Branch-and-Benders-Cut Algorithms for the Single Source Capacitated Facility Location Problem

Furkan Enderer[1,2], Claudio Contardo[1,3], Bernard Gendron[1,2]

[1] *CIRRELT* , Centre Interuniversitaire de Recherche sur les Réseaux d'Entreprise, la Logistique et le Transport
[2] Département d'informatique et de recherche opérationnelle, Université de Montréal
[3] Département de management et technologie, ESG UQAM

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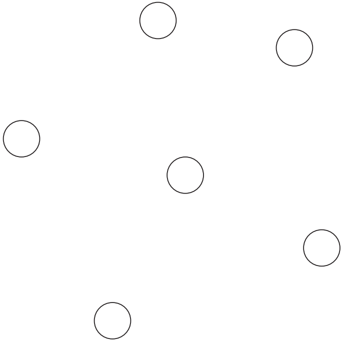
Facility Location Problems

- Given a set of potential facility locations and a set of customers, FLPs aim to determine the subset of facilities to open and serve customers so as to satisfy customer demands.



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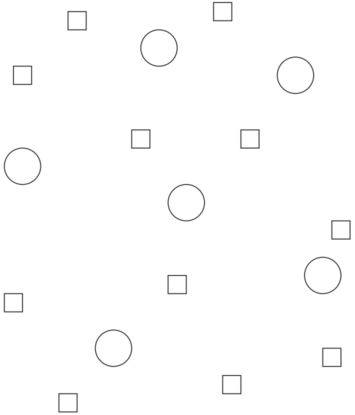

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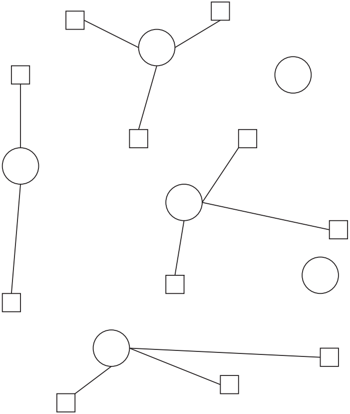

Facility Location Problems

- Given a set of potential facility locations and a set of customers, FLPs aim to determine the open a subset of facilities and serve customers so as to satisfy customer demands.
- CFLP: Capacitated facilities.
- SSCFLP: Each customer must be served by exactly one facility.
- Applications: Logistics, Transportation, Telecommunication.

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Solution Methodologies

- Lagrangian relaxation (Holmberg et al. [1999], Darby-Dowman and Lewis [1988])
- LR Heuristics (Agar and Salhi [1998], Klincewicz and Luss [1986], Sridharan [1993])
- Branch-and-Price (Díaz and Fernández [2002], Neebe and Rao [1983])
- Heuristics (Ahuja et al. [2004],Contreras and Díaz [2008])
- Cutting Plane methods (Yang et al. [2012])

Notation

Standard Formulation

- I : set of potential facilities
- J : set of customers
- f_i : fixed cost of opening a facility
- c_{ij} : cost of assigning customer j to facility i
- b_i : capacity of each facility
- d_j : demand of each customer

Standard Formulation

$$z(SSCFLP) = \quad \text{Minimize} \quad \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i \tag{1}$$

Subject to

$$\sum_{i \in I} x_{ij} = 1 \qquad j \in J \tag{2}$$

$$\sum_{j \in J} d_j x_{ij} \leq b_i y_i \qquad i \in I \tag{3}$$

$$x_{ij} \leq y_i \qquad i \in I, j \in J \tag{4}$$

$$x_{ij} \in \{0, 1\} \qquad i \in I, j \in J \tag{5}$$

$$y_i \in \{0, 1\} \qquad i \in I. \tag{6}$$

Notation

Set Partitioning Formulation

- Ω_i : set of feasible assignment patterns for each facility $i \in I$.
- c_p : total cost of assignment pattern $p \in \Omega_i$.
- a_{ij}^p : binary constant defining the assignment pattern $p \in \Omega_i$.

Set Partitioning Formulation

$$z(SSCFLP) = \text{Minimize} \quad \sum_{p \in \Omega} c_p \theta_p \tag{7}$$

Subject to

$$\sum_{i \in I} \sum_{p \in \Omega^i} a_{ij}^p \theta_p = 1 \qquad j \in J \tag{8}$$

$$\sum_{p \in \Omega^i} \theta_p \leq 1 \qquad i \in I \tag{9}$$

$$\theta_p \in \{0, 1\} \qquad p \in \Omega \tag{10}$$

Reformulation

$$\text{Minimize} \quad \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i$$

Subject to

$$\sum_{i \in I} x_{ij} = 1 \qquad j \in J \tag{11}$$

$$\sum_{j \in J} d_j x_{ij} \leq b_i y_i \qquad i \in I \tag{12}$$

$$\sum_{i \in I} x_{ij} = \sum_{i \in I} \sum_{p \in \Omega^i} a_{ij}^p \theta_p \qquad j \in J \tag{13}$$

$$\sum_{p \in \Omega^i} \theta_p \leq 1 \qquad i \in I \tag{14}$$

$$x_{ij} \in \{0, 1\} \qquad i \in I, j \in J \tag{15}$$

$$y_i \in \{0, 1\} \qquad i \in I \tag{16}$$

$$\theta_p \in \{0, 1\} \qquad p \in \Omega \tag{17}$$

Benders Master Problem I

$$z(M(x,y)) = \text{Minimize} \qquad \sum_{j\in J}\sum_{i\in I} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i \tag{18}$$

Subject to

$$\sum_{i\in I} x_{ij} = 1 \qquad j \in J \tag{19}$$

$$\sum_{j\in J} d_j x_{ij} \leq b_i y_i \qquad i \in I \tag{20}$$

$$x_{ij} \leq y_i \qquad i \in I, j \in J \tag{21}$$

$$x_{ij} \in [0,1] \qquad i \in I, j \in J \tag{22}$$

$$y_i \in \{0,1\} \qquad i \in I. \tag{23}$$

Benders Subproblem I

$$z(S(\theta_p, \bar{y})) = \text{Minimize} \qquad \sum_{p \in \Omega} c_p \theta_p \tag{24}$$

Subject to

$$\sum_{i \in I : \bar{y}_i = 1} \sum_{p \in \Omega^i} a_{ij}^p \theta_p = 1 \qquad j \in J \tag{25}$$

$$\sum_{p \in \Omega^i} \theta_p \leq 1 \qquad i \in I : \bar{y}_i = 1 \tag{26}$$

$$\theta_p \in \{0, 1\} \qquad p \in \Omega \tag{27}$$

Benders Cuts I

If the LP relaxation of the Benders subproblem is infeasible, we can identify an extreme ray (α_j^r, π_i^r) and generate the following feasibility cut:

$$\sum_{j \in J} \sum_{i \in I | \bar{y}_i = 1} \alpha_j^r x_{ij} + \sum_{i \in I | \bar{y}_i = 1} \pi_i^r y_i \leq 0. \tag{28}$$

If the LP relaxation of the Benders subproblem feasible, we can identify an extreme point (α_j^k, π_i^k) and generate the following optimality cut:

$$\sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij} - \sum_{j \in J} \sum_{i \in I | \bar{y}_i = 1} \alpha_j^k x_{ij} - \sum_{i \in I | \bar{y}_i = 1} \pi_j^k y_i \geq 0. \tag{29}$$

In this case, (30) is a valid lower bound on the optimal solution value of the original problem.

$$\sum_{i \in I} f_i \bar{y}_i + \sum_{j \in J} \sum_{i \in I | \bar{y}_i = 1} \alpha_j^k + \sum_{i \in I | \bar{y}_i = 1} \pi_i^k \tag{30}$$

Benders Cuts I

The cuts (28)-(29) are classical LP-duality based Benders cuts and are generated when the LP relaxation of the Benders subproblem is solved. On the other hand, if the integer subproblem ($S(\theta_p, \bar{y})$) is solved to optimality by Branch-and-Price, the cuts (31) can be added to the master problem.

$$\sum_{i \in I | \bar{y}_i = 0} y_i - \sum_{i \in I | \bar{y}_i = 1} (1 - y_i) \geq 1 \tag{31}$$

Outline of the Algorithm

- Branch on y variables in the Benders Master problem
- Solve the LP relaxation of the Benders subproblem with $\bar{y}_i \in \{0, 1\}$
- Add feasibility/optimality cuts
- If branching is complete, solve the Benders subproblem by B&P
- Add canonical cuts and restart B&B at the same node

Benders Master Problem II

$$z(M_2(x,y)) = \text{Minimize} \qquad \sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij} + \sum_{i \in I} f_i y_i$$

Subject to

$$\sum_{i \in I} x_{ij} = 1 \qquad j \in J \tag{32}$$

$$\sum_{j \in J} d_j x_{ij} \leq b_i y_i \qquad i \in I \tag{33}$$

$$x_{ij} \leq y_i \qquad i \in I, j \in J \tag{34}$$

$$x_{ij} \in [0,1] \qquad i \in I, j \in J \tag{35}$$

$$y_i \in \{0,1\} \qquad i \in I \tag{36}$$

Benders Subproblem II

$$z(S_2(\theta_p, \bar{x}, \bar{y})) = \text{Minimize} \quad \sum_{p \in \Omega} c_p \theta_p \tag{37}$$

Subject to

$$\sum_{p \in \Omega^i} a_{ij}^p \theta_p = \bar{x}_{ij} \qquad i \in I, j \in J : \bar{x}_{ij} > 0 \tag{38}$$

$$\sum_{p \in \Omega^i} \theta_p \leq \bar{y}_i \qquad i \in I : \bar{y}_i = 1 \tag{39}$$

$$\theta_p \in [0, 1] \qquad p \in \Omega \tag{40}$$

Benders Cuts II

If $S_2(\theta_p, \bar{x}, \bar{y})$ is infeasible, we identify an extreme ray (α^r_{ij}, π^r_i) defined by the region 38-39 and generate the following feasibility cut and add it to the Benders master problem $M_2(x, y)$:

$$\sum_{j \in J | x_{ij} > 0} \sum_{i \in I | \bar{y}_i = 1} \alpha^r_{ij} x_{ij} + \sum_{i \in I | \bar{y}_i = 1} \pi^r_i y_i \leq 0. \tag{41}$$

If $S_2(\theta_p, \bar{x}, \bar{y})$ is feasible, we can identify an extreme point (α^k_{ij}, π^k_i) defined by the region 38-39 and generate the following optimality cut:

$$\sum_{j \in J} \sum_{i \in I} c_{ij} x_{ij} - \sum_{j \in J | x_{ij} > 0} \sum_{i \in I | \bar{y}_i = 1} \alpha^k_{ij} x_{ij} - \sum_{i \in I | \bar{y}_i = 1} \pi^k_i y_i \geq 0. \tag{42}$$

Benders Cuts II

In this case,

$$z(S_2(\theta_p, \bar{x}, \bar{y})) = \sum_{p \in P} c_p \theta_p = \sum_{j \in J | x_{ij} > 0} \sum_{i \in I | \bar{y}_i = 1} \alpha_{ij}^k + \sum_{i \in I | \bar{y}_i = 1} \pi_i^k \qquad (43)$$

(43) is the optimal solution value to the Benders subproblem and (44) is a valid lower bound on the optimal solution value of the SSCFLP.

$$\sum_{i \in I} f_i \bar{y}_i + \sum_{j \in J | x_{ij} > 0} \sum_{i \in I | \bar{y}_i = 1} \alpha_{ij}^k + \sum_{i \in I | \bar{y}_i = 1} \pi_i^k \qquad (44)$$

Computational Results

Lower bounds obtained by solving the standard formulation by CPLEX and the lower bounds obtained by BBC algorithms. Two sets of benchmark instances:

- The first set: 71 problem instances introduced by Holmberg et al. [1999] and by Ahuja et al. [2004].
- The second set: 57 problem instances introduced by Delmaire et al. [1999].

Computational Results

Lower bounds obtained by solving the standard formulation by CPLEX and the lower bounds obtained by BBC algorithms. Two sets of benchmark instances:

- The first set: 71 problem instances introduced by Holmberg et al. [1999] and by Ahuja et al. [2004].
- The second set: 57 problem instances introduced by Delmaire et al. [1999].

The second BBC algorithm produced very fractional x_{ij} values after adding several cuts at a specic node. We set a tolerance of $\epsilon = 0.1$ in order to avoid solving the benders subproblem for very fractional solutions $(0 < x_{ij} < 0.1)$.

Table: Computational Results on Holmberg's instances

			CPLEX			BBC		BBC2	
Instance	$\|I\|$	$\|J\|$	time	gap_1	$time_1$	gap_2	$time_2$	gap_3	$time_3$
P1-P12	10	50	0.16	0.71	0.07	0.00	0.15	0.00	0.11
P13-P24	20	50	0.23	0.45	0.11	0.00	0.10	0.00	0.07
P25-P40	30	150	6.26	0.18	0.74	0.02	47.14	0.02	5.23
P41-P49	10-30	70-90	0.32	0.64	0.19	0.00	0.42	0.00	0.99
P50-P55	10-20	100	1.04	0.55	0.25	0.02	69.09	0.02	4.62
P56-P71	30	200	34.57	0.18	3.15	0.03	28.21	0.03	7.46

Results on Delmaire's Instances

Table: Computational Results on Delmaire's instances

			CPLEX			BBC		BBC2	
Instance	$\|I\|$	$\|J\|$	time	gap_1	$time_1$	gap_2	$time_2$	gap_3	$time_3$
A1-A6	10	20	2.69	4.24	0.12	0.31	0.35	0.43	16.66
A7-A17	15	30	8.26	2.46	0.15	0.69	4.26	0.84	7.70
A18-A25	20	40	32.50	1.15	0.42	0.35	14.02	0.47	4.86
A26-A33	20	50	321.31	1.04	0.49	0.20	56.79	0.42	8.55
A34-A41	30	60	73.90	0.38	0.84	0.02	145.81	0.15	9.35
A42-A49	30	75	1196.15	0.20	0.98	0.02	372.30	0.06	21.06
A50-A57	30	90	1314.96	0.20	0.92	0.01	420.93	0.08	23.44

Conclusion and Future Directions

- Combining the two proposed BBC algorithms in one lower bounding procedure.
- Embedding column generation in an enumeration tree for the subproblems.

Thank you for listening!

Solving the Benders Subproblem

$$(RMP) = \text{Minimize} \quad \sum_{p\in\Omega_t} c_p\theta_p \tag{45}$$

Subject to

$$\sum_{i\in I:\bar{y}_i=1} \sum_{p\in\Omega_t^i} a_{ij}^p\theta_p = 1 \qquad j\in J \tag{46}$$

$$\sum_{p\in\Omega_t^i} \theta_p \le 1 \qquad i\in I:\bar{y}_i=1 \tag{47}$$

$$\theta_p \in [0,1] \qquad p\in\Omega_t. \tag{48}$$

Let α_j and π_i be the dual values associated with constraints (46) and (47), respectively. Then, the reduced cost coefficient associated with facility i and assignment pattern p is:

$$\bar{C}_i^p = \sum_{j\in J} c_{ij}a_{ij}^p - \sum_{j\in J}\alpha_j a_{ij}^p - \pi_i. \tag{49}$$

Solving the Benders Subproblem

For each $i \in I$, given the optimal dual vector (α_j^t, π_i^t) of the RMP at iteration t, the pricing subproblem can be stated as follows:

$$(PP) = \text{Minimize} \qquad \sum_{j \in J} c_{ij} a_{ij} - \sum_{j \in J} \alpha_j^t a_{ij}^p - \pi_i^t$$

$$\text{Subject to}$$

$$\sum_{j \in J} a_{ij} d_j \leq b_i \tag{50}$$

$$a_{ij} \in \{0, 1\} \qquad j \in J.$$

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