Complete only for connected weak-hereditary Π
    2. Non-binary scheme
- **Infeasible $(s+1)$-path branching**
  GBR tailored to s-club
  - Pros
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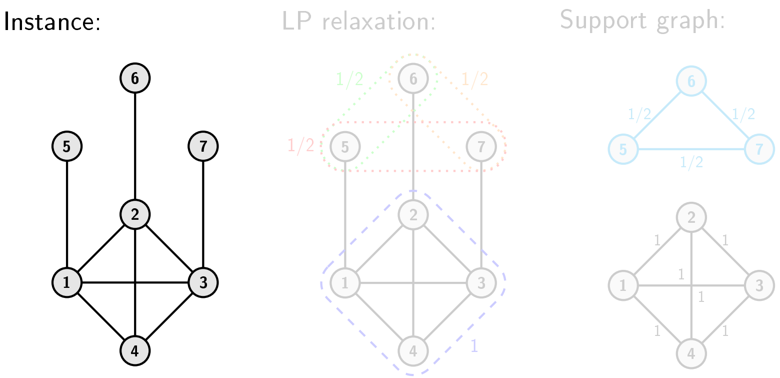

Non-integer part of the solution is disconnected in the original graph

⇒ All components of the support graph are 2-plex

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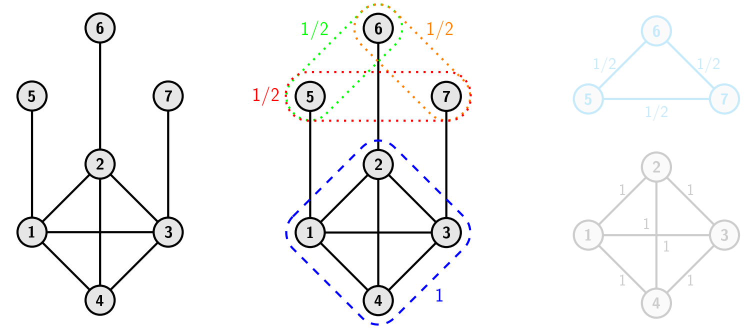

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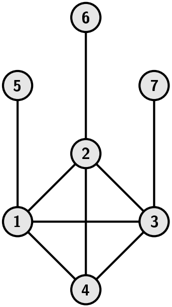

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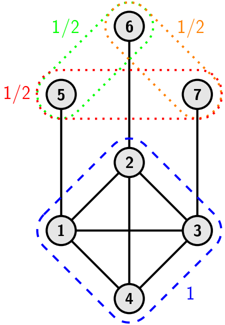
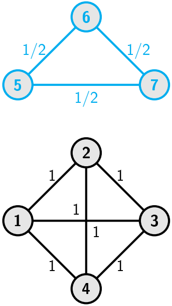


Support graph:



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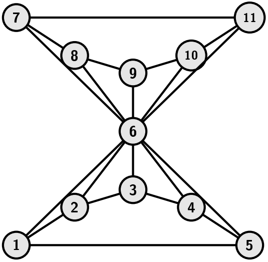

LP relaxation:

Support graph:

All connected components not fulfilling Π have a superset that satisfies Π.

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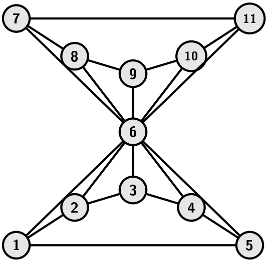

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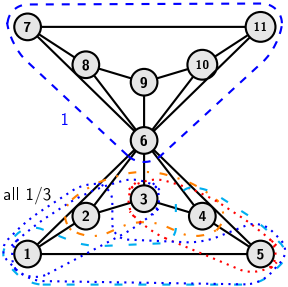

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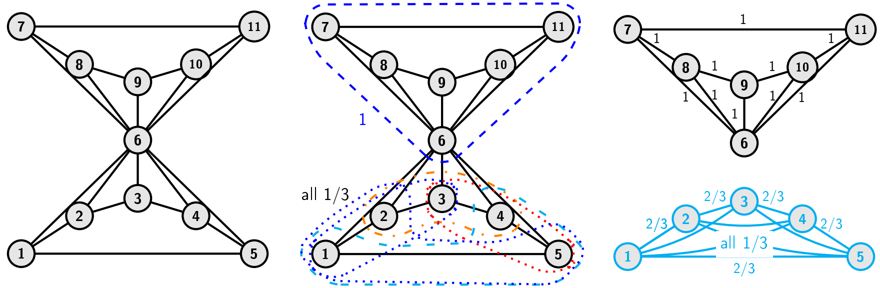

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Branching for Relaxed Clique Partitioning

Branching schemes: (Pricing problem non-preserving)

Ryan and Foster (1981) branching
Branching on whether two nodes $i, j \in V$ are in the same ($x_i = x_j$) or different ($x_i + x_j \leq 1$) relaxed cliques

Pros
1. Complete
2. Easy to understand
3. Effective for set partitioning in general
4. Simple to handle in MIP-based pricing algorithms

Cons
1. Destroys structure of MW-RC pricing problem
2. Substantially complicates combinatorial B&B-based pricing algorithms

Branching for Relaxed Clique Partitioning

Branching schemes: (Pricing problem non-preserving)

Ryan and Foster (1981) branching
Branching on whether two nodes $i, j \in V$ are in the same ($x_i = x_j$) or different ($x_i + x_j \leq 1$) relaxed cliques

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Branching for Relaxed Clique Covering

Let $g(T) = \sum_{S \in \Omega} |T \cap S| \lambda_S$ be the number of vertex contacts for subset $T \subseteq V$ and $g^\lambda(T)$ the value in the current solution.

Hierarchy of Branching Decisions:

1. Branch on vertex contacts for $T \subseteq V$ with branches $g(T) \leq \lfloor g^\lambda(T) \rfloor$ and $g(T) \geq \lceil g^\lambda(T) \rceil$
2. Fix vertex contacts of $P \subseteq V$ to its minimum ($g(P) = |P|$), alternative branch $g(P) \geq |P| + 1$
3. Apply branching for partitioning in $V_{=1}\{i \in V : g_i \text{ is fixed to } 1\}$
4. Vertex duplication for nodes in $V_{>1}\{i \in V : g_i \text{ is fixed to value } > 1\}$
   - Replace node i by g_i^λ copies $i_{(1)}, \ldots, i_{g_i^\lambda}$ of that node
   - Each of these node must be covered exactly once
   - No relaxed clique must contain two (or more) of these nodes
   - Separate constraints $x_{i_{(s)}} + x_{i_{(t)}} \leq 1$ for all pairs of copies $s \neq t$

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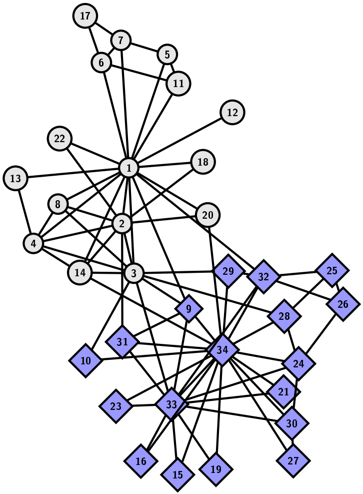
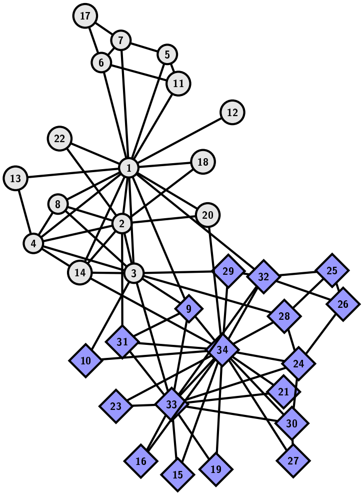
Zachary's Karate Club (Zachary (1977), also: 10th DIMACS challenge)



'Real' solution (Zachary, 1977)

3-club partitioning

Zachary's Karate Club (Zachary (1977), also: 10th DIMACS challenge)



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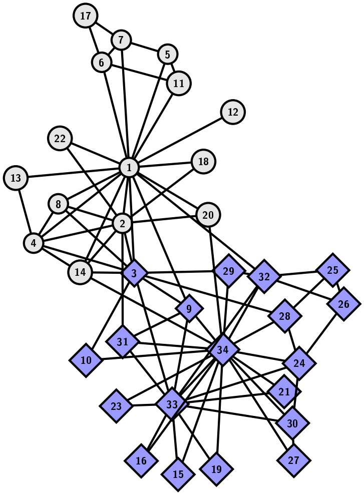
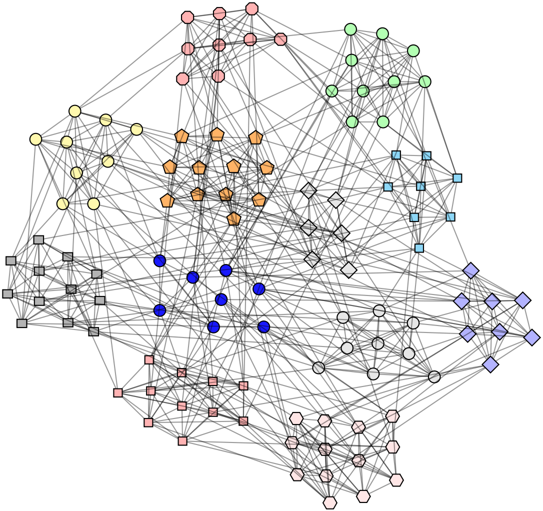


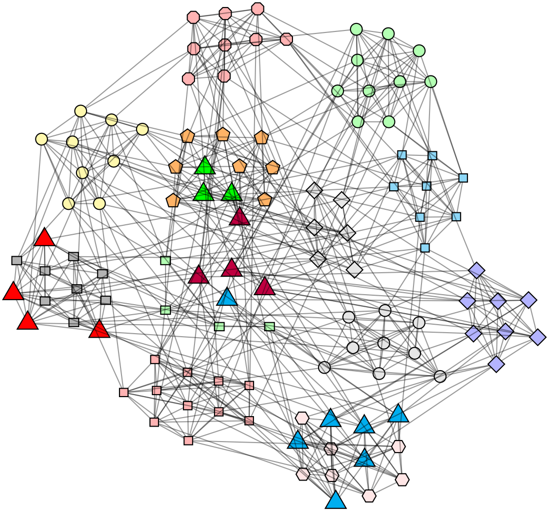
3-club partitioning

College Football (Girvan and Newman (2002), 10th DIMACS challenge)



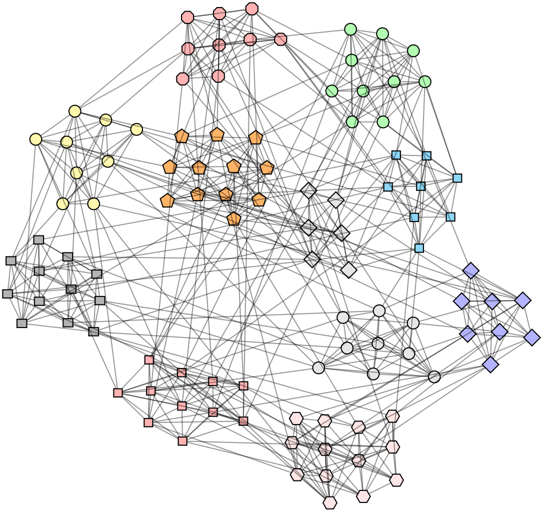
'Real' solution: 11 Conferences + 8 independent teams

College Football (Girvan and Newman (2002), 10th DIMACS challenge)



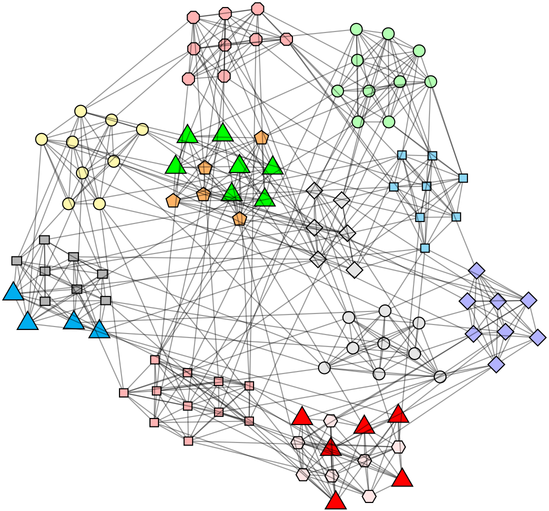
3-plex partitioning in 16 groups

College Football (no independents teams)



'Real' solution: 11 Conferences

College Football (no independents teams)



3-plex partitioning in 14 groups

Branch-and-Price insights:

1. Excellent lower bounds
2. Practical hardness increases with s (decreases with γ) and the density of the graph and depends on the type of relaxed clique
3. Covering vs. partitioning
   - Covering slightly easier than partitioning for solving LP-relaxation of the master program
   - Covering much harder when it comes to branching
4. Branching
   - Ryan/Foster branching seems to be much more effective
   - Pricing problems get harder when using Ryan/Foster branching
5. Subproblem solution
   - Combinatorial B&Bs much faster than MIPs for solving the pricing problem
   - Ryan/Foster branching complicates the pricing problems for the combinatorial B&Bs

Conclusions and Outlook

Conclusions:

- Introduced the relaxed clique covering/partitioning problem
- New approach for community detection
- Interesting components of branch-and-price
  - Branching
  - Solution of maximum weight relaxed clique pricing problem

Outlook:

- Finding a complete pricing problem preserving branching scheme
- Heuristics and metaheuristics can accelerate pricing

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Thank you for listening!

Any questions?!

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