Improved column generation for solving set partitioning problems

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Broment 13 June 2012

Linear programming

$$(LP) \qquad z^{\text{LP}} = \min_{x} \quad c^{\top} x \tag{1}$$

$$\text{s.t.} \quad Ax = b \tag{2}$$

$$x \geq 0 \tag{3}$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$

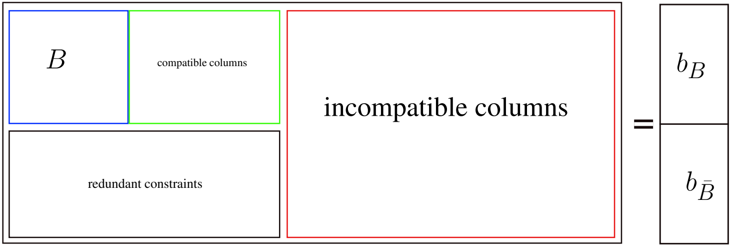

Figure 1: IPS decomposition of a linear program

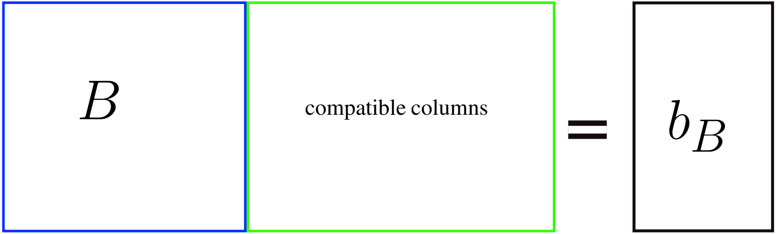

Figure 2: Reduced Problem RP_B

Motivation

We know that a solution x is an optimal solution of LP if and only if there exists a dual solution π to LP such that

$$\bar{c}_j := c_j - \pi^\top A_j = 0, \qquad \forall j \in \{1 \ldots d\} \tag{4}$$

$$\bar{c}_j := c_j - \pi^\top A_j \geq 0, \qquad \forall j \in \{d+1 \ldots n\} \tag{5}$$

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Complementarity problem

$$\max_{s,\ \pi} \quad s \tag{6}$$

$$\text{s.t.} \quad c_j - \pi^\top A_j = 0, \qquad \forall j \in \{1...d\} \tag{7}$$

$$c_j - \pi^\top A_j \geq s, \qquad \forall j \in I = \{d+1...n\} \tag{8}$$

Let $A_B = \begin{pmatrix} A_B^1 \\ A_B^2 \end{pmatrix}$ be a submatrix of A composed of columns indexed in B where A_B^1 is without loss of generality composed of the first $|S|$ linearly independent rows. A_B^2 is of course composed of redundant rows.

Duality of complementarity problem

$$(CP_B) \qquad z_B^{CP} = \min_{\lambda} \sum_{j \in I} \lambda_j \bar{c}_j \tag{9}$$

$$\text{s.t.} \quad MA_{\mathcal{I}}\lambda = 0 \tag{10}$$

$$\sum_{j \in I} \lambda_j = 1 \tag{11}$$

$$\lambda \geq 0. \tag{12}$$

with $M = (A_B^2(A_B^1)^{-1}, -I_{m-p})$ is a compatibility Matrix. $A_{\mathcal{I}}$ is a matrix of incompatible columns.

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A matrix M is said to be a compatibility matrix if and only if $M\ D = 0$ for every compatible column D.

Let x_B^* an optimal solution of RP_B and λ^* an optimal solution of CP_B

Proposition

- *If* $z_B^{CP} \geq 0$, *then* $(x_B^*, 0)$ *is an optimal solution to* LP.
- *If* $z_B^{CP} < 0$, *then* $(x_B^*, 0)$ *is not an optimal solution to* LP.

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- *If* $z_B^{CP} < 0$, *then* $(x_B^*, 0)$ *is not an optimal solution to* LP.
- *Add* $\left\{A_j \;/\; \lambda_j^* > 0\right\}$ *to* $RP_B \Rightarrow$ *the optimal value decreases.*

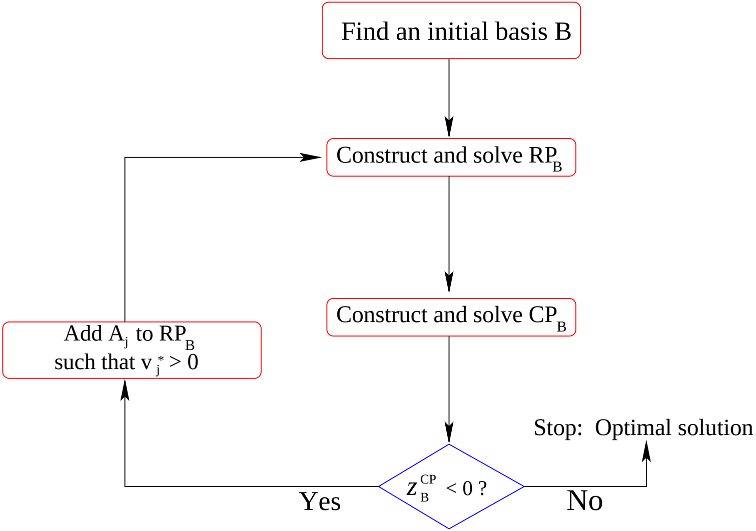

Figure 3: The Non-aggregated improved primal simplex algorithm (IPS-Non-aggregated)

Compatibility Matrix: open questions

There is an infinity of compatibility matrices: if P is an invertible matrix and M a compatibility matrix, then PM is also a compatibility matrix.

Some of the roles of the compatibility matrix is a distinction between the compatible columns and incompatible columns.

Question: How to find the best compatibility matrix?

- Good classification of incompatibility columns
- Good numerical results (sparsity of matrices)

- The same reduced problem RP_B
- The same complementarity problem:

$$(CP_B) \qquad z_B^{CP} = \min_{\lambda} \sum_{i \in I} \lambda_i \bar{c}_i \tag{13}$$

$$\text{s.t.} \quad MA_{\mathcal{I}} \lambda = 0 \tag{14}$$

$$\sum_{j \in I} \lambda_j = 1 \tag{15}$$

$$\lambda \geq 0. \tag{16}$$

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- Add the aggregated column $\omega = \sum_{j \in I} \lambda_j A_j$ in RP_B

Let $I = \{d+1, \ldots, n\}$ the index set of incompatible columns, $\omega = \sum_{i \in I} v_i A_i$ with $\sum_{i \in I} v_i = 1$, $\bar{c}_\omega = \sum_{i \in I} v_i \bar{c}_i$.

$$\Omega = \{\omega \;/\; \omega \text{ is compatible with } B \text{ having a negative reduced cost}\}$$

Let x_B optimal solution of RP_B

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Theorem

$$x_B \text{ is optimal for } LP$$

$$\Updownarrow$$

$$\Omega = \emptyset$$

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Theorem

$$x_B \text{ is optimal for } LP$$

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$$\Omega = \emptyset \iff z_B^{CP} \geq 0.$$

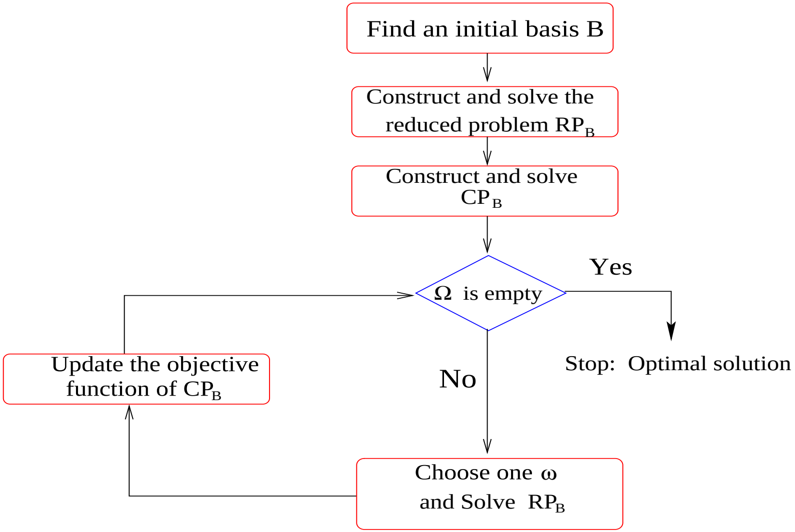

Figure 4: IPS-CG Algorithm

instance	nb const	nb var	deg (%)	instance	nb const	nb var	deg (%)
FA6	5067	17594	68	FA7	5159	20434	59
FA8	5159	21437	65	FA9	5159	23258	66
FA10	5159	24492	66	FA11	5159	24812	66
FA12	5159	24645	66	FA13	5159	25746	65
FA14	5159	22641	71	FA15	5182	23650	63
FA16	5182	23990	64	FA17	5182	24282	65
FA18	5182	24517	65	FA19	5182	24875	65
vcs1	2084	10343	44	vcs2	2084	6341	45
vcs3	2084	6766	45	vcs4	2084	7337	48
vcs5	2084	7837	48	vcs6	2084	8308	48
vcs7	2084	8795	47	vcs8	2084	9241	47
vcs9	2084	10150	50	vcs10	2084	6327	45

Table 1: Instance characteristics

vcs: vehicle and crew scheduling problem
fa: fleet assignment and aircraft routing

inst	CPLEX	IPS-N	IPS-A	IPS-M	$\frac{\text{CPLEX}}{\text{IPS-N}}$	$\frac{\text{CPLEX}}{\text{IPS-A}}$	$\frac{\text{CPLEX}}{\text{IPS-M}}$
vcs1	235.47	63.11	73.22	66.34	3.73	3.22	3.55
vcs2	97.71	36.73	38.83	34.80	2.66	2.52	2.81
vcs3	109.08	36.85	41.99	40.55	2.96	2.60	2.69
vcs4	134.25	40.17	49.29	43.23	3.34	2.72	3.11
vcs5	168.86	44.27	55.79	49.62	3.81	3.03	3.40
vcs6	164.57	45.74	57.44	52.57	3.60	2.87	3.13
vcs7	189.88	49.13	59.07	52.22	3.86	3.21	3.64
vcs8	206.60	49.45	62.17	50.21	4.18	3.32	4.11
vcs9	247.97	60.94	76.16	66.75	4.07	3.26	3.71
vcs10	102.26	31.12	38.64	33.21	3.29	2.65	3.08
Avg					3.55	2.94	3.32

Table 2: Computational results for vcs instances

- IPS-M: Start by IPS-A and finish by IPS-N.
- Cplex: Primal simplex

inst	CPLEX	IPS-N	IPS-A	IPS-M	$\frac{\text{CPLEX}}{\text{IPS-N}}$	$\frac{\text{CPLEX}}{\text{IPS-A}}$	$\frac{\text{CPLEX}}{\text{IPS-M}}$
FA6	273.59	41.75	49.20	21.02	6.55	5.56	13.02
FA7	363.60	53.28	55.64	42.79	6.82	6.53	8.50
FA8	405.50	52.88	45.47	36.08	7.67	8.92	11.24
FA9	517.40	59.94	51	48.09	8.63	10.15	10.76
FA10	697.85	52.98	58.94	42.59	13.17	11.84	16.39
FA11	553.56	54.99	57.43	49.23	10.07	9.64	11.24
FA12	610.05	58.98	60.04	54.19	10.34	10.16	11.26
FA13	694.48	64.70	63.74	54.86	10.73	10.90	12.66
FA14	646.32	77.89	60.53	49.40	8.30	10.71	13.08
FA15	571.60	41.30	44.14	36.10	13.84	12.95	15.83
FA16	927.65	47.03	46	39.64	19.72	20.17	23.40
FA17	602.55	63.11	65.24	63.75	9.55	9.24	9.45
FA18	649.26	59.42	58.33	49.48	10.93	11.13	13.12
FA19	624.51	45.68	45.58	43.85	13.63	13.67	14.24
Avg					10.71	10.83	13.16

Table 3: Computational results for FA instances

When columns are not known a priori

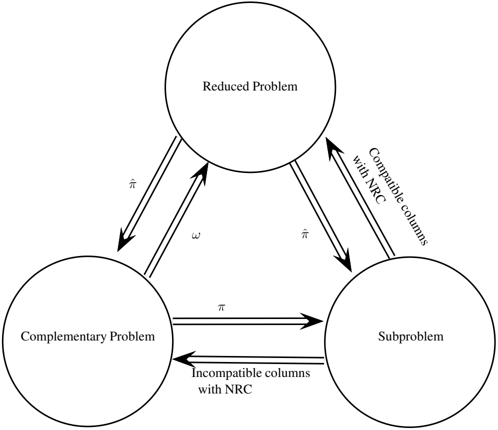

Figure 5: Column generation with three levels

A set partitioning model (SPP) is a special case of (LP) when x is binary, A is $0-1$ matrix and b is $e = (1, 1, ..., 1)^\top$. We can group the tasks into clusters

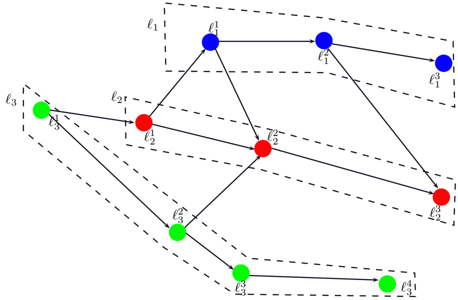

Figure 6: Clusters in (SPP)

In the reduced formulation of (SPP), we have one constraint for each cluster instead of a constraint for each task.

DCA and MPDCA algorithm

DCA Algorithm

- In DCA Algorithm, the dual variables for the discarded rows are computed by a heuristic procedure based on solving shortest path problems.
- They apply a change of variables to transform the sub-problem to a system of difference inequalities. This system corresponds to the dual of a shortest path problem in an oriented network G

MPDCA Algorithm

- In MPDCA Algorithm: only slightly incompatible variables are considered at each iteration (K-phase).
- More and more incompatible variables are priced out for the initial partition.
- This strategy avoids a fast increase of the basis size because only slightly incompatible variables are considered at each iteration.

Some results

- The cost of an arc in G corresponds to the partial reduced cost of its corresponding column in IPS.
- Each negative cycle in G is a convex combination having a negative reduced cost. When added this cycle to a nondegenerate aggregated problem, the objective function decreases.
- The cycle corresponding to $\min_i \frac{\tilde{C}_i}{|F_i|}$ is the cycle that the complementary problem CP_Q outputs.
- When we run a shortest path algorithm on G, we have two cases:
  1. detect a negative cycle (convex combination with a negative reduced cost) that decreases the objective value of a nondegenerate aggregated problem or
  2. obtain a dual solution that is feasible for the incompatible columns present in G.

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This results generalizes and improves MPDCA

Compatibility Matrix in SPP case

The compatibility matrix in SPP case:

$$M = \begin{pmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_{|L|} \end{pmatrix}$$

where M_l is the compatibility matrix of cluster l.

Bus driver scheduling or VCS case

The compatibility matrix in this case:

$$(M_l)_{\substack{1 \le i \le n_l - 1 \\ 1 \le j \le n_l}} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

where n_l is the number of tasks in cluster l. Consecutive tasks on initial path will probably be consecutive in the final solution.

Compatibility Matrix in SPP case

Facility location , MDVSP,...

The compatibility matrix in this case:

$$(M_l)_{\substack{1\leq i\leq n_l-1\\1\leq j\leq n_l}} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

where n_l is the number of tasks in cluster l.

Proposition

A column A_j is k-incompatible if and only if MA_j has k nonzero elements (1 or -1).

References

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Thank you for your attention