Selective pricing in branch-and-price algorithms for vehicle routing

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Outline

Vehicle routing

General definition

Given a set of demands (deliveries, pickups, etc.), find least-cost vehicle routes to fulfill these demands

- Large number of problem variants (CVRP, VRPTW, etc.)
- Most of the times, routes are subject to elementarity requirements (e.g., customers cannot be visited more than once)

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A simplified model for vehicle routing

E: set of feasible elementary routes

N: set of customers

c_r: cost of route r

a_{ri}: 1 if route r visits customer i, 0 otherwise

x_r: binary variable equal to 1 if route r is selected, 0 otherwise

$$\min \quad \sum_{r \in E} c_r x_r \tag{1}$$

$$\text{s.t.} \quad \sum_{r \in E} a_{ri} x_r = 1, \qquad \forall i \in N \tag{2}$$

$$x_r \in \{0, 1\}, \qquad \forall r \in E \tag{3}$$

Branch-price-and-cut (BPC)

BPC is the leading exact methodology for vehicle routing

- Branch-and-bound algorithm
- Linear relaxations are solved by column generation
  - Iterative method alternating between a restricted master problem and a pricing problem
- Cuts are added to strengthen these linear relaxations

Pricing problem

- Find feasible routes with negative reduced costs
- Corresponds to an elementary shortest path problem with resource constraints (ESPPRC)
- NP-hard in the strong sense

Path relaxations

Given the difficulty of solving the ESPPRC, a relaxation of the pricing problem is often used

- Allows non-elementary routes
  - SPPRC with k-cycle elimination (Irnich, Villeneuve, 2006)
  - ng-SPPRC (Baldacci et al., 2008)
- In model (1)–(3), set E is replaced by a larger set $R \supset E$
- Non-elementary routes can be part of linear relaxation solutions
  - Yield weaker lower bounds in the search tree
  - Removed from integer solutions through cutting and branching

Lower bounds

A lower bound $\underline{z}(R)$ is computed by solving

$$\begin{aligned} \underline{z}(R) = \min \quad & \sum_{r \in R} c_r x_r \\ \text{s.t.} \quad & \sum_{r \in R} a_{ri} x_r = 1, && \forall i \in N \\ & 0 \leq x_r \leq 1, && \forall r \in R \end{aligned}$$

Traditional stopping criterion

Reduced cost $\bar{c}_r \geq 0$ for $r \in R$

A valid lower bound is also achieved if $\bar{c}_r \geq 0$ for $r \in E$

Indeed, we can assume that ...

... this lower bound $\underline{z}(Q)$ is computed by replacing R with $Q = E \cup B$ where B is the set of routes in the basis

Remarks

- $\underline{z}(Q) \geq \underline{z}(R)$ because $E \subseteq Q \subseteq R$
- Q and thus $\underline{z}(Q)$ depend on the pricing and CG algorithms

Selective pricing

- Aims at pricing elementary paths among the relaxed paths
- Can discard a non-elementary path even if it has the least reduced cost
- Can still generate non-elementary paths

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Labeling algorithm

Pricing is performed using a labeling algorithm

- Label contains all information (resource values, including reduced cost) to represent a partial path
- Labels are extended in the network using resource extension functions
- A dominance rule is applied to avoid enumerating all paths

Here we focus only on certain label components and on the dominance rule

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Additional notation

- L: a label representing a path $p(L)$
- $n(p)$: last node of path p
- $\bar{c}(p)$: reduced cost of path p
- w: a path extension
- $p \oplus w$: path resulting from the concatenation of path p and extension w (assuming that w starts in node $n(p)$)

Dominance definition without path relaxation

Let L_1 be a label such that $p(L_1) \in E$

Definition

L_1 is dominated if,
for every extension w_1 such that $p(L_1) \oplus w_1 \in E$,
there exists a label L_2 and an extension w_2 such that

1. $n(p(L_2)) = n(p(L_1))$
2. $n(p(L_2) \oplus w_2) = n(p(L_1) \oplus w_1)$
3. $p(L_2) \oplus w_2 \in E$
4. $\bar{c}(p(L_2) \oplus w_2) \leq \bar{c}(p(L_1) \oplus w_1)$

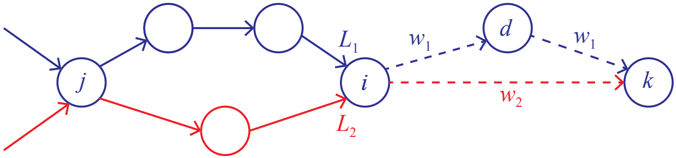
Remark

Often $w_2 = w_1$ is chosen

Example where $w_2 \neq w_1$ (Ropke and Cordeau, 2009)

Pickup and delivery problem

- $O(L)$: Requests on board at node $n(L)$ along path $p(L)$
- Triangle inequality on every resource holds at delivery nodes
- L_2 can dominate L_1 if $O(L_2) \subseteq O(L_1)$
- Here, $O(L_1) \setminus O(L_2)$ contains a single request whose delivery node is d



Dominance definitions with path relaxation

The previous definition holds when E is replaced by R

Let's see a stronger definition

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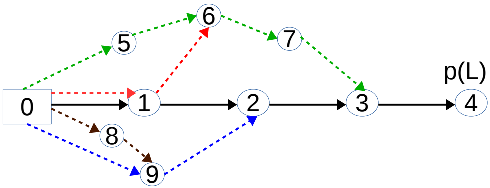
Let's see a stronger definition

Set $S(L)$

- $S(L)$: contains $p(L)$ and all paths identified by the algorithm as dominated by $p(L)$
- It includes extensions of paths that were dominated by a subpath of $p(L)$ and, recursively, the extensions of the paths dominated by them



S(L) = {(0,1,2,3,4),
(0,5,6,7,3,4),
(0,1,6,7,3,4),
(0,9,2,3,4),
(0,8,9,2,3,4)}

New definition (selective pricing)

L_1 is dominated if,
for every extension w_1 such that $p(L_1) \oplus w_1 \in R$, EITHER

1. there exists a label L_2 and an extension w_2 such that
    1. $n(p(L_2)) = n(p(L_1))$
    2. $n(p(L_2) \oplus w_2) = n(p(L_1) \oplus w_1)$
    3. $p(L_2) \oplus w_2 \in R$
    4. $\bar{c}(p(L_2) \oplus w_2) \leq \bar{c}(p(L_1) \oplus w_1)$

OR

2. $q \oplus w_1 \notin E, \qquad \forall q \in S(L_1)$

Remark

If the second case occurs for an extension w_1, then $p(L_1) \oplus w_1 \notin E$ and label L_1 might be discarded without identifying a path dominating $p(L_1) \oplus w_1$

ng-path

- For each customer i, let $NG(i) \subseteq N$ be the neighborhood of i, its λ closest customers
- An *ng*-route can contain a cycle $i_0 - i_1 - i_2 - \ldots - i_k = i_0$ only if there exists $j \in \{1, 2, \ldots, k-1\}$ such that $i_0 \notin NG(i_j)$

$M(L)$: set of customers (memory) to which L cannot be extended according to the NG-sets

Traditional dominance rule

A label L_1 is dominated by a label L_2 if

1. $\bar{c}(p(L_2)) \leq \bar{c}(p(L_1))$
2. $M(L_2) \subseteq M(L_1)$
3. other conditions depending on the problem

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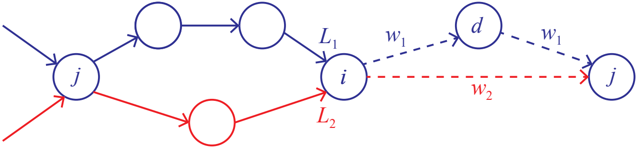
Example from Cherkesly et al. (2015)

Pickup and delivery problem

- Same example as before except with *ng*-routes and $k = j$
- $j \in M(L_2) \subseteq M(L_1)$
- $j \notin NG(d)$
- $p(L_1) \oplus w_1 \in R$
- $p(L_2) \oplus w_2 \notin R$ but $q \oplus w_1 \notin E, \forall q \in S(L_1)$



ng-paths with selective pricing

$C(L)$: set of nodes belonging to all paths in $S(L)$

A utopian labeling algorithm

- When extending label L_i along an arc (i,j) to create a label L_j: $C(L_j) = C(L_i) \cup \{j\}$
- When label L_2 dominates label L_1: $C(L_2) = C(L_2) \cap C(L_1)$
- Dominance rule (L_2 dominates L_1):
  - $M(L_2) \subseteq M(L_1) \cup C(L_1)$
- Do not extend label L to any node in $C(L)$

Impossible because ...

$S(L)$ and thus $C(L)$ are not fully known before the end of the algorithm

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An applicable labeling algorithm

- When extending label L_i along an arc (i, j) to create a label L_j: $C(L_j) = C(L_i) \cup \{j\}$
- When label L_2 dominates label L_1: $C(L_2) = C(L_2) \cap C(L_1)$
- Dominance rule (L_2 dominates L_1):
  - $M(L_2) \subseteq M(L_1) \cup C(L_1)$
  - $C(L_2) \subseteq C(L_1)$ if L_2 has been extended
- Do not extend label L to any node in $C(L)$

Heuristic pricing

Heuristic

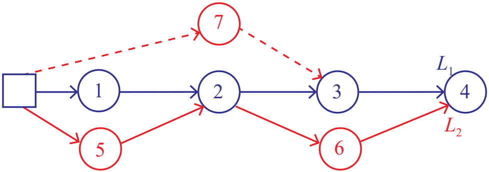
Use standard pricing except that $M(L_2) \subseteq M(L_1)$ is replaced by

$$M(L_2) \subseteq V(L_1)$$

in the dominance rule ($V(L_1)$: set of customers visited in $p(L_1)$)

Example showing why it is not exact

$M(L_1) = \{4\}$, $M(L_2) = \{2, 4\}$, L_2 dominates L_1 and cannot be extended to 2, while $0 - 7 - 3 - 4 - 2$ would be feasible



Computational experiments

Instances

- Vehicle routing problem with time windows (VRPTW)
- A few Solomon instances (100 customers) and Gehring and Homberger instances (200 customers)
- Hard-to-solve instances
  - Number of customers per route is large
  - Many possibilities to cycle

Experiments

- Comparison of
  1. Pecin et al. (2016) algorithm with standard *ng*-path pricing
  2. Pecin et al. (2016) algorithm with selective *ng*-path pricing (heuristic followed by exact algorithm)
- Root node only with rounded capacity cuts only

Results

Instance	Time (s) with SelectiveP	Time (s) with StandardP	Gain %
R208	1250	1539	18,7%
R210-10%	293	352	16,7%
R210-20%	401	416	3,6%
R211-10%	432	467	7,5%
R211-20%	498	635	21,5%
C2-2-3	8195	8487	3,4%
C2-2-4	33014	33614	1.8%
R2-2-4	31925	40012	20,2%
R2-2-8	39795	64602	38.4%
RC2-2-3	12585	14795	14,9%
RC2-2-4	57046	144783	60,6%
RC2-2-7	4210	4553	7,5%
RC2-2-8	7847	7902	0.7%
RC2-2-9	10338	11493	10.0%

Conclusion

- New paradigm for very-hard-to-solve instances
- May be applicable to other problem types
- Preliminary results show some potential

Future work

- Improve the proposed algorithm
  - Refine dominance rule
  - Filter out some non-elementary routes during column generation
- Apply principle to other problems with relaxed pricing

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