Pricing problems

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Column Generation 2016
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- Primal Simplex (PS, 1945) for LPs : many degenerate pivots.
- Column Generation (CG, 1960) for *huge* LPs : CG $\equiv$ PS
  Scheduling with thousands of flight legs/month
  $\Rightarrow$ **gazillion of aircraft and crew schedules.**
  Degeneracy level above 90% in integer solutions.
  Perturbation of the master problem (1985-2000).

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Efficient tools within a column generation framework

- **Dual Variable Stabilization and Dual-Optimal Inequalities**
  du Merle et al. (1999), Valerio de Carvalho (2003), Irnich et al. (2014)
- **Dynamic Constraint Aggregation** (DCA) **for Set Partitioning Problems**
  Elhallaoui, Metrane, Desaulniers, Soumis (2005-08)
- **Improved Primal Simplex** (IPS) : *Non degenerate pivots at every iteration.*
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- **Minimum Mean Cycle-Canceling** (MMCC) for network flow problems.
  Goldberg and Tarjan (1989). *Strongly polynomial*.

Generic pricing framework

MMCC (1989), IPS (2008), DCA (2005), and PS (1945)
are all **special cases** of the **generic pricing framework** VSD($\bullet$)
by selecting a subset $P \subseteq B$ of the *basic columns*.

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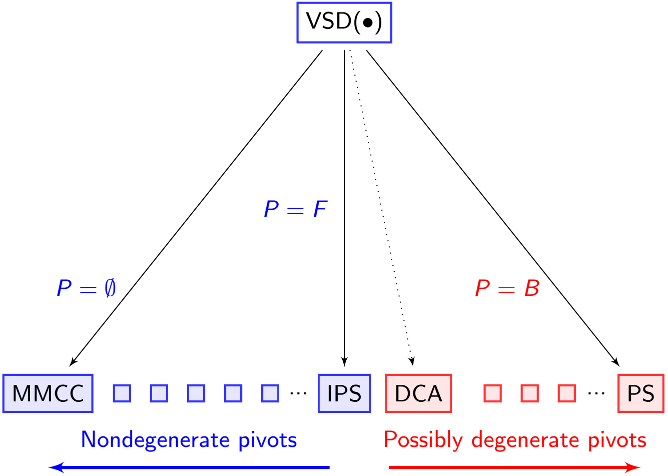
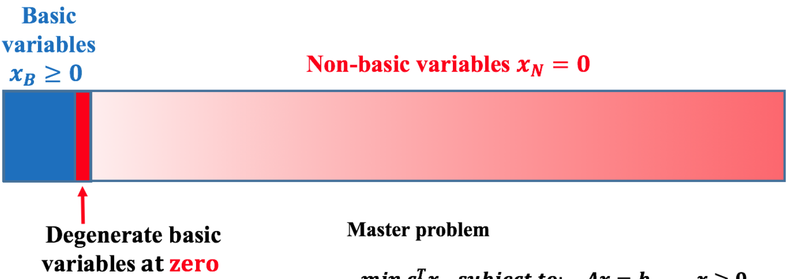


Primal Degeneracy

Definition

A degenerate solution occurs if some basic variables are at one of their bounds.



Master problem

$$min\ c^T x \quad subject\ to: \quad Ax = b, \quad x \geq 0$$

Part 1

VSD($\bullet$) for network flow problems.

Capacitated min cost flow (CMCF)

$$z^*_{\mathbf{CMCF}} := \min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$\text{st.} \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i \quad [\pi_i] \quad \forall i \in N$$

$$0 \le \ell_{ij} \le x_{ij} \le u_{ij} \qquad \forall (i,j) \in A,$$

where $\boldsymbol{\pi} := [\pi_i]_{i\in N}$ is the vector of dual variables.

When vector $\mathbf{b} := [b_i]_{i\in N} = \mathbf{0}$,
one faces a *circulation problem*.

N : set of vertices
A : set of arcs
$\mathbf{c}$: arc cost vector

Cost vs. Reduced cost of a cycle

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Let $\boldsymbol{\pi} = [\pi_i]_{i \in N}$ and define the arc reduced costs as
$\bar{c}_{ij} := c_{ij} - \pi_i + \pi_j, \quad (i,j) \in A.$

Property. **** The reduced cost of a cycle is equal to the cost of that cycle. ****

Cycle $W : 1 \to 2 \to 3 \to 1$

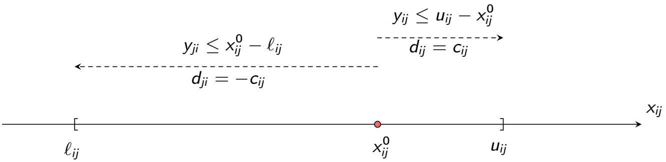
$$\begin{aligned} \bar{c}(W) \quad &= c_{12} - \pi_1 + \pi_2 \\ &+ c_{23} - \pi_2 + \pi_3 \\ &+ c_{31} - \pi_3 + \pi_1 \quad = c(W) \end{aligned}$$

Residual network $G(\mathbf{x}^0) = (N, A(\mathbf{x}^0))$

Change of variables at $\mathbf{x}^0$

Each arc $(i,j) \in A$ is replaced by two arcs representing possible variations :

- upward residual flow $0 \leq y_{ij} \leq r_{ij}^0$
- downward residual flow $0 \leq y_{ji} \leq r_{ji}^0$



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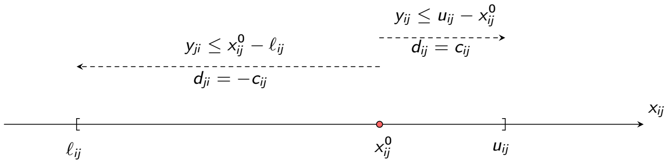
Arc set $A(\mathbf{x}^0)$

Among the **arc support** $A' := \{(i,j) \cup (j,i) \mid (i,j) \in A\}$, only those with strictly positive residual capacities are of interest :

$A(\mathbf{x}^0) := \{(i,j) \in A' \mid r^0_{ij} > 0\}$.



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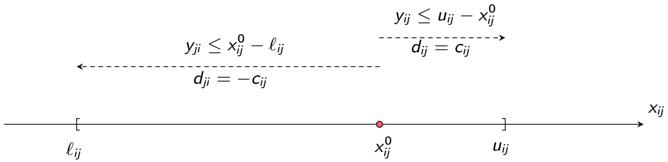

Change of variables at $\mathbf{x}^0$

$$x_{ij} = x^0_{ij} + y_{ij} - y_{ji}, \; \forall (i, j) \in A$$

RESIDUAL NETWORK $G(\mathbf{x}^0) = (N, A(\mathbf{x}^0))$

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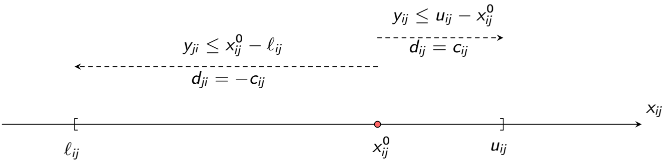

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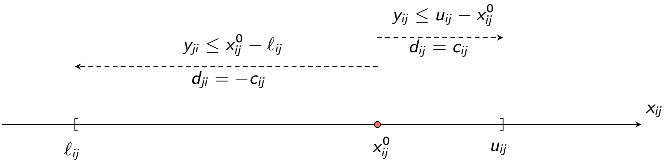


Change of variables at $\mathbf{x}^0$ **Direction at $\mathbf{x}^0$**

$$x_{ij} = x_{ij}^0 + y_{ij} - y_{ji}, \ \forall (i,j) \in A$$

$$x_{ij} - x_{ij}^0 = y_{ij} - y_{ji}, \ \forall (i,j) \in A$$

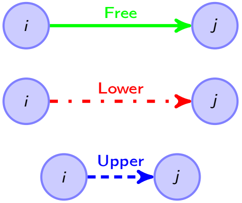
$$\vec{v}_{ij} = x_{ij} - x_{ij}^0 = y_{ij} - y_{ji}, \ \forall (i,j) \in A$$

Index sets F, L, U at $\mathbf{x}^0$

Free $F = \{(i,j) \in A \mid \ell_{ij} < x^0_{ij} < u_{ij}\}$

Lower $L = \{(i,j) \in A \mid \ell_{ij} = x^0_{ij}\}$

Upper $U = \{(i,j) \in A \mid x^0_{ij} = u_{ij}\}$



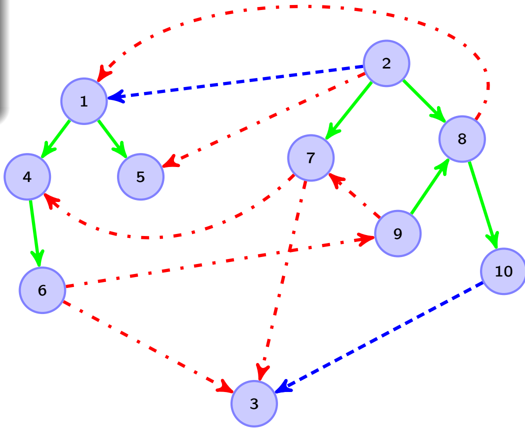


Figure : Arc status on the original network G

F : INDEX SET OF *FREE* VARIABLES

When an arc is free, flow can be carried in either direction thus meaning the presence of two residual arcs.

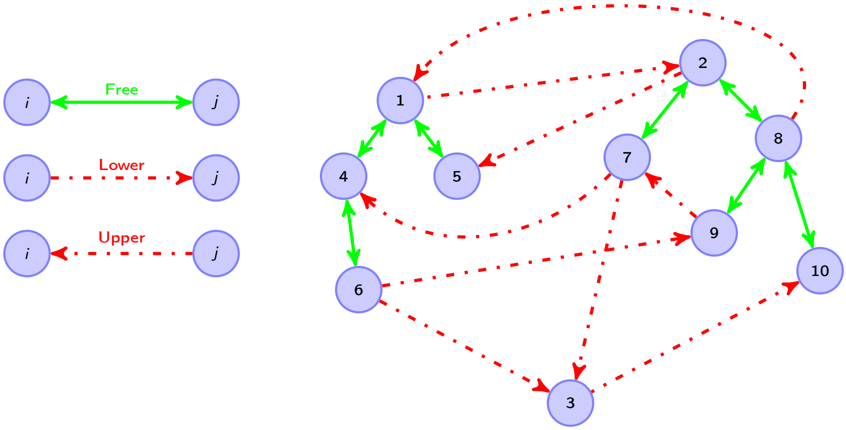


FIGURE : Arc status on the residual network $G(\mathbf{x}^0)$

LET $z^0 := \mathbf{c}^\intercal \mathbf{x}^0$ **AND** $x_{ij} = x_{ij}^0 + y_{ij} - y_{ji}, \ \forall (i,j) \in A$

$$z^*_{\mathbf{CMCF}} := z^0 + \min \sum_{(i,j)\in A(\mathbf{x^0})} d_{ij} y_{ij}$$

$$\text{st.} \sum_{j:(i,j)\in A(\mathbf{x^0})} y_{ij} - \sum_{j:(j,i)\in A(\mathbf{x^0})} y_{ji} = 0 \quad [\pi_i] \quad \forall i \in N$$

$$0 \leq y_{ij} \leq r_{ij}^0 \qquad \forall (i,j) \in A(\mathbf{x}^0).$$

N : set of vertices
$A(\mathbf{x}^0)$: set of arcs
$\mathbf{d}$: arc cost vector

The reformulation corresponds to a **circulation problem** on $G(\mathbf{x}^0)$.

Klein algorithm (1967)

1. Find a *negative cost directed cycle* $\mathbf{y}^0$ on $G(\mathbf{x}^0)$.
2. Determine step-size $\rho := \min_{(i,j)\in W} \frac{r_{ij}^0}{y_{ij}^0} > 0$.
3. Compute $\mathbf{x}^0 + \rho \vec{\mathbf{v}}^0$, update $G(\bullet)$ and repeat until optimality.

LET $z^0 := \mathbf{c}^\mathsf{T}\mathbf{x}^0$ **AND** $x_{ij} = x^0_{ij} + y_{ij} - y_{ji},\ \forall (i,j) \in A$

$$z^*_{\mathbf{CMCF}} := z^0 + \min \sum_{(i,j)\in A(\mathbf{x^0})} d_{ij} y_{ij}$$

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***** How many variables are involved in a cycle ?**

9.19. Show that if we apply the cycle-canceling algorithm to the minimum cost flow problem shown in Figure 9.24, some sequence of augmentations requires 2×10^6 iterations to solve the problem.

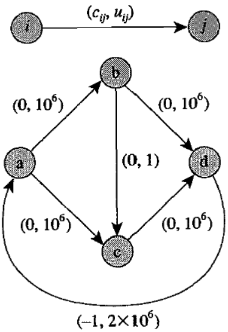

Figure 9.24 Network where cycle canceling algorithm performs 2×10^6 iterations.

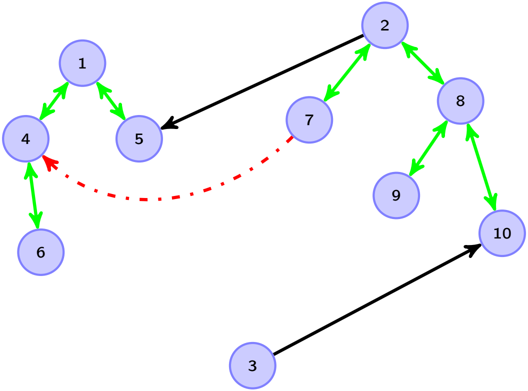

FIGURE : Non-basic arc $(7, 4)$ induces a unique cycle of zero step size.

Occurrence of degenerate pivots in the Primal Network Simplex

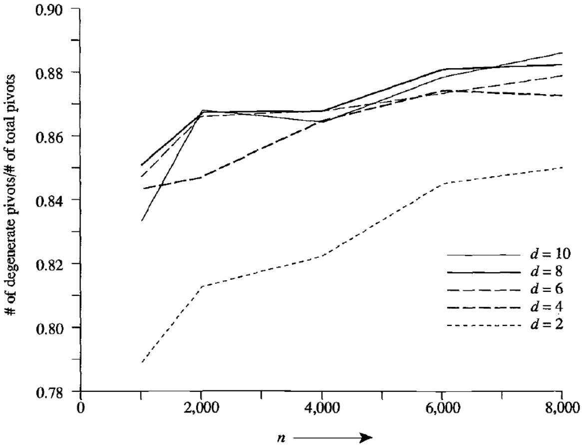

Figure 18.7 Occurance of degenerate pivots.

d= NETWORK DENSITY, AHUJA, MAGNANTI, ORLIN (1993)

75%–90% of the Primal Simplex pivots are degenerate on network problems.

ARC REDUCED COSTS

Let $\boldsymbol{\pi} = [\pi_i]_{i \in N}$.

$$\bar{c}_{ij} := c_{ij} - \pi_i + \pi_j, \qquad (i,j) \in A$$

$$\bar{d}_{ij} := d_{ij} - \pi_i + \pi_j, \qquad (i,j) \in A(\mathbf{x}^0).$$

Equivalent conditions. $\mathbf{x}^0$ is optimal

$\Leftrightarrow$ **Primal.** *No negative cost directed cycle on* $G(\mathbf{x}^0)$.

$\Leftrightarrow$ **Dual.** $\exists \boldsymbol{\pi}$ such that $\bar{d}_{ij} \geq 0, \forall (i,j) \in A(\mathbf{x}^0)$.

$\Leftrightarrow$ **Complementary slackness.** $\exists \boldsymbol{\pi}$ such that $\forall (i,j) \in A$,

$$\bar{c}_{ij} > 0 \Rightarrow (i,j) \in L\,;$$

$$\bar{c}_{ij} < 0 \Rightarrow (i,j) \in U\,;$$

$$(i,j) \in F \Rightarrow \bar{c}_{ij} = 0.$$

Arc reduced costs

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G vs. $G(\mathbf{x}^0)$

Primal and dual conditions on $G(\mathbf{x}^0)$.

Complementary slackness conditions on G.

Partition of the variables for MMCC, IPS, and PS

Framework

Find a cycle $\mathbf{y}^0 = \begin{bmatrix} \mathbf{y}_H^0 \\ \mathbf{y}_V^0 \end{bmatrix}$ (a ray)

Given $P \subseteq B$, partition of the set of arcs of $A(\mathbf{x}^0)$ into $\{H_P^0, \mathbf{V_P^0}\}$.

MMCC □ □ □ □ □ ··· IPS   □ □ □ ··· PS

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Visible arcs $\mathbf{y}_V^0$ computed first (pricing) ; *Hidden arcs* $\mathbf{y}_H^0$ computed second.

MMCC ☐ ☐ ☐ ☐ ☐ ··· IPS ☐ ☐ ☐ ··· PS

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Compute direction $\vec{\mathbf{v}}^0$.

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VSD($\bullet$)

MMCC □ □ □ □ □ ··· IPS   □ □ □ ··· PS

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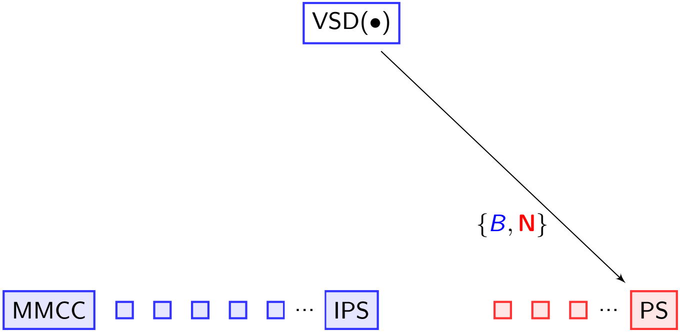

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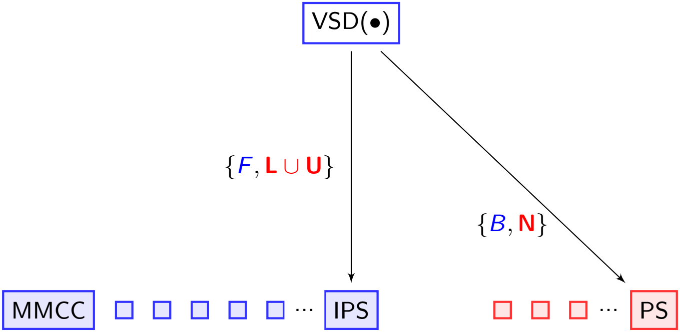

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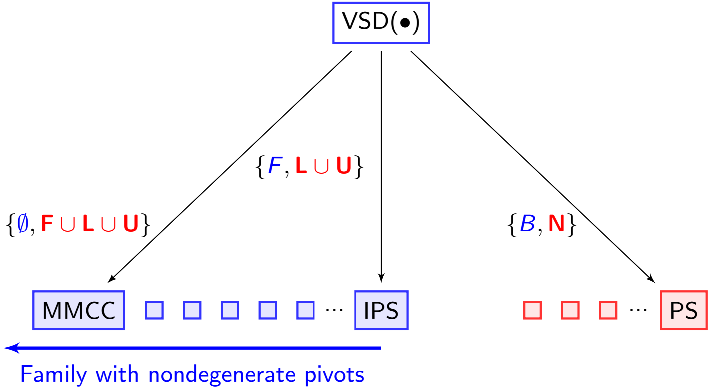

Algorithmic process for network flow problems

- Given a feasible flow solution $\mathbf{x}^0$ and the residual network $G(\mathbf{x}^0)$.
- Select $P \subseteq B$.
- **Hide a subset $H_P = \{(i,j),(j,i) \mid (i,j) \in P\}$ of the arcs.**
  Apply a cost transfer by manipulating the dual vector π
  such that the reduced cost of the hidden arcs becomes zero.
- **Pricing** : On the set of visible arcs, find a *directed cycle* $\mathbf{y}_V^0$
  of minimum **average** *reduced cost* μ_V.
- If $\mu_V \geq 0$, **terminate** with an optimality certificate for $\mathbf{x}^0$.
- Recover $\mathbf{y}_H^0$ and the uniquely extended cycle $\mathbf{y}^0 = \begin{bmatrix} \mathbf{y}_H^0 \\ \mathbf{y}_V^0 \end{bmatrix}$.
- Determine the step-size $\rho = \min_{(i,j)\in W} \frac{r_{ij}^0}{y_{ij}^0} \geq \mathbf{0}$.
- Compute $\mathbf{x}^1 = \mathbf{x}^0 + \rho \vec{\mathbf{v}}^0$, update $G(\mathbf{x}^1)$ and repeat until optimality.

Pricing of MMCC on $A(\mathbf{x}^0)$

MMCC (1989) $P = \emptyset$

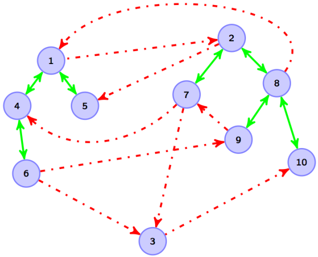

Figure : Arcs of $G(\mathbf{x}^0)$

Find a *directed cycle* of **minimum average (reduced) cost** μ_V : $\rho > 0$

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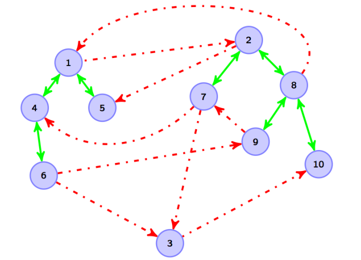

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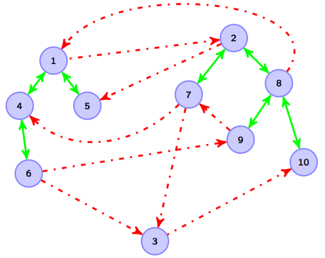


Figure : Arcs of $G(\mathbf{x}^0)$

Find a *directed cycle* of **minimum average (reduced) cost** μ_V : $\rho > 0$

Pricing : Find $\boldsymbol{\pi}$ such that the smallest reduced cost μ is as large as possible

$$\max \ \mu_V$$

$$\text{st.} \ \mu_V \leq d_{ij} - \pi_i + \pi_j \qquad [y_{ij}] \ \forall (i,j) \in A(\mathbf{x}^0)$$

Optimize $\boldsymbol{\pi}$ to prove optimality ($\mu \geq 0$), that is, **maximize** the smallest reduced cost value.

$\mathbf{x}^0$ optimal $\Leftrightarrow$

Dual : $\exists \boldsymbol{\pi}$ such that $\bar{d}_{ij} \geq 0$, $\forall (i,j) \in A(\mathbf{x}^0)$.

MMCC (1989) $P = \emptyset$

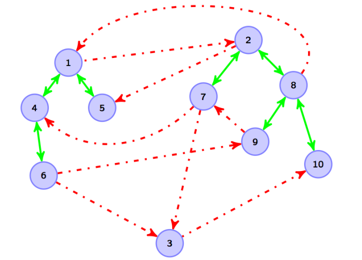


Figure : Arcs of $G(\mathbf{x}^0)$

Find a *directed cycle* of **minimum average (reduced) cost** μ_V : $\rho > 0$

Pricing : Find $\boldsymbol{\pi}$ such that the smallest reduced cost μ is as large as possible

$$\max \ \mu_V$$

$$\text{st.} \quad \mu_V \leq d_{ij} - \pi_i + \pi_j \qquad [y_{ij}] \quad \forall (i,j) \in A(\mathbf{x}^0)$$

Optimize $\boldsymbol{\pi}$ to prove optimality ($\mu \geq 0$), that is, **maximize** the smallest reduced cost value.

$\mathbf{x}^0$ optimal $\Leftrightarrow$

Primal : $G(\mathbf{x}^0)$ contains *no negative cost directed cycle*.

Dual : $\exists \boldsymbol{\pi}$ such that $\bar{d}_{ij} \geq 0$, $\forall (i,j) \in A(\mathbf{x}^0)$.

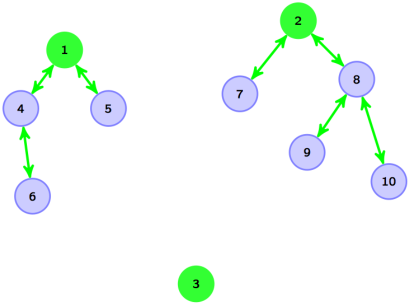
$$\mu_V := \min \sum_{(i,j)\in A(\mathbf{x}^0)} d_{ij} y_{ij}$$

$$\text{st.} \quad \sum_{j:(i,j)\in A(\mathbf{x}^0)} y_{ij} - \sum_{j:(j,i)\in A(\mathbf{x}^0)} y_{ji} = 0 \qquad [\pi_i] \quad \forall i \in N$$

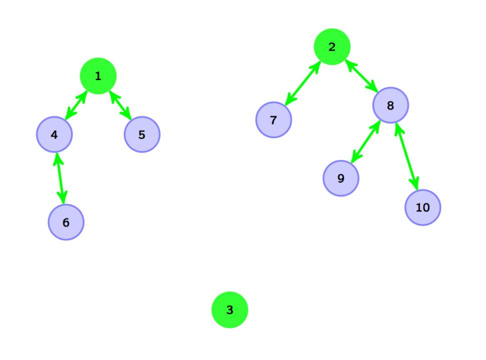
$$\sum_{(i,j)\in A(\mathbf{x}^0)} y_{ij} = 1 \qquad [\mu_V]$$

$$y_{ij} \geq 0 \qquad \forall (i,j) \in A(\mathbf{x}^0)$$

Improved Primal Simplex (2008) $P = F$



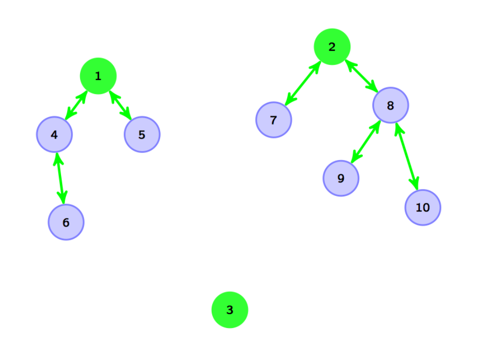
Improved Primal Simplex (2008) $P = F$



- Select a root node for each tree of F.
- Set to zero the reduced cost of free arcs.
- Compute reduced costs of arcs $(i, j) \notin F$.
- Shrink the trees of free arcs.
- Remove dominated arcs.

Find a *directed cycle* of **minimum** average **reduced cost** $\mu_V : \rho > 0$

Improved Primal Simplex (2008) $P = F$



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- Remove dominated arcs.

Find a *directed cycle* of **minimum** average reduced cost $\mu_V : \rho > 0$

$\mu_V \geq 0 \Leftrightarrow \mathbf{x}^0$ optimal

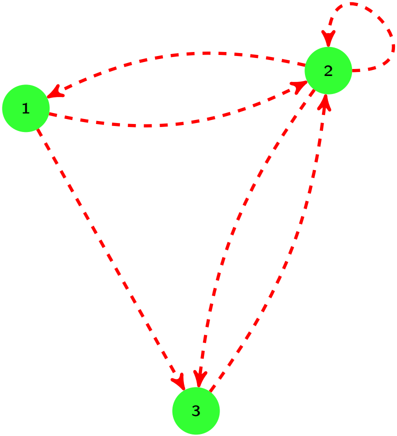
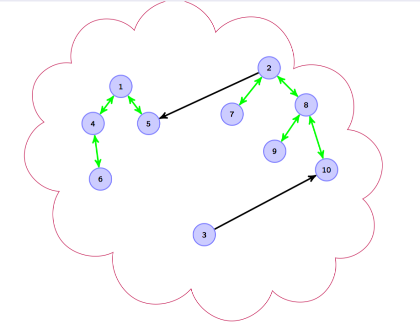
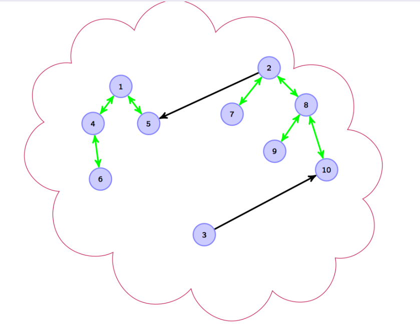


Figure : **Contracted network $F(\mathbf{x}^0)$**

Primal Simplex (1945) $P = B$

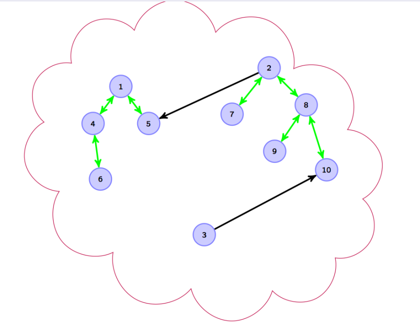


Primal Simplex (1945) $P = B$



- Select a root node for the spanning tree.
- **Set to 0 the reduced cost of basic arcs.**
- Compute reduced costs of arcs $(i, j) \notin B$.
- **Shrink the spanning tree of basic arcs**.
  Orientation of some hidden arcs is lost.

PRIMAL SIMPLEX (1945) $P = B$



- Select a root node for the spanning tree.
- **Set to 0 the reduced cost of basic arcs.**
- Compute reduced costs of arcs $(i,j) \notin B$.
- **Shrink the spanning tree of basic arcs.**
  Orientation of some hidden arcs is lost.

Find a loop (a single arc) of **minimum average reduced cost** $\mu_V : \rho \geq 0$

** *Extended cycle* can be infeasible on $G(\mathbf{x}^0)$.

$\mu_V \geq 0 \Rightarrow \mathbf{x}^0$ optimal

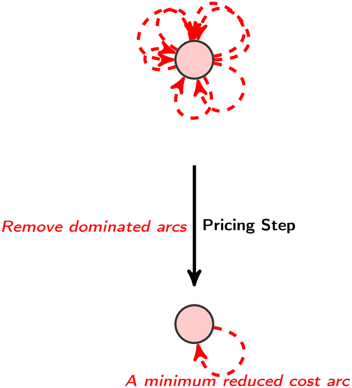
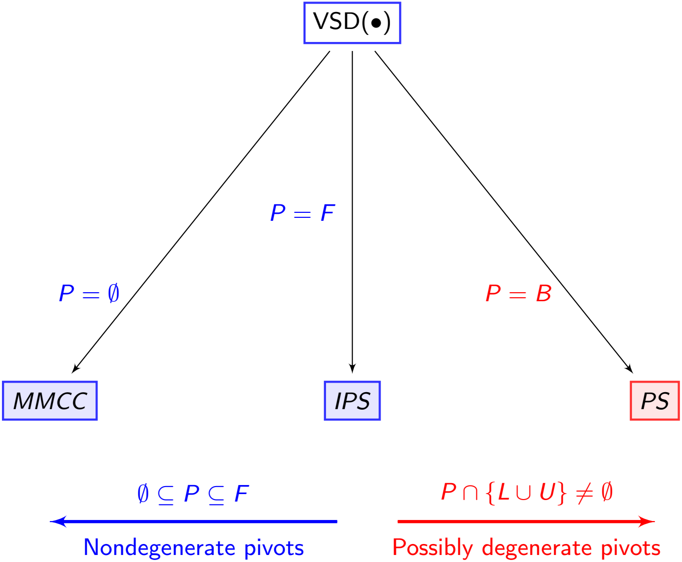


FIGURE : **Single node network** $B(\mathbf{x}^0)$

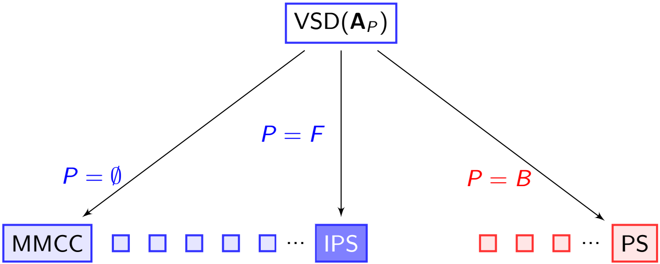
Pricing : directed cycle of minimum mean reduced cost μ_V on the visible graph...



Part 2

VSD($\mathbf{A}_P$) for linear problems, where $\mathbf{A}_P$ is a submatrix of basis A_B.

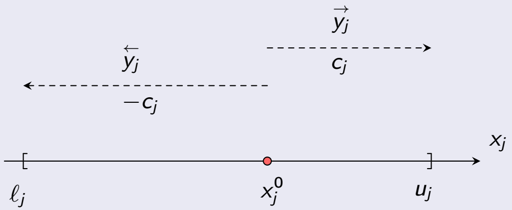
Nonsingular transformation $\mathrm{T}^{-1}(\mathrm{Ax} = \mathrm{b})$.



LINEAR PROGRAM LP

$$z^\star_{LP} := \min \ \mathbf{c}^\mathsf{T}\mathbf{x}$$
$$\text{st.} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \quad [\boldsymbol{\pi}]$$
$$\boldsymbol{\ell} \le \mathbf{x} \le \mathbf{u},$$

where $\mathbf{x} \in \mathbb{R}^n_+$ and $\boldsymbol{\pi} \in \mathbb{R}^m$.



LET $\mathbf{x}^0$ BE FEASIBLE

$$\mathbf{x} := \mathbf{x}^0 + (\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) \quad \Rightarrow \quad \cancel{\mathbf{A}\mathbf{x}^0} + \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \cancel{\mathbf{b}}$$

$$z^\star_{LP} = \mathbf{c}^\mathsf{T}\mathbf{x}^0 + \min \quad \mathbf{c}^\mathsf{T}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}})$$
$$\text{st.} \quad \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{0} \quad [\boldsymbol{\pi}]$$
$$\mathbf{0} \le \vec{\mathbf{y}} \le \vec{\mathbf{r}}^0$$
$$\mathbf{0} \le \overleftarrow{\mathbf{y}} \le \overleftarrow{\mathbf{r}}^0$$

$G(\mathbf{x}^0)$	$LP(\mathbf{x}^0)$
Circulation network	**Homogeneous system**
Directed cycles	*Combinations of the $\vec{\mathbf{y}}$ and $\overleftarrow{\mathbf{y}}$ variables*

$LP(\mathbf{x}^0)$

$$z^\star_{LP} := z^0 + \min \quad \mathbf{c}^\mathsf{T}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}})$$

$$\text{st.} \quad \mathbf{A}(\vec{\mathbf{y}} - \overleftarrow{\mathbf{y}}) = \mathbf{0} \quad [\pi]$$

$$\mathbf{0} \leq \vec{\mathbf{y}} \leq \vec{\mathbf{r}}^0$$

$$\mathbf{0} \leq \overleftarrow{\mathbf{y}} \leq \overleftarrow{\mathbf{r}}^0$$

$\mathbf{x}^0$ OPTIMAL

$\Leftrightarrow$ **Primal** : $LP(\mathbf{x}^0)$ contains no negative cost combination of the $\vec{y}, \overleftarrow{y}$-variables.

$\Leftrightarrow$ **Dual** : $\exists \boldsymbol{\pi}$ such that $\bar{d}_j \geq 0$,

$$\forall \vec{y}_j, \overleftarrow{y}_j \text{ such that } \vec{r}^0_j > 0, \overleftarrow{r}^0_j > 0.$$

$\Leftrightarrow$ **Complementary slackness conditions** verified on LP.

Transformation matrix $\mathbf{T}^{-1}$ (a special case is the inverse of the simplex basis)

$$\mathbf{T} = \begin{bmatrix} \mathbf{A}_{RP} & \mathbf{0} \\ \mathbf{A}_{SP} & \mathbf{I}_{m-p} \end{bmatrix} \quad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{RP}^{-1} & \mathbf{0} \\ -\mathbf{A}_{SP}\mathbf{A}_{RP}^{-1} & \mathbf{I}_{m-p} \end{bmatrix}$$

$$\bar{\mathbf{A}} = \mathbf{T}^{-1} \begin{bmatrix} \mathbf{A}_{RP} & \mathbf{A}_{RQ} \\ \mathbf{A}_{SP} & \mathbf{A}_{SQ} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_p & \bar{\mathbf{A}}_{RQ} \\ \mathbf{0} & \bar{\mathbf{A}}_{SQ} \end{bmatrix}$$

Residual problem $LP(\mathrm{x}^0)$ after transformation

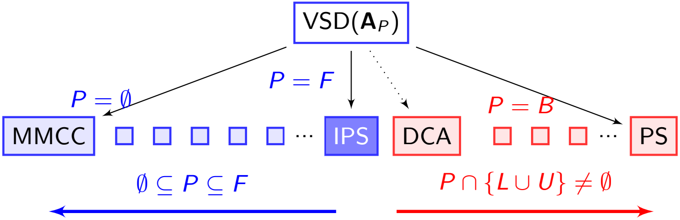
$$\begin{array}{lrcrcll}
\mathbf{c}^\intercal \mathbf{x}^0 + \min & \mathbf{c}_P^\intercal \mathbf{y}_H & + & \mathbf{c}_Q^\intercal \mathbf{y}_V & & & \\
 & \mathbf{y}_H & + & \bar{\mathbf{A}}_{RQ}\mathbf{y}_V & = & \mathbf{0} & [\psi_R = \mathbf{c}_P] \\
 & & & \bar{\mathbf{A}}_{SQ}\mathbf{y}_V & = & \mathbf{0} & [\psi_S] \\
 & -\overleftarrow{\mathbf{r}}_P^0 \le \mathbf{y}_H \le \overrightarrow{\mathbf{r}}_P^0 & & -\overleftarrow{\mathbf{r}}_Q^0 \le \mathbf{y}_V \le \overrightarrow{\mathbf{r}}_Q^0 & & &
\end{array}$$

Interpretation

Dynamic Dantzig-Wolfe decomposition based on the actual value of $\mathbf{x}^0$.

Transformation $\mathbf{T}^{-1}$ can be changed at every iteration, or kept forever.

Variants



SET P SELECTION

- $P = B$, primal simplex
- $P \supseteq F$, dynamic constraint aggregation
- $P = F$, improved primal simplex (*all moves are on edges*)
- $P = \emptyset$, minimum mean cycle-canceling

Proposition.

When $P \subseteq F$, the step size $\rho > 0$.

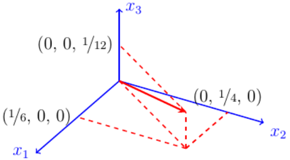
Properties

Generic framework	MMCC	IPS	DCA	PS
	Nondegenerate pivots		*Possibly degenerate pivots*	
$\emptyset \subseteq P \subseteq B$	$\emptyset$	F	$F \subseteq P$	B
z	$z^0 > z^1$	$z^0 > z^1$	$z^0 \geq z^1$	$z^0 \geq z^1$
μ	$\mu^0 \leq \mu^1$	*oscillations*	*oscillations*	*oscillations*

$$\begin{array}{lrrrl} \max & 130x_1 & +80x_2 & +60x_3 & \\ \text{s.t.} & 2x_1 & -x_2 & +2x_3 & \le 21 \\ & -x_1 & +x_2 & -x_3 & \le 8 \\ & 2x_1 & -x_2 & -x_3 & \le 15 \\ & -x_1 & -x_2 & +2x_3 & \le 32 \\ & x_1, & x_2, & x_3 & \ge 0 \end{array}$$

Variant	c_j coefficient / Direction & Solution	130 / x_1	80 / x_2	60 / x_3	s_1	s_2	s_3	s_4	z^1	μ_V^0	ρ^0
$P = F$	$\vec{\mathbf{v}}^0$ (edge)	1	0	0	-2	1	-2	1		130	7.5
	$\mathbf{x}^1$	7.5	0	0	6	15.5	0	39.5	975		
$P = \emptyset$	$\vec{\mathbf{v}}^0$ (interior) $w_j = 1$	1/6	1/4	1/12	-1/4	0	0	1/4		140/3	84
	$\mathbf{x}^1$	14	21	7	0	8	15	53	3920		
$P = \emptyset$	$\vec{\mathbf{v}}^0$ (face) $w_j = \lVert\mathbf{a}_j\rVert^2$	1/22	1/11	0	0	-1/22	0	3/22		145/11	176
	$\mathbf{x}^1$	8	16	0	21	0	15	56	2320		

Figure 6: Directions found at $\mathbf{x}^0$ in pricing for $P = F$ and $P = \emptyset$



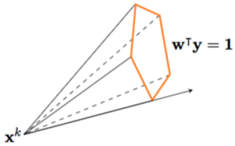


Figure 2: At $\mathbf{x}^k$, the cone $\{\mathbf{y} \geq \mathbf{0} \mid \mathbf{K}\mathbf{y} = \mathbf{0}\}$ cut by $\mathbf{w}^\intercal\mathbf{y} = \mathbf{1}$

$$\begin{aligned} &\max \; \mu_V \\ &\text{s.t.} \quad \mu_V \leq \frac{\bar{d}_j}{w_j} = \frac{1}{w_j}(d_j - \boldsymbol{\psi}_R^\intercal \mathbf{k}_{Rj} - \boldsymbol{\psi}_S^\intercal \bar{\mathbf{k}}_{Sj}) \quad [y_j] \quad \forall j \in V_P, \end{aligned} \tag{4}$$

where the vector $\boldsymbol{\psi}_R^\intercal = \mathbf{c}_P^\intercal \mathbf{A}_{RP}^{-1}$ is fixed by (24) whereas the vector $\boldsymbol{\psi}_S^\intercal$ is part of the optimization so as to maximize the minimum reduced cost.

Dualizing (25), we obtain the primal form of the oracle which comprises $m - p + 1$ constraints and writes as

$$\begin{aligned} \min \; & \sum_{j \in V_P} \tilde{d}_j y_j \\ \text{s.t.} \; & \sum_{j \in V_P} \bar{\mathbf{k}}_{Sj} y_j = \mathbf{0} && [\boldsymbol{\psi}_S] \\ & \sum_{j \in V_P} w_j \, y_j = 1 && [\mu_V] \\ & y_j \geq 0, \quad \forall j \in V_P. \end{aligned} \tag{5}$$

where $\tilde{d}_j := d_j - \boldsymbol{\psi}_R^\intercal \mathbf{k}_{Rj} = d_j - \mathbf{c}_P^\intercal \mathbf{A}_{RP}^{-1} \mathbf{k}_{Rj}, \forall j \in V_P$.

Optimize π with stabilization intervals

$$
\begin{aligned}
\max \quad & \mu \\
\text{st.} \quad & \mu \le d_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j \quad \forall j \\
& \pi_i \in [inf_i, sup_i] \quad \forall i
\end{aligned}
$$

Adding information to the pricing problem

Optimize π with stabilization intervals

$$\begin{aligned} \max\ & \mu \\ \text{st. } & \mu \leq d_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j \quad \forall j \\ & \pi_i \in [inf_i, sup_i] \quad \forall i \end{aligned}$$

Optimize π with a link with the current objective value

$$\begin{aligned} \max\ & \mu \\ \text{st. } & \mu \leq d_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j \quad \forall j \\ & \boldsymbol{\pi}^\intercal \mathbf{b} \geq z^k \end{aligned}$$

Adding information to the pricing problem

Optimize π with stabilization intervals

$$\begin{aligned} \max\ & \mu & & \\ \text{st.}\ & \mu & \leq d_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j & \quad \forall j \\ & \pi_i & \in [inf_i, sup_i] & \quad \forall i \end{aligned}$$

Optimize π with a link with the current objective value

$$\begin{aligned} \max\ & \mu & & \\ \text{st.}\ & \mu & \leq d_j - \boldsymbol{\pi}^\intercal \mathbf{a}_j & \quad \forall j \\ & \boldsymbol{\pi}^\intercal \mathbf{b} & \geq z^k & \end{aligned}$$

Properties : $z^0 > z^1$ $\mu^0 < \mu^1$ $s^0 < s^1$ **Interior directions**

1. About the Minimum Mean Cycle-Canceling Algorithm
   Discrete Applied Mathematics, 2015.
   doi :10.1016/j.dam.2014.07.005
   Primal/dual formulations of the pricing problem ; new complexity result.
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   European Journal of Transportation and Logistics, 2015.
   doi :10.1007/s13676-015-0077-5
   VSD(•) links DCA, IPS, and PE (positive edge rule).
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   Operations Research Letters. 2014.
   doi :10.1016/j.orl.2014.10.001
   Theorems on Networks $\mapsto$ Linear programs.
4. A Strongly Polynomial Contraction-Expansion Algorithm for Networks
   Understanding degeneracy on network flow problems.
5. Vector Space Decomposition for Linear and Network Flow Problems
   Family of algorithms with nondegenerate pivots + interior directions.
6. A linear fractional pricing problem for solving linear programs.
   Interior directions.