Stabilized Dynamic Constraint Aggregation (SDCA) for Solving Set Partitioning Problems

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1. Dual Variable Stabilization (DVS)
2. Dynamic Constraint Aggregation (DCA)
3. Stabilized Dynamic Constraint Aggregation (SDCA)

Primal-Dual pair of a linear problem

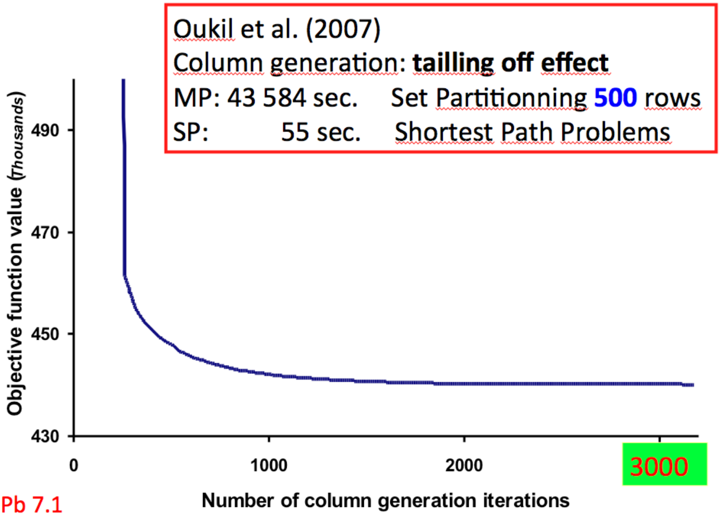
$$
\begin{aligned}
(P) \qquad & \min_{\lambda} && \sum_{x\in\Delta} c_x \lambda_x && && (1)\\
& \text{s.t.} && \sum_{x\in\Delta} \mathbf{a}_x \lambda_x = \mathbf{b} && [\boldsymbol{\pi} \in \mathbb{R}^m] && (2)\\
& && \lambda_x \geq 0, && \forall x \in \Delta && (3)
\end{aligned}
$$

$$
\begin{aligned}
(D) \qquad & \max_{\pi} && \boldsymbol{\pi}^\top \mathbf{b} && && (4)\\
& \text{s.t.} && \boldsymbol{\pi}^\top \mathbf{a}_x \leq c_x, && \forall x \in \Delta && (5)
\end{aligned}
$$

Δ is a finite set of combinatorial objects.

Rows (or *tasks*) in (2) are indexed by $t \in T := \{1, \ldots, m\}$.

Motivation



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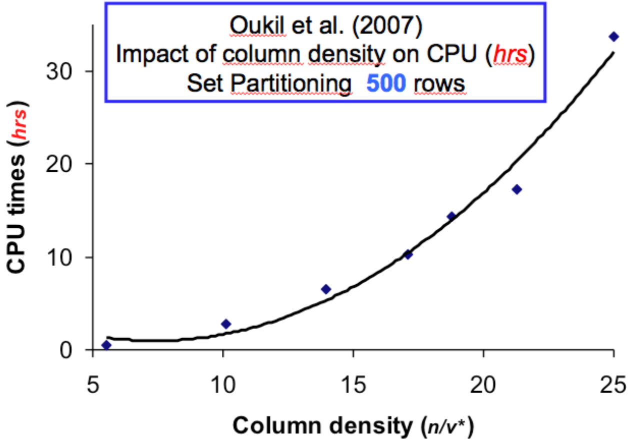
Motivation

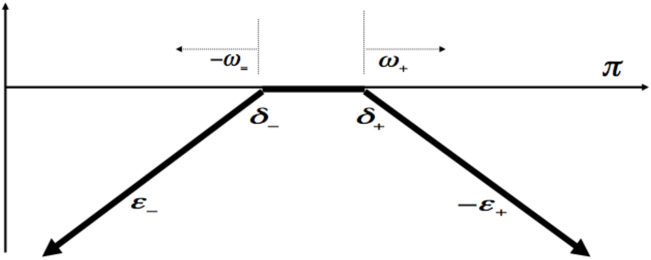

Oukil et al. (2007)
Impact of column density on CPU (hrs)
Set Partitioning 500 rows
CPU times (hrs)
0
10
20
30
5
10
15
20
25
Column density (n/v*)

1.- Dual Variable Stabilization (DVS)

du Merle et al. (1999), Oukil et al. (2007), Ben Amor et al. (2009)

DVS penalizes a large portion of the dual solution space.

A penalty function $f_t(\pi_t)$, $\forall t \in T$:
continuous, concave, symmetric, and piecewise linear with 3 pieces.



$$\delta_- - \omega_- \leq \pi \leq \delta_+ + \omega_+$$
$$\omega_- \geq 0, \omega_+ \geq 0$$

Stabilization in the primal space

$P^{\varepsilon\delta}$: A relaxation of the primal problem P_Δ

$$(P^{\varepsilon\delta}) \quad \min_{\lambda, y} \quad \sum_{x \in \Delta} c_x \lambda_x - \boldsymbol{\delta}_-^\top \mathbf{y}_- + \boldsymbol{\delta}_+^\top \mathbf{y}_+ \tag{6}$$

$$\text{s.t.} \quad \sum_{x \in \Delta} \mathbf{a}_x \lambda_x - \mathbf{y}_- + \mathbf{y}_+ = \mathbf{b} \quad [\boldsymbol{\pi} \in \mathbb{R}^m] \tag{7}$$

$$\mathbf{0} \leq \mathbf{y}_- \leq \boldsymbol{\varepsilon}_- \quad [\boldsymbol{\omega}_- \in \mathbb{R}^m] \tag{8}$$

$$\mathbf{0} \leq \mathbf{y}_+ \leq \boldsymbol{\varepsilon}_+ \quad [\boldsymbol{\omega}_+ \in \mathbb{R}^m] \tag{9}$$

$$\lambda_x \geq 0, \qquad \forall x \in \Delta \tag{10}$$

To solve P, a sequence of linear approximations (relaxations), i.e., stabilized problems P^s $(s \geq 1)$ are solved with ε^s and $\boldsymbol{\delta}^s$ yielding objective values $z^1 < z^2 < \ldots < z^s < \ldots \leq z^*$.

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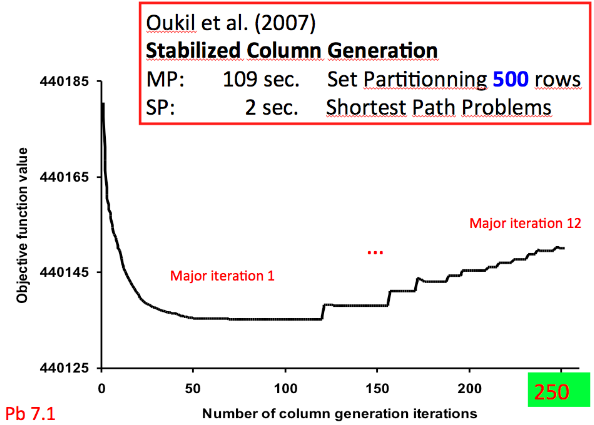

DVS algorithm

1: Set $s := 1$ and choose stabilization parameters ε^s and δ^s.
2: Solve stabilized problem P^s: $(\boldsymbol{\lambda}^s, \mathbf{y}^s)$.
3: **if** $\mathbf{y}^s = \mathbf{0}$ **then**
4: Stop. $\boldsymbol{\lambda}^s$ is optimal to P.
5: **else**
6: $s := s + 1$, update ε^s and δ^s, and return to Step 2.

2.- Dynamic Constraint Aggregation (DCA) for set partitioning problems

Let $Q := \{T_\ell\}_{\ell \in L}$ be **a partition of the rows** of T into clusters.

$$(P_Q) \quad \min_{\lambda} \quad \sum_{x \in \Delta_Q} c_x \lambda_x \tag{11}$$

$$\text{s.t.} \quad \sum_{x \in \Delta_Q} \mathbf{a}_{Qx} \lambda_x = \mathbf{1} \qquad [\boldsymbol{\pi}_Q \in \mathbb{R}^{|L|}] \tag{12}$$

$$\lambda_x \geq 0, \qquad \forall x \in \Delta_Q \tag{13}$$

column $\mathbf{a}_{Qx} \in \{0,1\}^{|L|}$ is a clustered version of $\mathbf{a}_x \in \{0,1\}^m$:

component $a_{Qx\ell} = 1$ if object x selects cluster $\ell \in L$, 0 otherwise.

Each subset of identical rows defines a cluster in Q

From an integer solution

1	1	1		
1			=	1
1			=	1
1			=	1
1			=	1
	1		=	1
	1		=	1
	1		=	1
		1	=	1
		1	=	1
		1	=	1

1	1	1		
1			=	1
	1		=	1
		1	=	1

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	1		=	1
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	1		=	1
		1	=	1
		1	=	1
		1	=	1

1	1	1		
1			=	1
	1		=	1
		1	=	1

From a fractional solution

0.50	0.50	0.25	0.25	0.75		
1	1				=	1
1	1				=	1
1		1	1		=	1
1		1	1		=	1
			1	1	=	1
			1	1	=	1
			1	1	=	1

0.50	0.50	0.25	0.25	0.75		
1	1				=	1
1		1	1		=	1
			1	1	=	1

To solve P, a *sequence* of row-aggregated problems P_{Q_h} $(h \geq 1)$ are solved with various row-partitions Q_h, yielding objective values $z_1 \geq z_2 \geq ... \geq z_h \geq ... \geq z^*$.

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A column generation process

P is solved using various clusters until an equivalence with the original problem.

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A column generation process

P is solved using various clusters until an equivalence with the original problem.

λ_{Q_h} is optimal to P if

there is no negative reduced cost variables with respect to **some disaggregated** dual vector $\pi \in \mathbb{R}^m$.

DCA algorithm for set partitioning problems

1: Select a row-partition Q.
2: Solve row-aggregated problem P_Q: $(\boldsymbol{\lambda}_Q; \boldsymbol{\pi}_Q)$.
3: Disaggregate the dual vector $\boldsymbol{\pi}_Q$ to compute $\boldsymbol{\pi} \in \mathbb{R}^m$.
4: Check if there exist negative reduced cost variables.
5: **if** no such variables exist **then**
6: Stop. $\boldsymbol{\lambda}_Q$ is optimal for P.
7: **else**
8: Update Q using a few negative reduced cost variables, and return to Step 2.
(*Each subset of identical rows defines a cluster.*)

Dual variable disaggregation for set partitioning problems

A sufficient condition: $$\sum_{t \in T_\ell} \pi_t = \pi_{Q\ell}, \qquad \forall \ell \in L. \qquad (14)$$

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$$\sum_{t \in T_\ell} \pi_t = \pi_{Q\ell}, \qquad \forall \ell \in L. \tag{14}$$

Incompatible variables in $\Delta_{\bar{Q}_1}$ and $\Delta_{\bar{Q}_2}$ (Elhallaoui et al. 2005)

Since the sum of the dual variables in cluster ℓ is known by (14), the non-negativity constraints on the reduced costs of variables λ_x, $x \in \Delta_{\bar{Q}_1} \cup \Delta_{\bar{Q}_2}$, can be transformed into difference inequalities, i.e., the disaggregation is the solution of a shortest path problem.

An incompatible column in $\Delta_{\bar{Q}_1}$ partially covers a single cluster whereas a column in $\Delta_{\bar{Q}_2}$ is incompatible with two.

Results-1: **DVS** dominates standard **CG** and **DCA**

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Tasks	Days	$\frac{\text{cpu DCA}}{\text{cpu DVS}}$	$\frac{\text{cpu CG}}{\text{cpu DVS}}$
500	2	5.53	6.06
1000		5.84	8.26
	Average	**5.68**	**7.16**

Tasks	Days	$\frac{\text{cpu DCA}}{\text{cpu DVS}}$	$\frac{\text{cpu CG}}{\text{cpu DVS}}$
500	5	61.28	48.67
1000		67.45	463.17
	Average	**64.37**	**255.92**

Table: CPU time ratios of standard CG and DCA over DVS

3.- Stabilized Dynamic Constraint Aggregation: integration of DCA within DVS to reduce MP cpu

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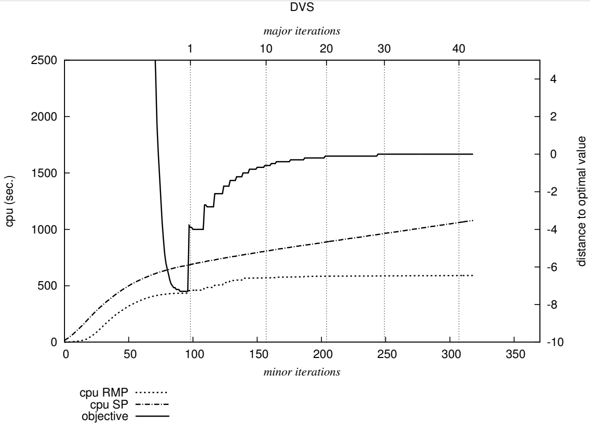

Figure: **DVS** for a 2500-task, 5-day scenario

SDCA: Stabilized & row-aggregated problem

$$P^s \quad \rightarrow \quad P^s_Q$$

$$\varepsilon^s_-,\ \varepsilon^s_+,\ \delta^s_-,\ \delta^s_+,\ y^s_-, y^s_+ \quad \rightarrow \quad \varepsilon^s_{Q-},\ \varepsilon^s_{Q+},\ \delta^s_{Q-},\ \delta^s_{Q+},\ y^s_{Q-}, y^s_{Q+}$$

$$P^s_Q \quad \rightarrow \quad P^s$$

$$(\boldsymbol{\lambda}^s_Q, \mathbf{y}^s_Q, z^s) \quad \rightarrow \quad (\boldsymbol{\lambda}^s, \mathbf{y}^s, z^s)$$

$$(\boldsymbol{\pi}^s_Q, \boldsymbol{\omega}^s_Q, z^s) \quad \rightarrow \quad (\boldsymbol{\pi}^s, \boldsymbol{\omega}^s, z^s)$$

SDCA: Stabilized & row-aggregated problem

Proposition 2

Row-aggregated parameters: $P^s \quad \rightarrow \quad P^s_Q$

For all $\ell \in L$

$$\varepsilon^s_{Q-,\ell} := \min_{t \in T_\ell}\{\varepsilon^s_{-,t}\}, \qquad \varepsilon^s_{Q+,\ell} := \min_{t \in T_\ell}\{\varepsilon^s_{+,t}\};$$

$$\delta^s_{Q-,\ell} := \sum_{t \in T_\ell} \delta^s_{-,t}, \qquad \delta^s_{Q+,\ell} := \sum_{t \in T_\ell} \delta^s_{+,t}$$

Stabilized primal solution: $P^s_Q \quad \rightarrow \quad P^s$

$$\lambda^s_x := \lambda^s_{Qx}, \forall x \in \Delta_Q \qquad \lambda^s_x := 0, \; \forall x \in \Delta \setminus \Delta_Q$$

$$y^s_{-,t} := y^s_{Q-,\ell} \;\text{ and }\; y^s_{+,t} := y^s_{Q+,\ell}, \quad \forall t \in T_\ell, \ell \in L,$$

SDCA: Dual variable disaggregation

$$\text{Define} \quad \beta^s_{\ell,i} := \sum_{j=1}^{i} \pi^s_{\ell,j}, \qquad \forall \ell \in L, i \in \{1, \ldots, |T_\ell|\}.$$

The system of difference inequalities takes into account, for each constraint $t \in T$, the trust region carried out by the δ-parameters.

$$\begin{aligned}
\beta^s_{\ell,|T_\ell|} &= \pi^s_{Q,\ell}, && \forall \ell \in L \\
\beta^s_{\ell,i} - \beta^s_{\ell,i-1} &\geq \delta^s_{-,t}, && \forall t = (\ell, i) \in T \text{ such that } y^s_{-,t} = 0 \\
\beta^s_{\ell,i} - \beta^s_{\ell,i-1} &= \delta^s_{-,t}, && \forall t = (\ell, i) \in T \text{ such that } 0 < y^s_{-,t} < \varepsilon^s_{-,t} \\
\beta^s_{\ell,i} - \beta^s_{\ell,i-1} &\leq \delta^s_{-,t}, && \forall t = (\ell, i) \in T \text{ such that } y^s_{-,t} = \varepsilon^s_{-,t} \\
\beta^s_{\ell,i} - \beta^s_{\ell,i-1} &\leq \delta^s_{+,t}, && \forall t = (\ell, i) \in T \text{ such that } y^s_{+,t} = 0 \\
\beta^s_{\ell,i} - \beta^s_{\ell,i-1} &= \delta^s_{+,t}, && \forall t = (\ell, i) \in T \text{ such that } 0 < y^s_{+,t} < \varepsilon^s_{+,t} \\
\beta^s_{\ell,i} - \beta^s_{\ell,i-1} &\geq \delta^s_{+,t}, && \forall t = (\ell, i) \in T \text{ such that } y^s_{+,t} = \varepsilon^s_{+,t}.
\end{aligned}$$

Still solvable as a shortest path problem.

Results-3: DVS vs. SDCA

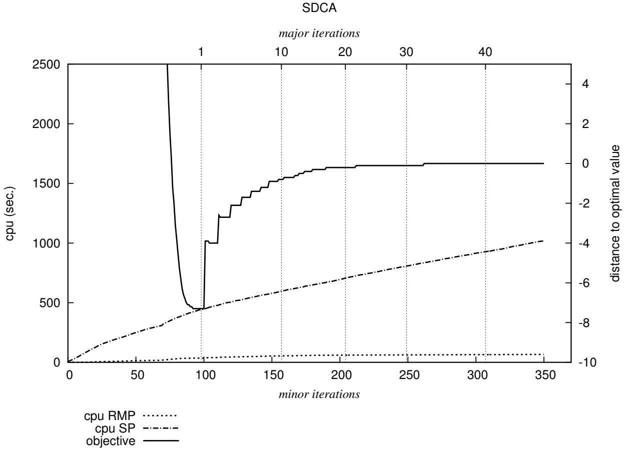

Figure: **SDCA** for a 2500-task, 5-day scenario

Computational Experiments

Stabilized Dynamic Constraint Aggregation (SDCA) within Column Generation

Set Partitioning model for the MDVSP.

Random instances using the procedure of Carpaneto et al. (1989), modified by Oukil et al. (2007) to adjust the level of degeneracy.

3 depots and between 500 and 3000 tasks spread over 2 or 5 days.

5-day instances are highly degenerate.

For each row-size, reported results are average over 6 instances.

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** **Initial primal & dual solutions**: MDVSP relaxed as a SDVSP.

** **Barycenter initial dual estimate** (Rousseau et al. 2007)

Results-3: DVS vs. SDCA (MP CPU)

	2-day scenario			5-day scenario		
Trips	DVS	**SDCA**	*Rows* (%)	DVS	**SDCA**	*Rows* (%)
500	7	4	84.2	7	3	71.9
1000	31	13	85.3	33	10	69.7
1500	109	44	85.6	109	20	68.8
2000	225	70	85.3	278	70	67.4
2500	471	136	85.2	575	89	69.7
3000	838	224	87.1	1160	159	68.9
Total	**1681**	**491**		**2162**	**350**	

Table: Average **Master Problem** CPU time (sec)

Master Problem CPU times reduced by factors **3** and **7** vs. DVS.
Row reduction to around 85% and 70%.
SDCA robust with regard to increasing levels of degeneracy.

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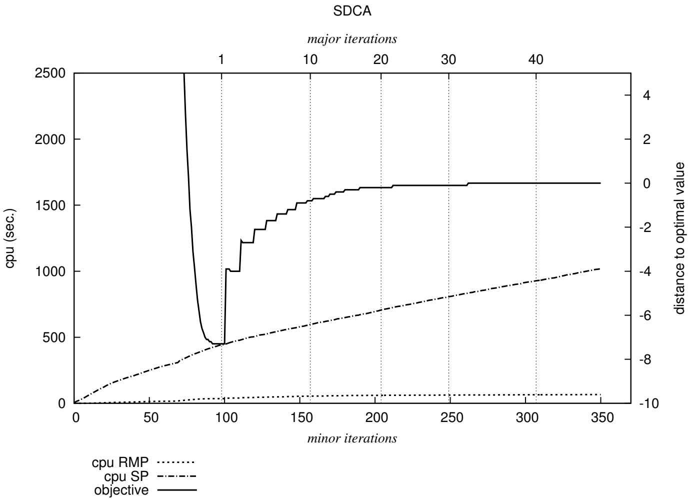

Figure: **SDCA** for a 2500-task, 5-day scenario

Results-3: DVS vs. SDCA (Total CPU)

Tasks	Days	DVS	SDCA
500	2	20	21
1000		126	126
1500		380	342
2000		776	677
2500		1403	1138
3000		2401	1942
Total		**5106**	**4246**

Tasks	Days	DVS	SDCA
500	5	20	18
1000		113	95
1500		376	280
2000		859	653
2500		1683	1138
3000		3035	1914
Total		**6086**	**4098**

Table: Average **Total** CPU time (sec)