Simultaneous Column-and-Row Generation for Solving Large-Scale Linear Programs with Column-Dependent-Rows

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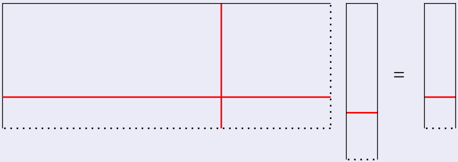
2012

Main Motivation: Column-and-Row Generation

minimize ...

subject to



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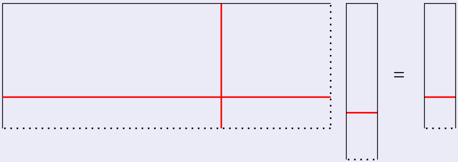

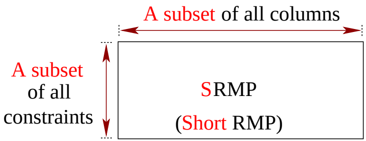

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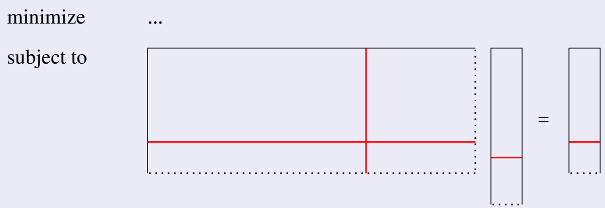

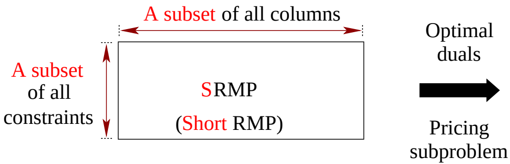

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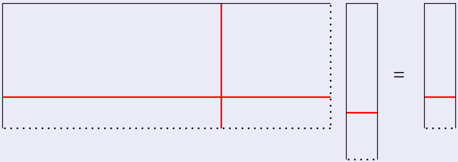

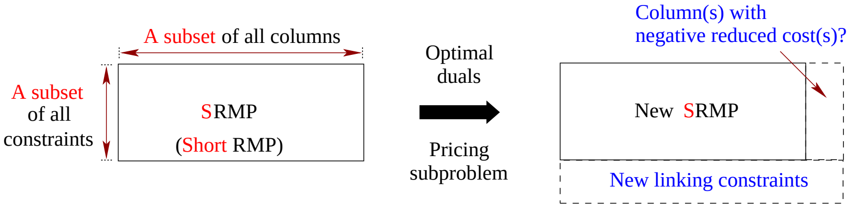

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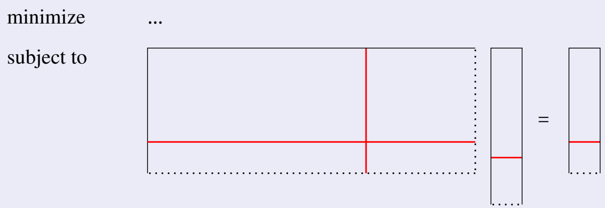

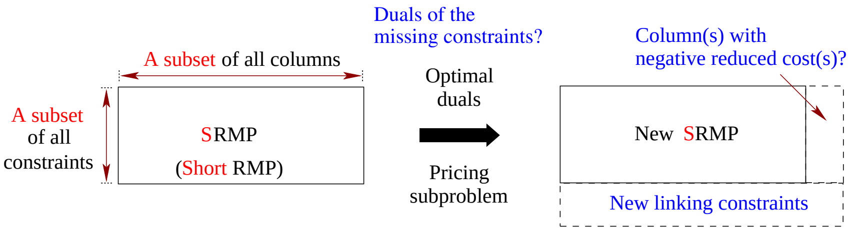

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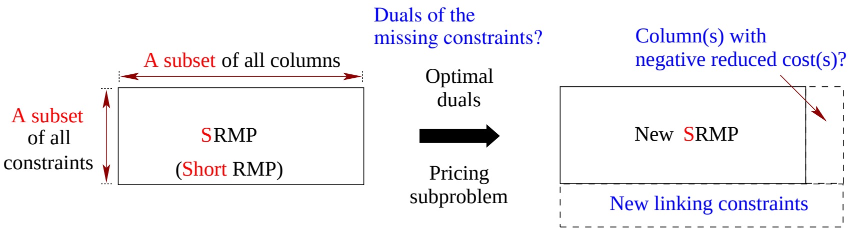

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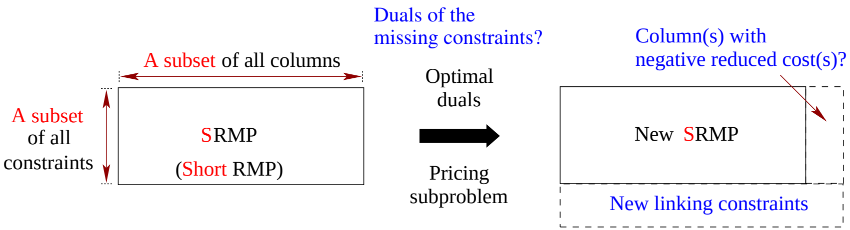

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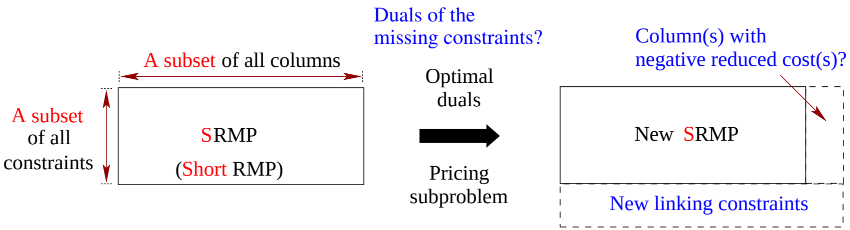

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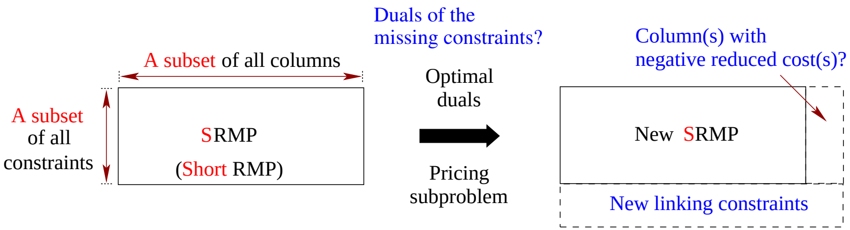

- Primal feasibility ✓ ? ✗ ?
- Dual feasibility ✓ ? ✗ ?
- Complementary slackness ✓ ? ✗ ?

Main Motivation: An Example

A set covering formulation, where the binary variable y_k is set to 1, if column k is selected. The parameters f_k and f_l are the individual contributions from columns k and l, whereas the parameter f_{kl} captures the cross-effects of having columns k and l simultaneously in the solution. That is, in the objective function we have

$$\cdots + f_k y_k + f_l y_l + f_{kl} y_k y_l + \cdots$$

A common linearization:

$$\begin{array}{ll} \text{minimize} & \cdots + f_k y_k + f_l y_l + f_{kl} x_{kl} + \cdots \\ \text{subject to} & \cdots \\ & y_k + y_l - x_{kl} \leq 1, \quad y_k - x_{kl} \geq 0, \quad y_l - x_{kl} \geq 0, \\ & 0 \leq y_k, y_l, x_{kl} \leq 1. \\ & \cdots \end{array}$$

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- A generic column-and-row generation algorithm for CDR-problems is presented. The optimality of this algorithm is proved.

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- The problems that we refer to as CDR-problems are formulated.
- A set of assumptions charactarizing CDR-problems are defined.
- A generic column-and-row generation algorithm for CDR-problems is presented. The optimality of this algorithm is proved.
- The proposed approach is applied to the quadratic set covering, the multi-stage cutting stock and the time-constrained routing problems.

CDR Problems: Generic Mathematical Model

$$
\begin{array}{llll}
(\mathbf{MP}) & \text{minimize} & \sum_{k\in K} c_k y_k + \sum_{n\in N} d_n x_n, & \\
& \text{subject to} & \sum_{k\in K} A_{jk} y_k \geq a_j, \quad j \in J, & \text{(MP-y)} \\
& & \sum_{n\in N} B_{mn} x_n \geq b_m, \quad m \in M, & \text{(MP-x)} \\
& & \sum_{k\in K} C_{ik} y_k + \sum_{n\in N} D_{in} x_n \geq r_i, \quad i \in I, & \text{(MP-yx)} \\
& & y_k \geq 0, k \in K, \quad x_n \geq 0, n \in N. &
\end{array}
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Short Restricted Master Problem (SRMP)

$$\text{minimize} \quad \sum_{k\in\bar{K}} c_k y_k + \sum_{n\in\bar{N}} d_n x_n,$$

$$\text{subject to} \quad \sum_{k\in\bar{K}} A_{jk} y_k \geq a_j, \quad j \in J, \qquad \text{(SRMP-y)}$$

$$\sum_{n\in\bar{N}} B_{mn} x_n \geq b_m, \quad m \in M, \qquad \text{(SRMP-x)}$$

$$\sum_{k\in\bar{K}} C_{ik} y_k + \sum_{n\in\bar{N}} D_{in} x_n \geq r_i, \quad i \in I(\bar{K}, \bar{N}), \qquad \text{(SRMP-yx)}$$

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- $I(\bar{K}, \bar{N}) \subset I$: the set of linking constraints formed by $\{y_k | k \in \bar{K}\}$, and $\{x_n | n \in \bar{N}\}$
- The set of rows is dependent on the set of variables.

- $(S_K \subset (K \setminus \bar{K}), S_N \subset (N \setminus \bar{N}))$: the set of y and x variables to be added to the SRMP during the column generation phase.

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- $\Delta(S_K, S_N) = I(\bar{K} \cup S_K, \bar{N} \cup S_N) \setminus I(\bar{K}, \bar{N})$: the new set of linking constraints induced by S_K and S_N

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- The restricted master problem grows both vertically and horizontally during column generation.

Problem Characterization

Assumption 1

The generation of a new set of variables $\{y_k | k \in S_K\}$ prompts the generation of a new set of variables $\{x_n | n \in S_N(S_K)\}$. Furthermore, a variable $x_{n'}, n' \in S_N(S_K)$, does not appear in any linking constraints other than those indexed by $\Delta(S_K)$ and introduced to the SRMP along with $\{y_k | k \in S_K\}$ and $\{x_n | n \in S_N(S_K)\}$.

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$$C_{ik} y_k + \sum_{n \in S_N(S_K)} D_{in} x_n \geq 0,$$

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Some Example Problems

- Multi-Stage Cutting Stock Problem (Zak, 2002)
- P-Median Problem (Avella et al., 2007)
- Time-Constrained Routing Problem (Avella et al., 2006)
- Robust Crew Pairing Problem (Muter et al., 2010)
- Quadratic Set Covering Problem
- Multi-Commodity Capacitated Network Design Problem (Frangioni and Gendron, 2009)
- Integrated Airline Recovery Problem (S. J. Maher, 2 days ago)

MULTI-STAGE CUTTING STOCK (MSCS)

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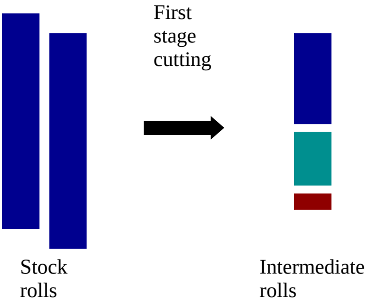

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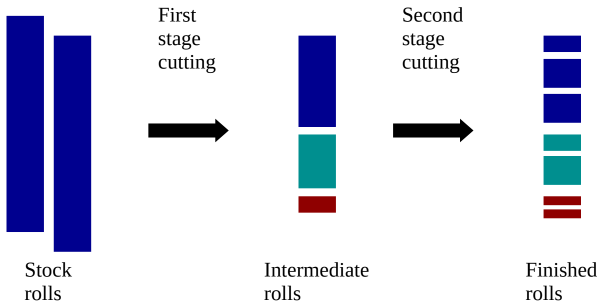

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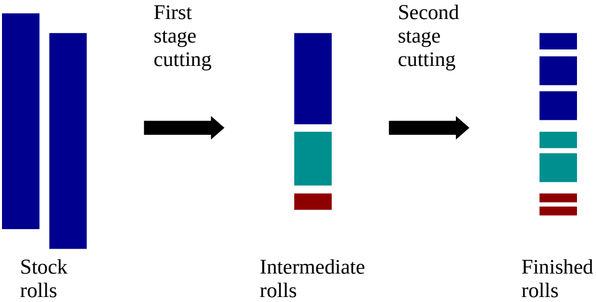

By using the minimum number of stock rolls satisfy the demand for finished rolls after cutting the rolls in two stages. Do not consume more intermediate rolls than produced.

MSCS PROBLEM

$$
\begin{aligned}
\text{minimize} \quad & \sum_{k\in K} y_k, \\
\text{subject to} \quad & \sum_{n\in N} B_{mn} x_n \ge b_m, \quad m \in M, \\
& \sum_{k\in K} C_{ik} y_k + \sum_{n\in N} D_{in} x_n \ge 0, \quad i \in I, \\
& y_k \ge 0, k \in K, \quad x_n \ge 0, n \in N,
\end{aligned}
$$

- I: the set of intermediate rolls
- M: the set of finished rolls
- K: the set of cutting patterns for the first stage
- N: the set of cutting patterns for the second stage
- $D_{in} = -1$ if a cutting pattern n for the second stage is cut from the intermediate roll i

MSCS Problem: SRMP

$$\begin{aligned}
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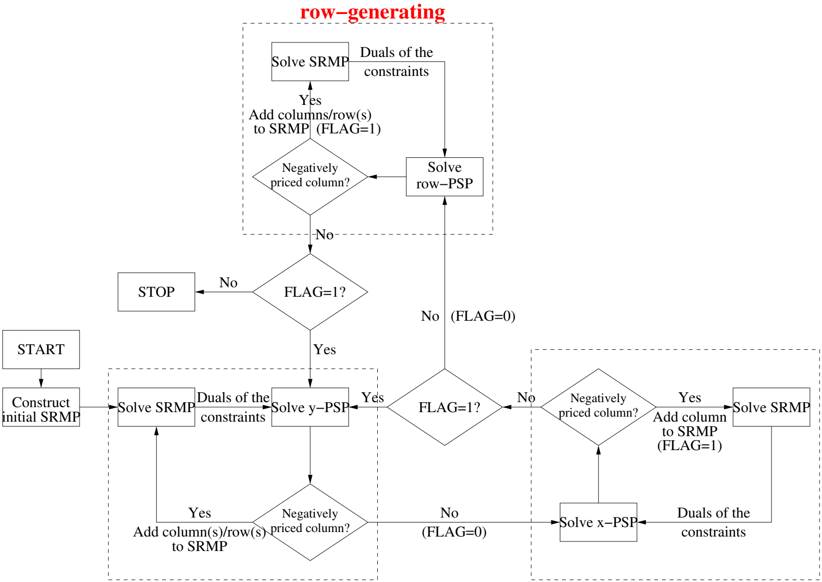
Generic CDR Model: The Dual Problem

$$\begin{array}{llllll}
\text{maximize} & \sum_{j\in J} a_j u_j + & \sum_{m\in M} b_m v_m + & \sum_{i\in I} r_i w_i, & & \\
\text{subject to} & \sum_{j\in J} A_{jk} u_j & + & \sum_{i\in I} C_{ik} w_i & \leq c_k, & k\in K, \\
 & & \sum_{m\in M} B_{mn} v_m + & \sum_{i\in I} D_{in} w_i & \leq d_n, & n\in N, \\
\end{array}$$

$$u_j \geq 0, j\in J, \quad v_m \geq 0, m\in M, \quad w_i \geq 0, i\in I,$$

- y- and x-pricing subproblems
- row-generating pricing subproblem

Flowchart: Column-and-Row Generation



Y-PRICING SUBPROBLEM (Y-PSP)

- The objective is to determine a variable y_k, $k \in (K \setminus \bar{K})$ with a negative reduced cost.

$$\zeta_y = \min_{k \in (K \setminus \bar{K})} \quad \{c_k - \sum_{j \in J} A_{jk} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{ik} w_i\},$$

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- If ζ_y is nonnegative, we move to the next subproblem.

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- The dual variables $\{u_j | j \in J\}$ and $\{w_i | i \in I(\bar{K}, \bar{N})\}$ are obtained from the optimal solution of the current SRMP.
- If ζ_y is nonnegative, we move to the next subproblem.
- Otherwise, there exists y_k with $\bar{c}_k < 0$, and SRMP grows by a single variable by setting $\bar{K} \leftarrow \bar{K} \cup \{k\}$.

Y-PRICING SUBPROBLEM (Y-PSP)

- The objective is to determine a variable y_k, $k \in (K \setminus \bar{K})$ with a negative reduced cost.

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- The dual variables $\{u_j | j \in J\}$ and $\{w_i | i \in I(\bar{K}, \bar{N})\}$ are obtained from the optimal solution of the current SRMP.
- If ζ_y is nonnegative, we move to the next subproblem.
- Otherwise, there exists y_k with $\bar{c}_k < 0$, and SRMP grows by a single variable by setting $\bar{K} \leftarrow \bar{K} \cup \{k\}$.
- MSCS problem: y-PSP generates a cutting pattern for the first stage that uses only the existing intermediate rolls.

X-PRICING SUBPROBLEM (X-PSP)

- $\Delta(\emptyset) = \emptyset$

$$\zeta_x = \min_{n \in N_{\bar{K}}} \quad \{d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in I(\bar{K},\bar{N})} D_{in} w_i\},$$

x-pricing subproblem (x-PSP)

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- The dual variables $\{v_m | m \in M\}$ and $\{w_i | i \in I(\bar{K}, \bar{N})\}$ are retrieved from the optimal solution of the current SRMP.

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- The dual variables $\{v_m | m \in M\}$ and $\{w_i | i \in I(\bar{K}, \bar{N})\}$ are retrieved from the optimal solution of the current SRMP.
- $N_{\bar{K}}$ is the index set of all $x-$variables that may be induced by the set of variables $\{y_k | k \in \bar{K}\}$ in the current SRMP.

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- If $\zeta_x < 0$, $\bar{N} \leftarrow \bar{N} \cup \{n\}$,
- Otherwise, the column-and-row generation algorithm continues with the appropriate subproblem.
- In the MSCS problem, the $x-$PSP identifies cutting patterns for the second stage that only consume intermediate rolls that are produced by the cutting patterns for the first stage in the current SRMP.

Row-Generating PSP

ROW-GENERATING PSP

- Note that before invoking the row-generating PSP, no negatively priced variables exist with respect to the current set of constraints in the SRMP.

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- Note that before invoking the row-generating PSP, no negatively priced variables exist with respect to the current set of constraints in the SRMP.
- The objective of this PSP is to identify new columns that price out favorably only after adding new linking constraints currently absent from the SRMP.
- The primary challenge here is to properly account for the values of the dual variables of the missing constraints, and thus be able to determine which linking constraints should be added to the SRMP together with a set of variables.

Row-Generating Pricing Subproblem (cont'd)

- The optimal solution of the row-generating PSP is a family $\mathcal{F}_k$ of index sets S_K^k.

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$$\mathcal{F}_k = \{\{k,l\},\{k,m\}\}, \quad \Sigma_k = \{k,l,m\}$$

- If the reduced cost of the variables y_k corresponding to the optimal family $\mathcal{F}_k$ is negative, then SRMP grows both horizontally and vertically:

$$\text{SRMP}(\bar{K},\bar{N},I(\bar{K},\bar{N})) \leftarrow \text{SRMP}(\bar{K}\cup\Sigma_k,\bar{N}\cup S_N(\Sigma_k),I(\bar{K},\bar{N})\cup\Delta(\Sigma_k))$$

ROW-GENERATING PRICING SUBPROBLEM (CONT'D)

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Row-Generating Pricing Subproblem (cont'd)

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For any given y_k, an associated $\mathcal{F}_k$, and $S^k_K \in \mathcal{F}_k$, we have

$$\bar{c}_k = c_k - \sum_{j \in J} A_{jk} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{ik} w_i - \sum_{i \in \Delta(\Sigma_k)} C_{ik} w_i,$$

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$$\bar{d}_n = d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in I(\bar{K},\bar{N})} D_{in} w_i - \sum_{i \in \Delta(\Sigma_k)} D_{in} w_i, \; n \in S_N(\Sigma_k)$$

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&= d_n - \sum_{m \in M} B_{mn} v_m - \sum_{i \in \Delta(S_K^k)} D_{in} w_i, \; n \in S_N(S_K^k),
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- $\{w_i | i \in \Delta(\Sigma_k)\}$ are unknown.
- Thinking-ahead approach: The ensuing analysis computes the optimal values of $\{w_i | i \in \Delta(\Sigma_k)\}$ without solving the SRMP explicitly under the presence of the currently missing associated set of linking constraints $\Delta(\Sigma_k)$.

THINKING-AHEAD: GUIDELINE

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- Construct an optimal basis for $\mathrm{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$

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  1. Primal feasibility (by Assumption 2)

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    2. Dual feasibility

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    3. Complementary slackness

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  1. Primal feasibility (by Assumption 2)
  2. Dual feasibility
  3. Complementary slackness
- Calculate the correct reduced cost of y_k under the condition that the reduced costs of the variables in $\mathrm{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ and $\Sigma_k \backslash \{k\}$ do not change.

Basis Augmentation

$$\mathbf{B} = \begin{pmatrix} \mathbf{A_1} & \mathbf{0} & \mathbf{E_1} \\ \mathbf{0} & \mathbf{B_1} & \mathbf{E_2} \\ \mathbf{C_1} & \mathbf{D_1} & \mathbf{E_3} \end{pmatrix}$$

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- New basic x-variables ($\subseteq S_N(\Sigma_k)$)

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- New basic x-variables ($\subseteq S_N(\Sigma_k)$)
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- The dual variables should not change

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- New basic x-variables ($\subseteq S_N(\Sigma_k)$)
- New basic surplus variables ($\subseteq \Delta(\Sigma_k)$)
- The dual variables should not change
- The dual variables may be positive ($\Delta_+(\Sigma_k)$, $C_{ik}y_k + \sum_{n \in S_N(S_K)} D_{in}x_n \geq 0$)

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- The dual variables may be positive ($\Delta_+(\Sigma_k)$, $C_{ik}y_k + \sum_{n \in S_N(S_K)} D_{in}x_n \geq 0$)
- The dual variables should be zero ($\Delta_0(\Sigma_k)$)

Row-Generating PSP (Two-Level Problem)

$$\zeta_{yx} = \min_{k \in (K \setminus \bar{K})} \left\{ c_k - \sum_{j \in J} A_{jk} u_j - \sum_{i \in I(\bar{K}, \bar{N})} C_{ik} w_i - \max_{\mathcal{F}_k \in \mathcal{P}_k} \left(\sum_{S_K^k \in \mathcal{F}_k} \alpha_{S_K^k} \right) \right\}, \text{ where}$$

$$\alpha_{S_K^k} = \text{maximize} \quad \sum_{i \in \Delta(S_K^k)} C_{ik} w_i, \tag{1a}$$

$$\text{subject to} \quad \sum_{i \in \Delta(S_K^k)} D_{in} w_i \leq d_n - \sum_{m \in M} B_{mn} v_m, \quad n \in S_N(S_K^k), \tag{1b}$$

$$w_i = 0, \quad i \in \Delta_0(S_K^k), \tag{1c}$$

$$w_i \geq 0, \quad i \in \Delta_+(S_K^k), \tag{1d}$$

$|\Delta(S_K^k)|$ many linearly independent tight constraints among (1b)-(1d). (1e)

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2. The values assigned to the dual variables $\{w_i | i \in \Delta(\Sigma_k)\}$ in the row-generating PSP are optimal for $\mathrm{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$. For $i \in \Delta_s(\Sigma_k)$, we have $w_i = 0$ at optimality.

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4. The reduced cost $\bar{c}_k$ computed in the row-generating PSP for any $y_k, k \notin \bar{K}$ and $\mathcal{F}_k \in \mathcal{P}_k$ is equal to the reduced cost of y_k with respect to the optimal solution of $\text{SRMP}(\bar{K}, \bar{N} \cup S_N(\Sigma_k), I(\bar{K}, \bar{N}) \cup \Delta(\Sigma_k))$.

Optimality Theorem

Given an optimal basis $\mathbf{B}$ for $\mathrm{SRMP}(\bar{K}, \bar{N}, I(\bar{K}, \bar{N}))$ and a set of associated optimal values for the dual variables $\{u_j | j \in J\}$, $\{v_m | m \in M\}$, and $\{w_i | i \in I(\bar{K}, \bar{N})\}$, the proposed column-and-row generation algorithm terminates with an optimal solution for the master problem (MP) if $\zeta_y \geq 0$, $\zeta_x \geq 0$, and $\zeta_{yx} \geq 0$ in three consecutive calls to the $y-$, $x-$, and the row-generating PSPs, respectively.

Conclusions and Future Research

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- We devised a generic methodology for solving CDR-problems to optimality

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- We plan to develop computationally efficient implementations of our generic methodology:
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  - Embedding the column-and-row generation algorithm into branch-and-price or branch-and-cut-and-price framework to solve large-scale integer programming problems with column-dependent-rows.

Related Research

- Muter, İ., Ş. İ. Birbil, K. Bülbül. Simultaneous Column-and-Row Generation for Large-Scale Linear Programs with Column-Dependent-Rows, *Mathematical Programming*, 2012, To appear, DOI: 10.1007/s10107-012-0561-8.
- Muter, İ., Ş. İ. Birbil, K. Bülbül, G. Şahin. A Note on A LP-based Heuristic for a Time-Constrained Routing Problem, *European Journal of Operational Research*, 2010, 221, 2, 306-307.
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