POLYTECHNIQUE MONTRÉAL

LE GÉNIE EN PREMIÈRE CLASSE

GERAD Groupe d'études et de recherche en analyse des décisions

# Integral Simplex Using Decomposition


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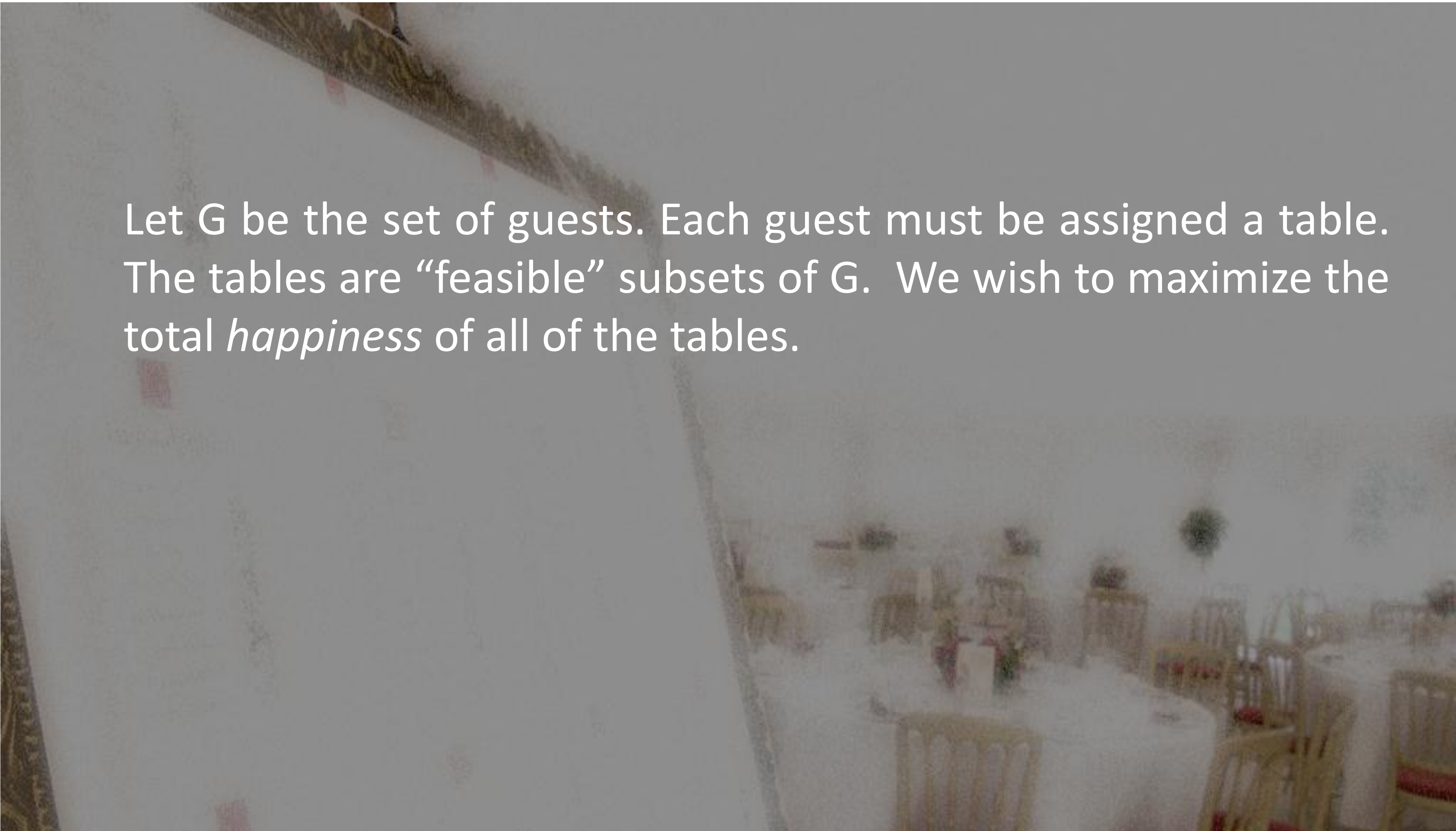
http://www.coin-or.org/PuLP/CaseStudies/a_set_partitioning_problem.html

# The set partitioning problem (SPP)

## Happy example: wedding



Let G be the set of guests. Each guest must be assigned a table. The tables are “feasible” subsets of G.  We wish to maximize the total *happiness* of all of the tables.

# The set partitioning problem (SPP)

## Mathematical formulation

Minimize $c\,x$ ← Unhappiness

Constraint matrix
with binary coefficients →

$A\,x = \mathbb{1}$

$x_j \in \{0, 1\}$

Usually solved by **branch** and **bound** and **cut** and **price** and …

# The set partitioning problem (SPP)

## Applications

- Weddings all over the world
- Transportation

Pairings

Locomotives

Bus driver scheduling

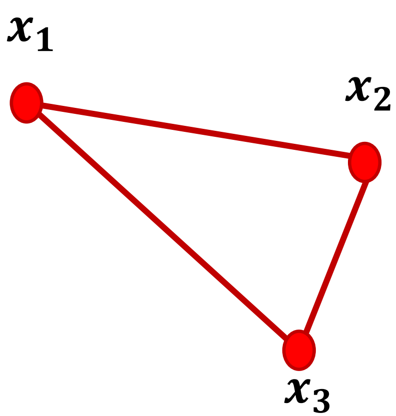
Trucking

- Clustering: data clustering, sensor clustering, …

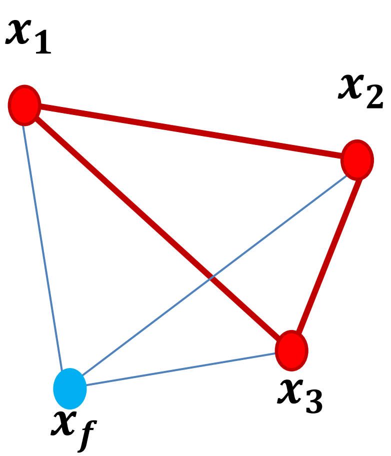
# The story

- The story began in 1969 with Trubin when he observed that the polytope Q of SPP is ***quasi-integral.***

**Meaning that:** every edge of conv(Q') is also an edge of Q where Q' is the set of its integer points.



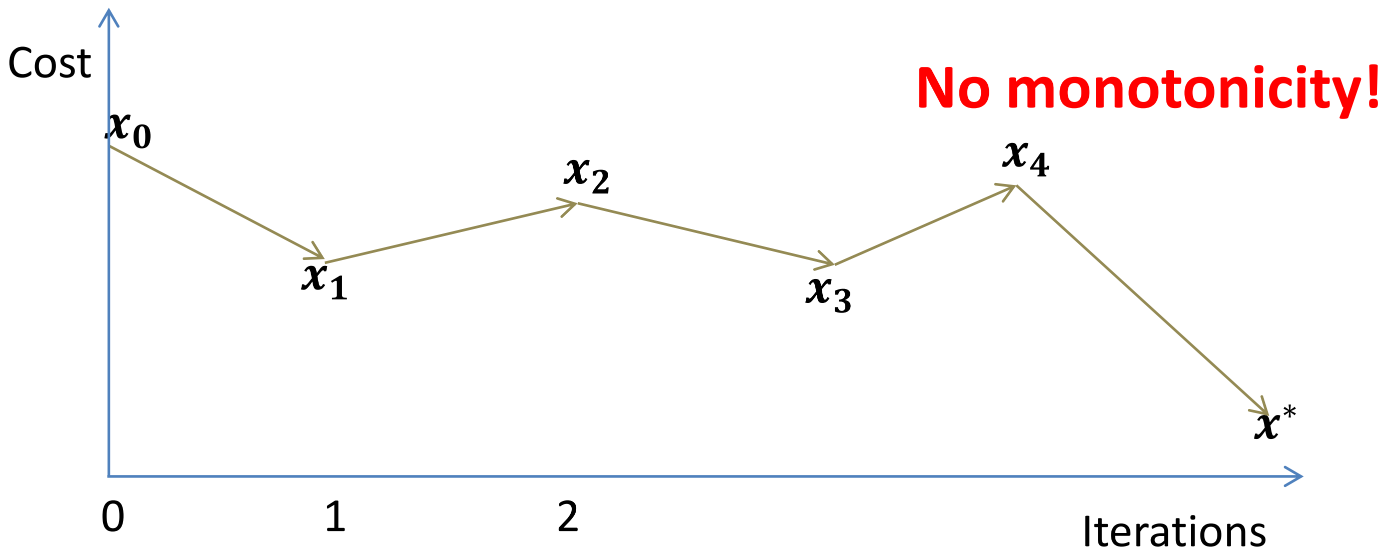

**conv(Q')**




**Q**

# The story

- **Interpretation:** there exists a path from $\boldsymbol{x_0}$ to $\boldsymbol{x}^*$ where vertices $\boldsymbol{x_j}$ are all integer.




- **Trubin** (1969) proposed a variant of the simplex algorithm to solve this kind of problems.

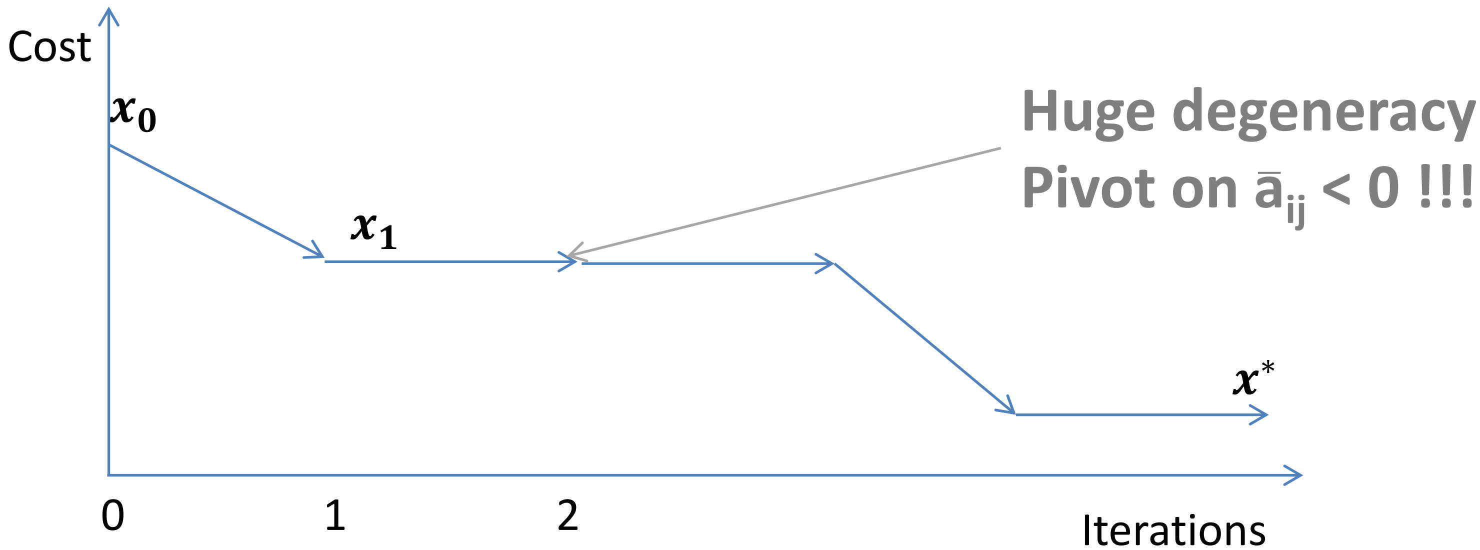
# The story

**Balas and Padberg quoted in 1972 criticizing Trubin:**

*Trubin, using a completely different line of reasoning, shows that all edges of the convex hull of the feasible integer solutions to Q are also edges of the feasible set of Q'. This property (interesting in itself) then implies the existence of a path containing only integer vertices between any two integer vertices of the feasible set. However, neither the upper bound on the number of pivots required to get from one integer vertex to the other nor the existence of a minimum-length path whose associated objective function values form a* ***monotonic*** *sequence follows directly from Trubin's result.*

# The story

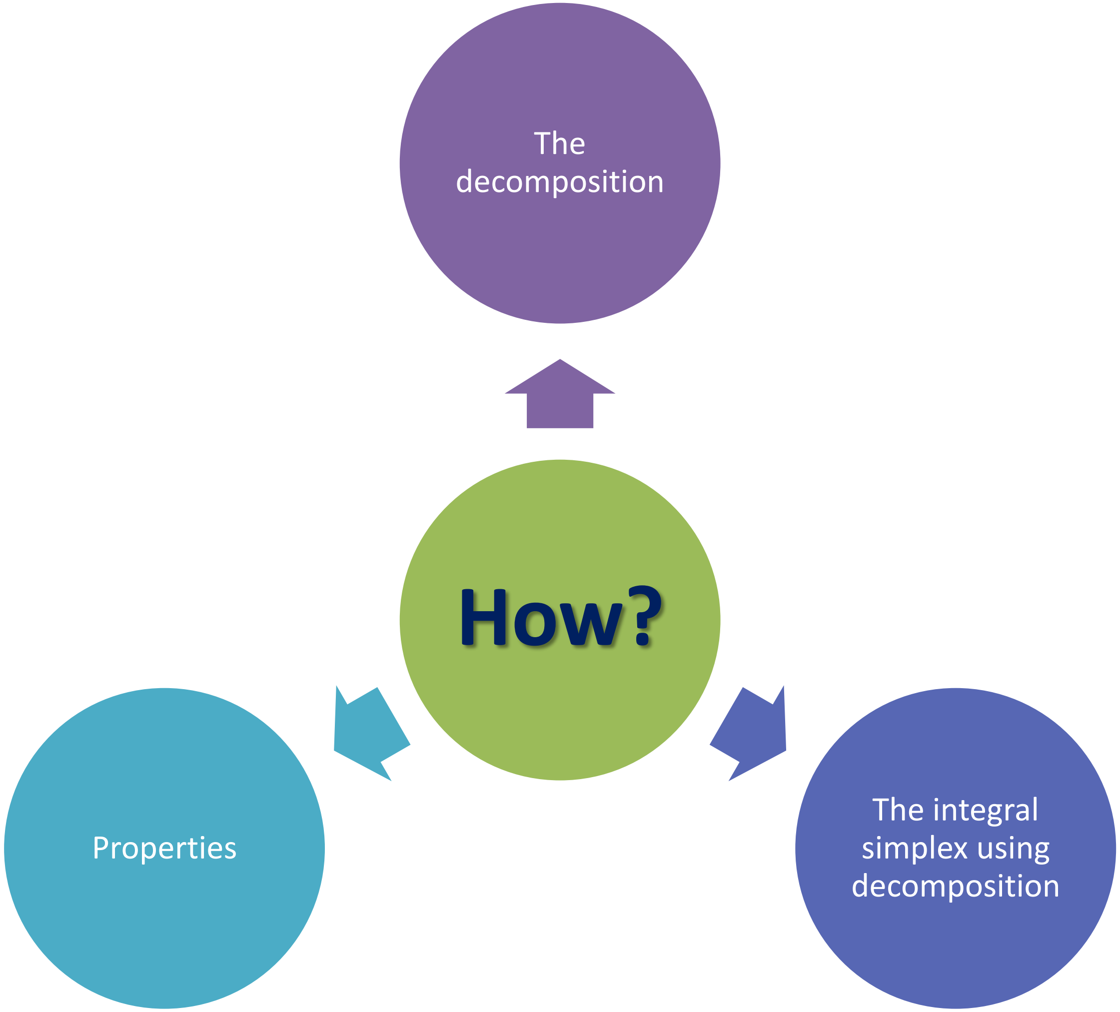
- Balas and Padberg (1972) proved the existence of a decreasing sequence of integer solutions leading to the optimal solution with at most ***m*** pivots.
- **<u>But in practice</u>...**



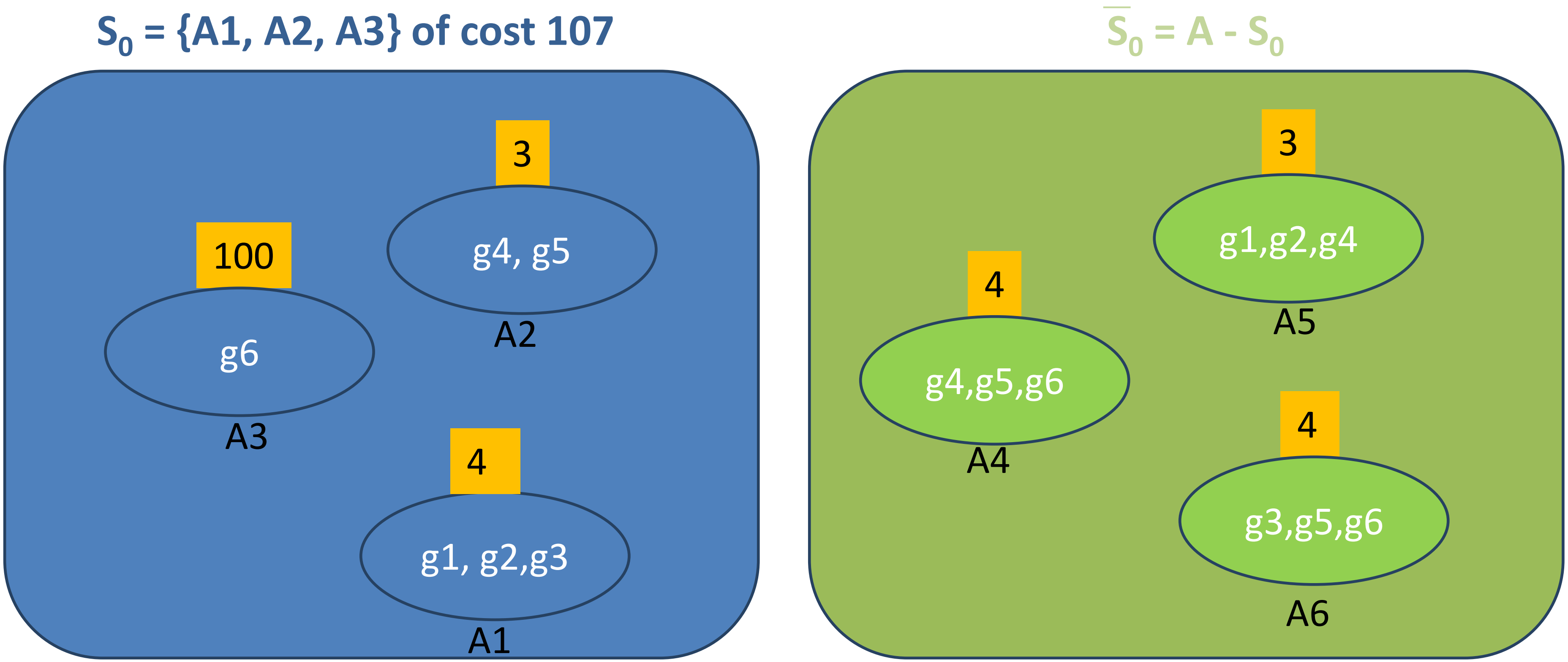

**Pivot on $\bar{a}_{ij} = 1$**

# The story

- Yemelichev, Kovalev and Kravtsov referred in 1984 (translation) to the algorithm as **Integral Simplex** for the first time.
- Thompson used in 2002 the integral simplex to solve instances with up to **163** constraints.
- Rönnberg and Larsson developed in 2009 an extension of the integral simplex method to the column generation context.


The decomposition
How?
Properties
The integral simplex using decomposition

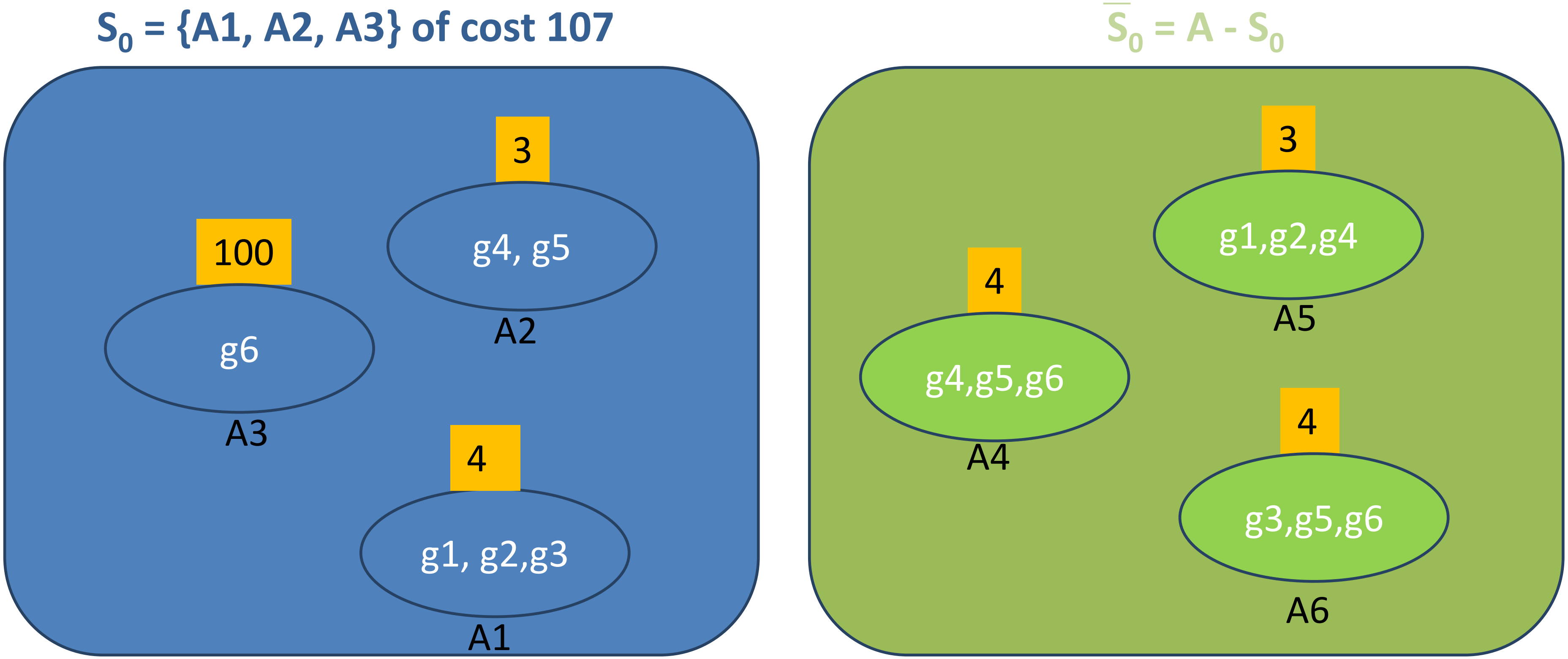
# The decomposition




Example: G = {g1, g2, g3, g4, g5, g6}

- A4 = A2 U A3
- A4 is compatible of lower cost 4 ( < 2 + 100)

# The decomposition



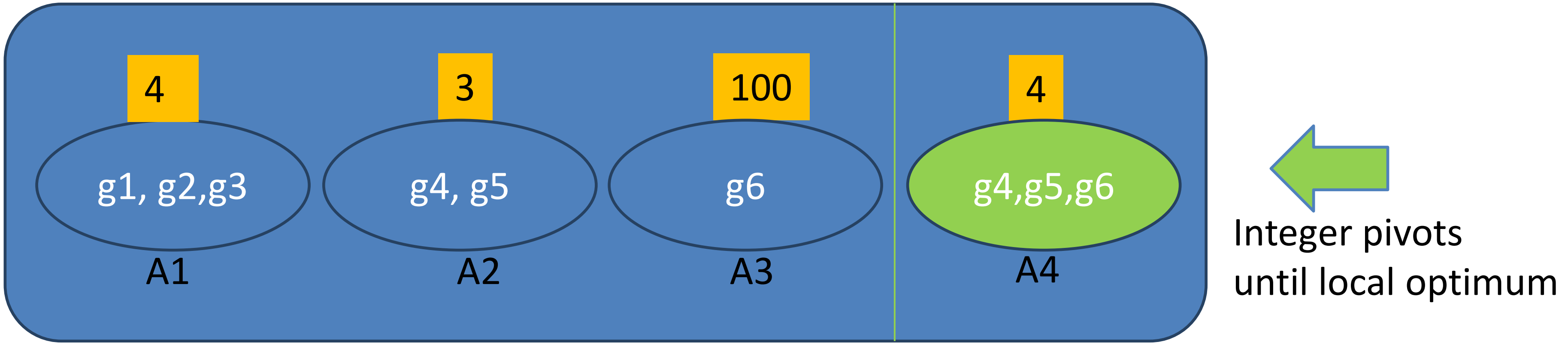

Example: G = {g1, g2, g3, g4, g5, g6}

**Definition:** if Ak is union of some columns of S, Ak is said to be compatible with S.

# The decomposition

## The reduced problem:

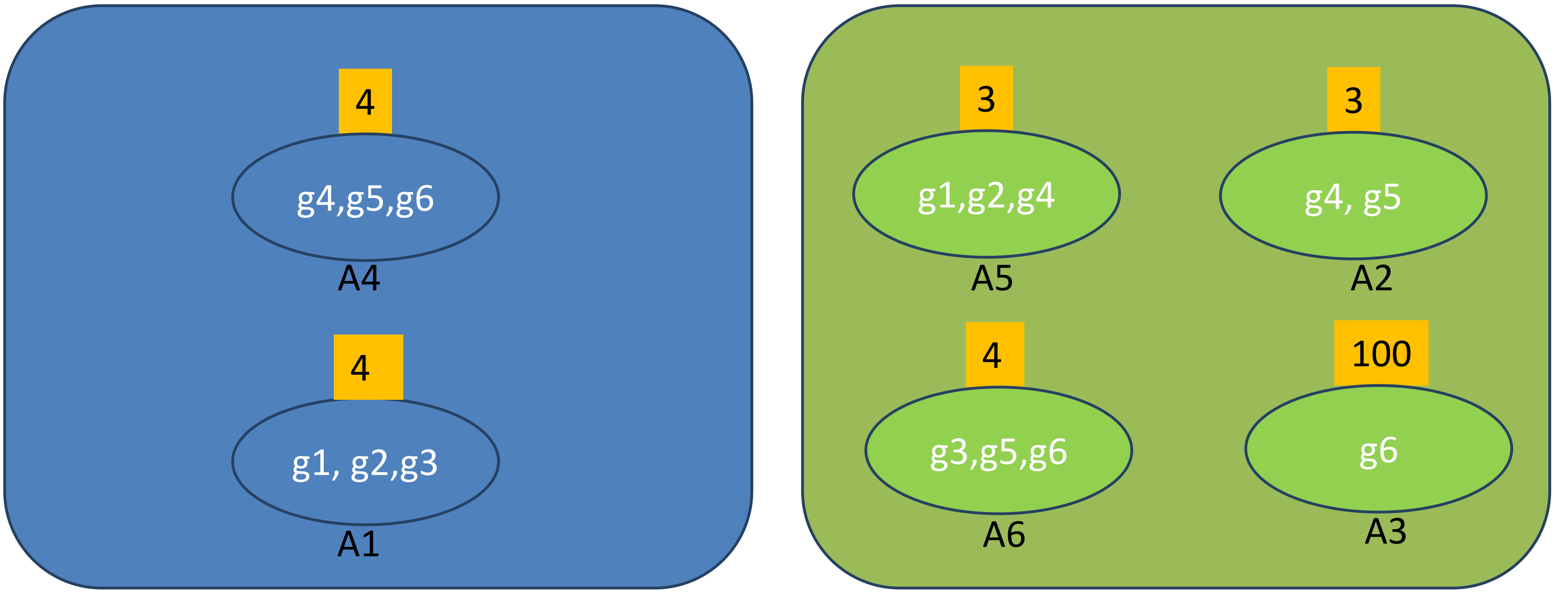
contains columns ***compatible*** with the partition




The reduced problem improves the current integer solution in polynomial time.

# The decomposition

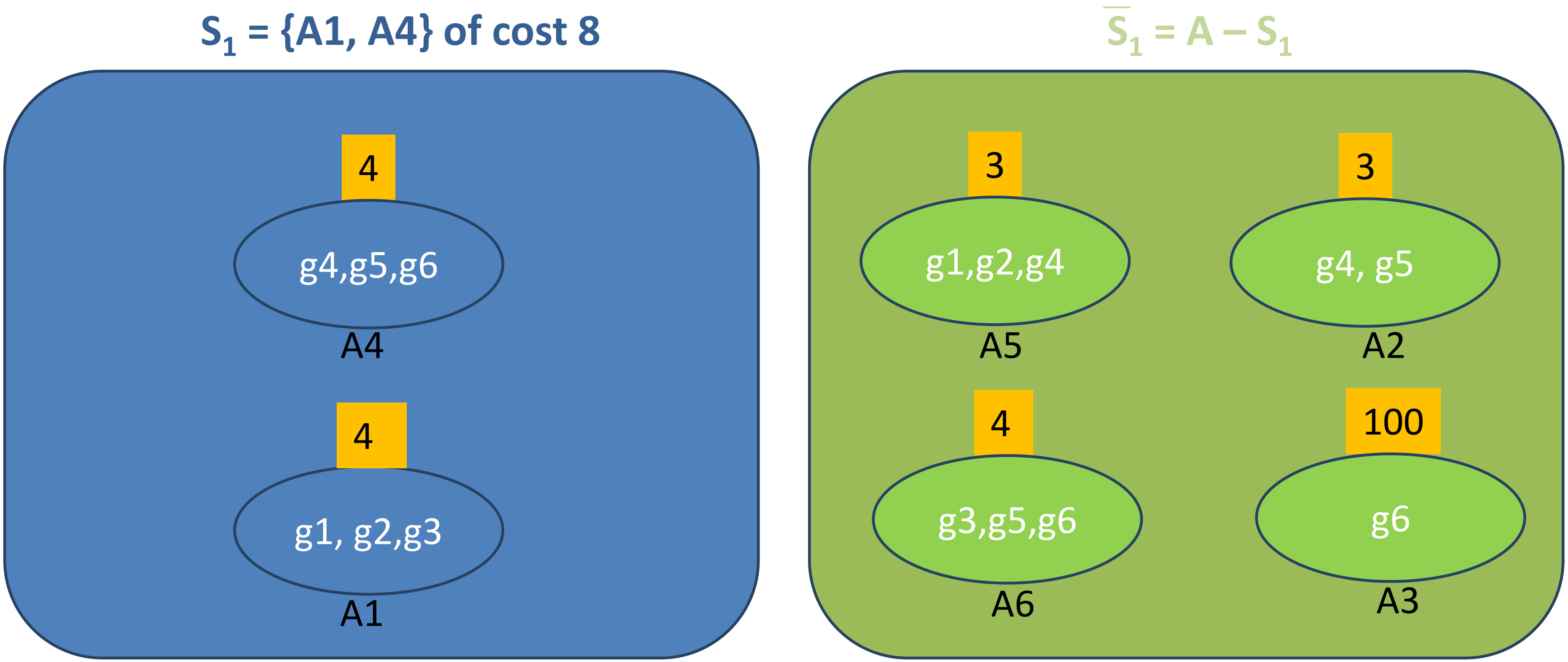
$S_1$ = {A1, A4} of cost 8 $\qquad \overline{S_1} = A - S_1$




**How to escape the local optimum?**

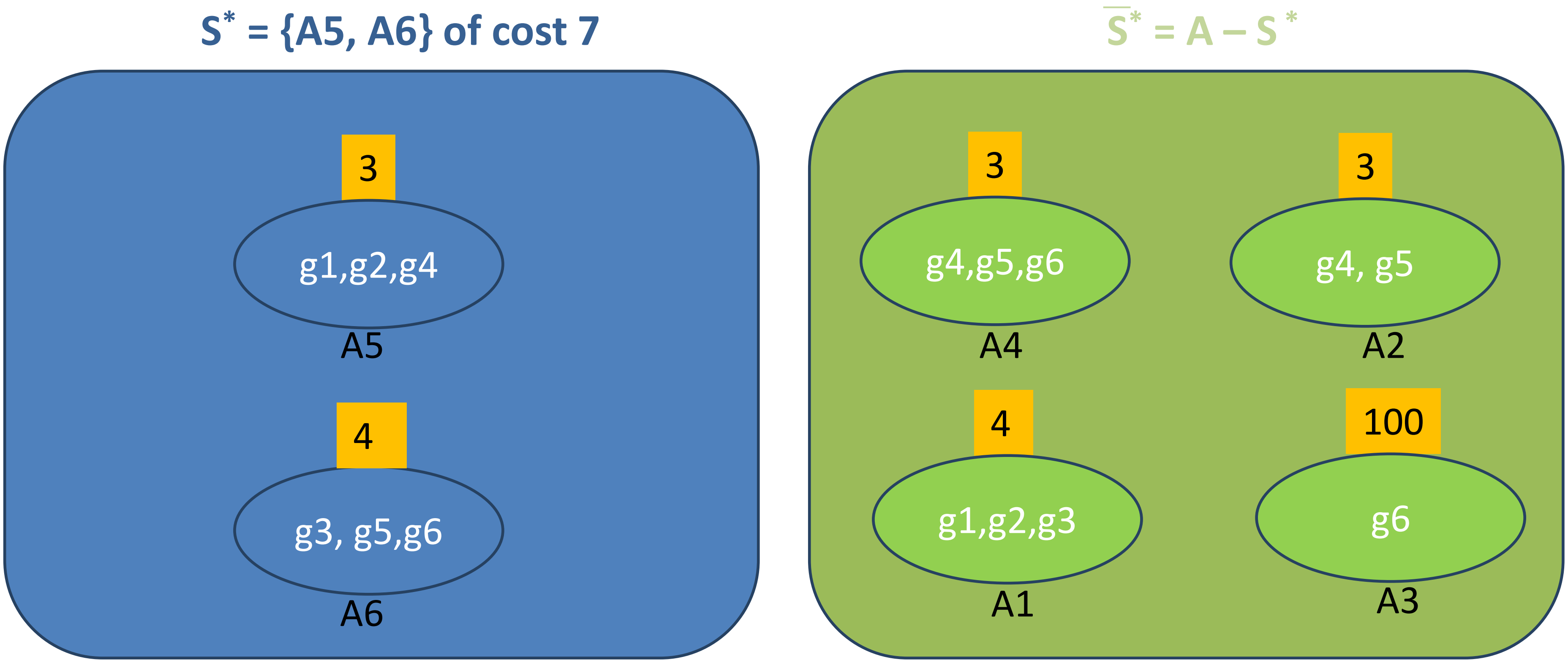
- Balas and Padberg *branch or pivot on negative coefficients* (degenerate variables) !!!

# The decomposition




- A5 U A6 = A1 U A4
- The combination $C_1$ = {A5, A6} is compatible and minimal.

# The decomposition



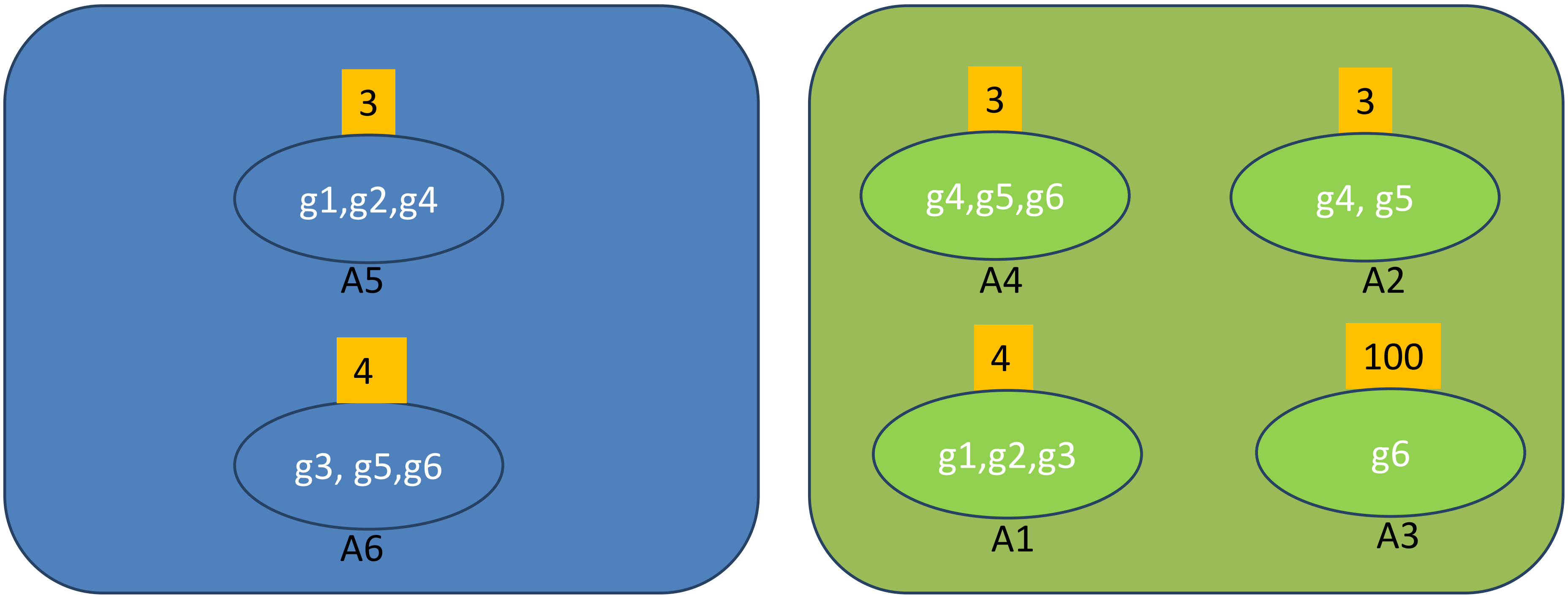

**Definition:**

- A combination $C_k$ is said to be compatible if it is union of some columns of S.
- $C_k$ is minimal if it becomes incompatible by removing any column from it.

# The decomposition

$S^* = \{A5, A6\}$ of cost 7

$\overline{S}^* = A - S^*$




How could we find such combinations ?

# The decomposition

## The complementary problem (CP):

contains columns ***incompatible*** with the partition

Mean reduced cost

$$Minimize \ \bar{c}\, x$$

Compatibility Matrix
$C_k \in$ solution subspace

$$M\,x = 0$$

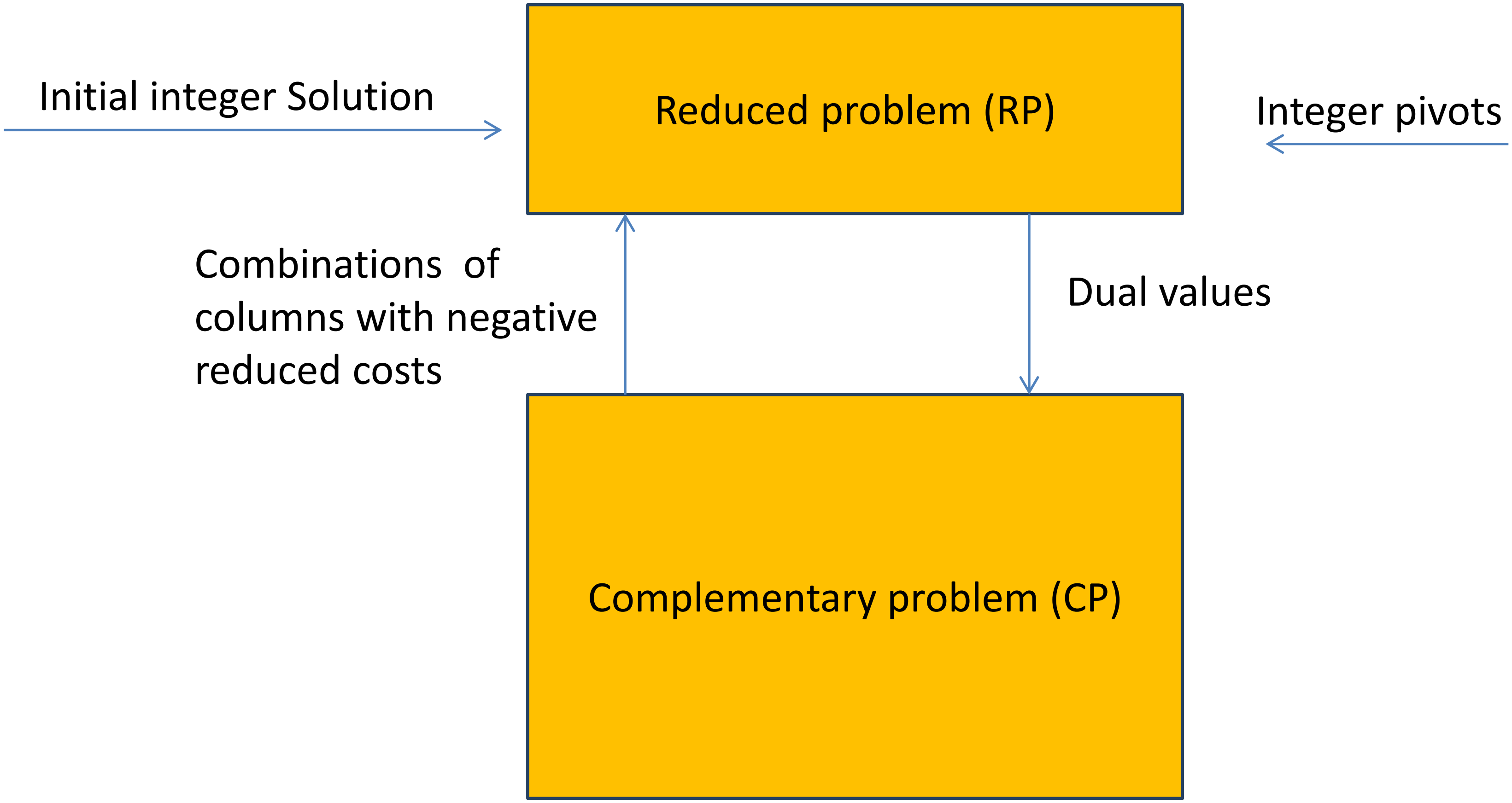
$$\sum_j \boldsymbol{x_j} = 1$$

Normalization constraint
to close the cone

$$\boldsymbol{x_j} \geq 0$$

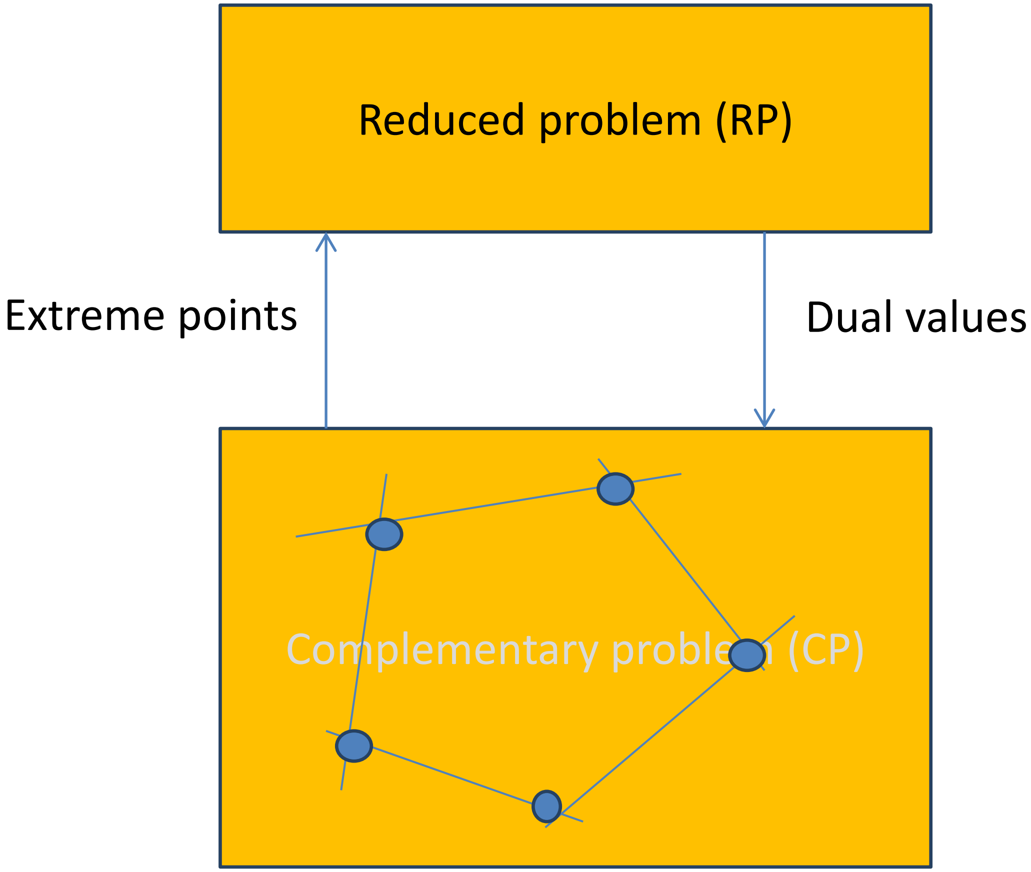
$A_j$ such that $x_j > 0$ are **disjoint**

# Integral simplex using decomposition algorithm (ISUD)



# Properties

## DW decomposition

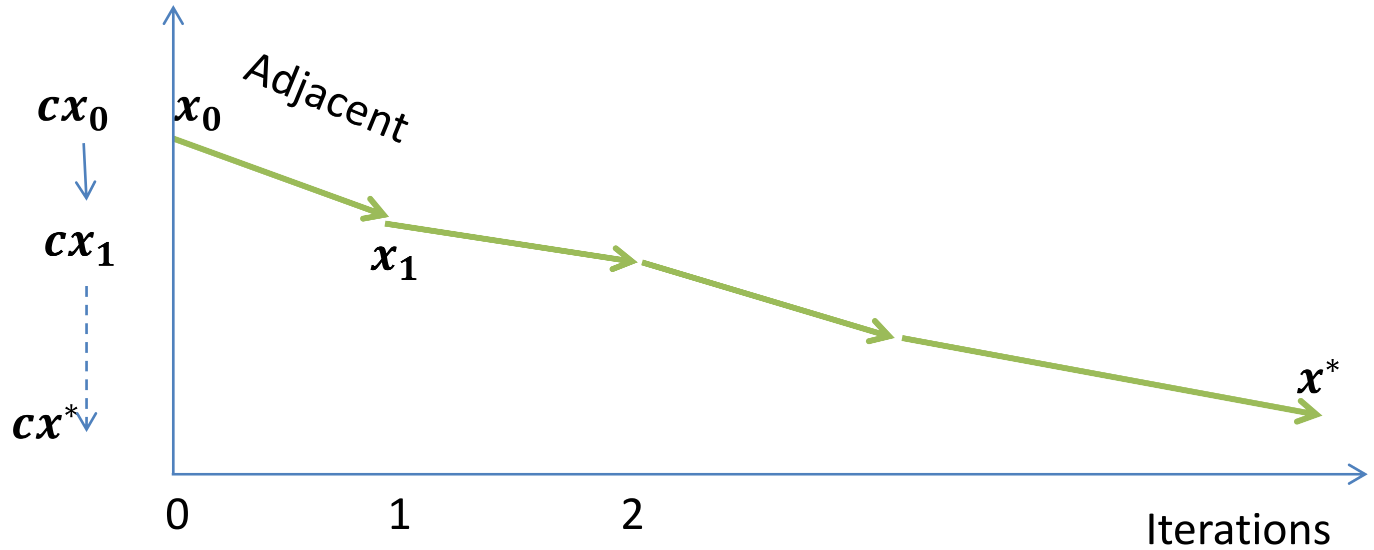



**Proposition:** ISUD is a Dantzig-Wolfe decomposition of the set partitioning problem with $Z^* = Z_{DW}$

# Properties

## Convergence

**Theorem:** ISUD is exact and guarantees a decreasing sequence of integer solutions leading to the optimal solution.

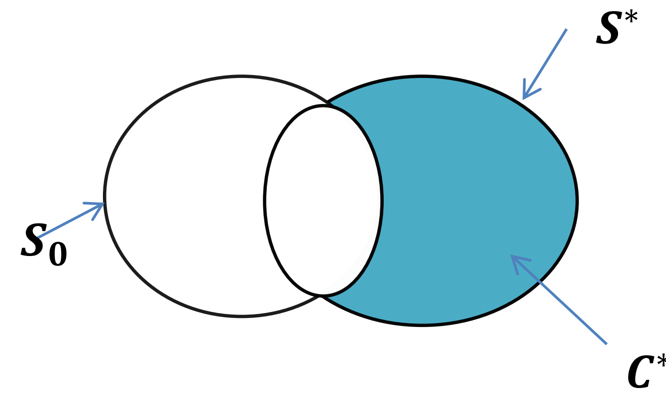



**No degeneracy, no pivoting on negative coefficients**

# Properties

## Minimal combinations

$\boldsymbol{S_0}$ : Set of columns of initial solution
$\boldsymbol{S^*}$ : Set of columns of optimal solution
$\boldsymbol{C^*}$ : Optimal combination ($\boldsymbol{C^*} = \boldsymbol{S^*}$ - $\boldsymbol{S_0}$ )



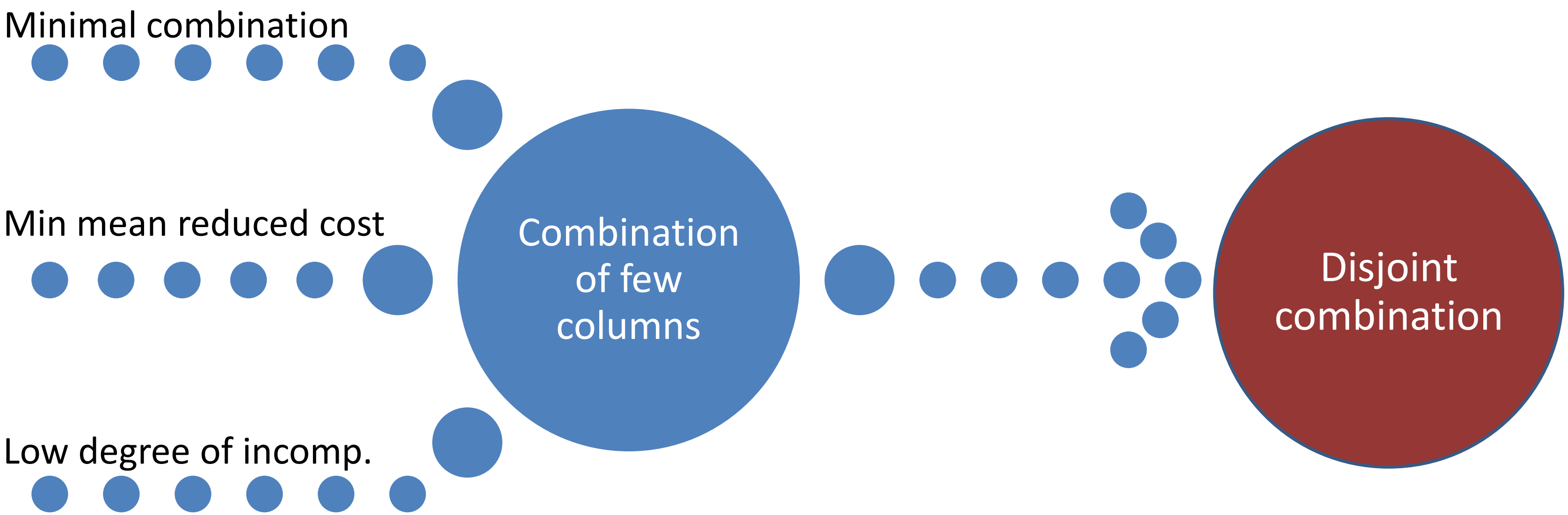

## Proposition

- $C^* = \bigcup_{i=1}^{k} C_i$ is union of minimal combinations
- The complementary problem (CP) finds minimal combinations

**ISUD is intrinsically parallelizable**

# Properties

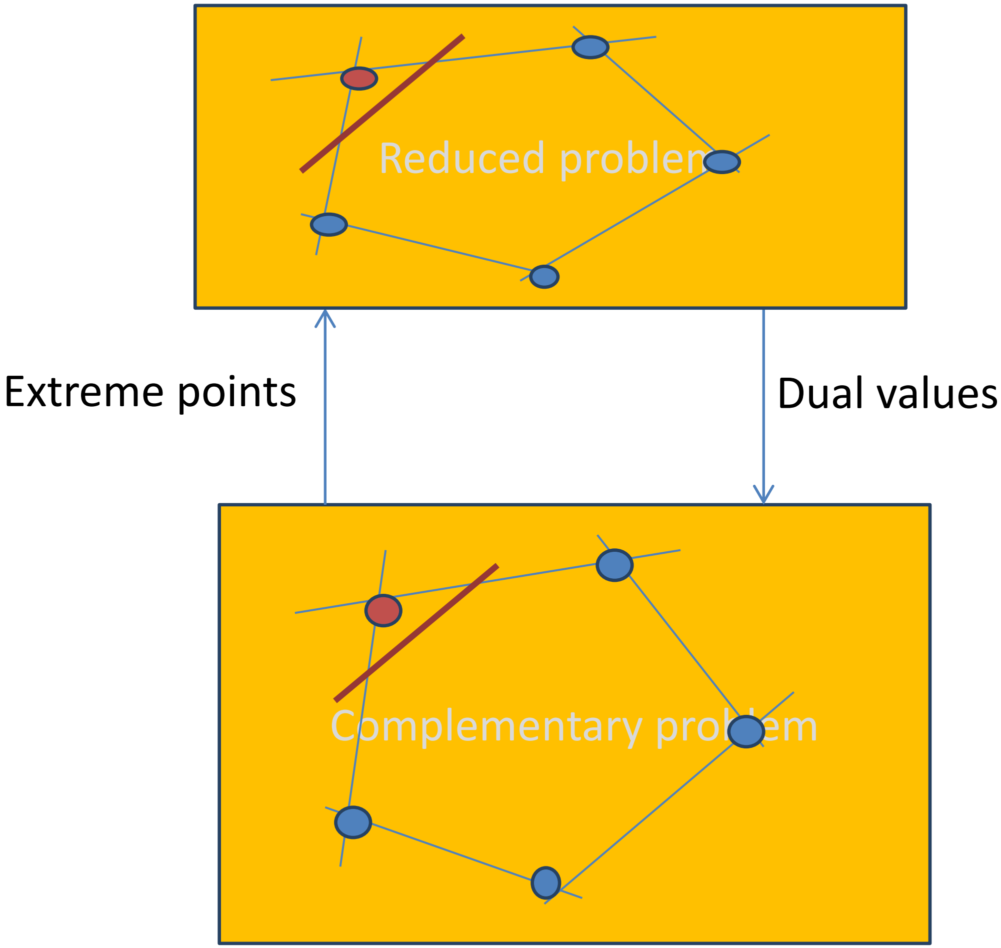
## Integrality of combinations




# ISUD favors integrality

# Properties

## Local cutting




If the current solution changes

↓

The undesired combination (non orthogonal ) becomes incompatible

↓

It cannot be generated by CP

↓

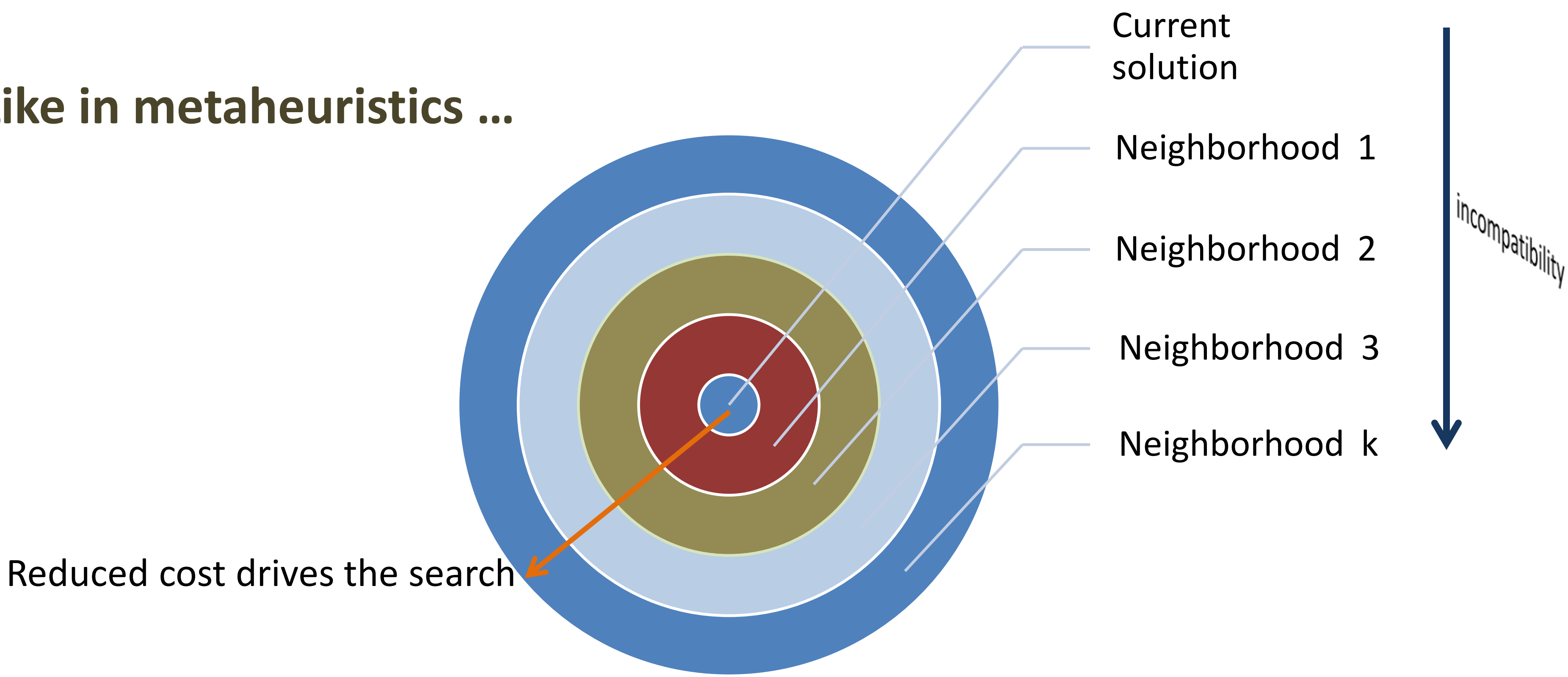
The cut is no longer needed

**Easy handling and less cutting**

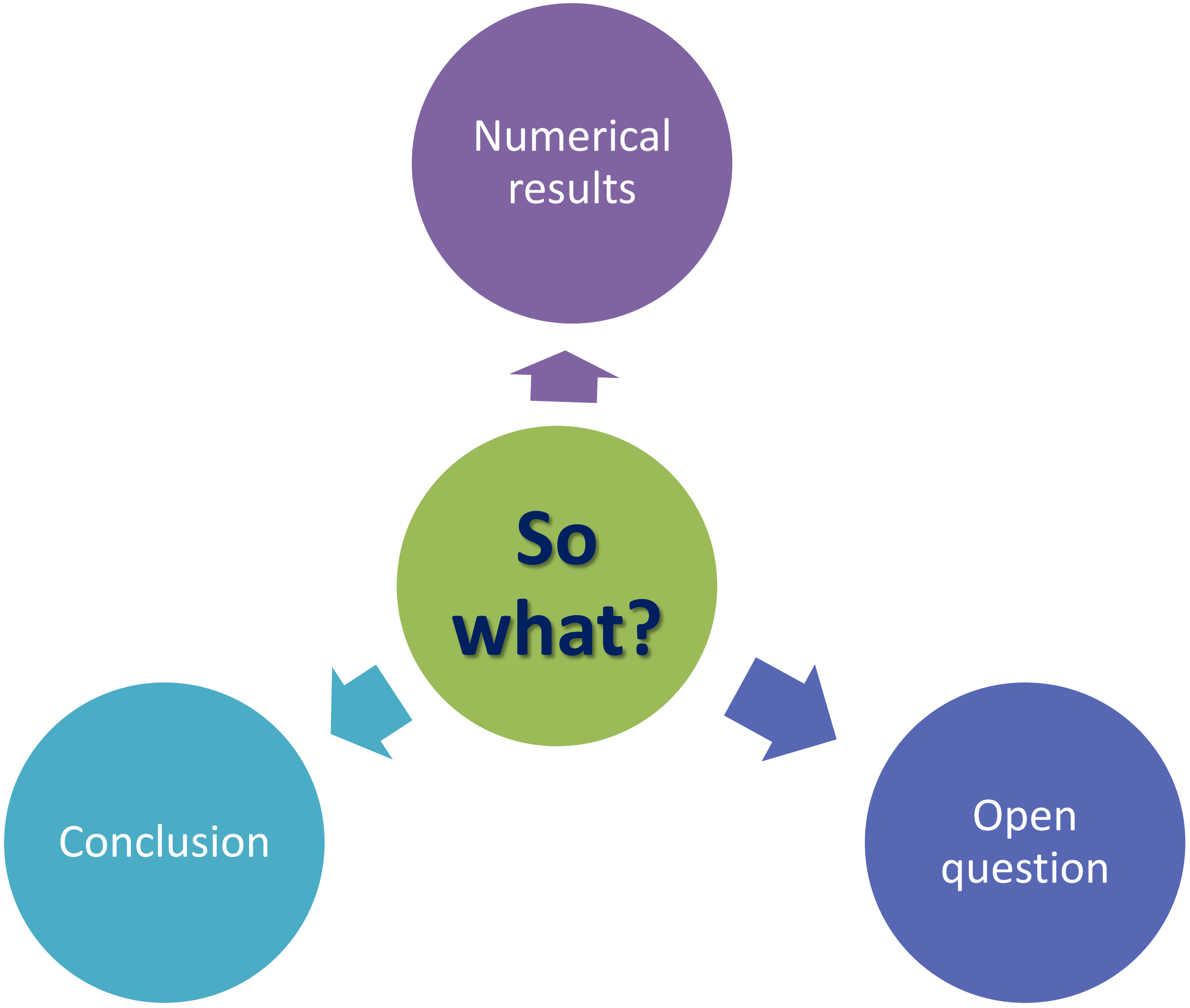
# Properties

## Local improvement

**Like in metaheuristics …**




**The know how developed in metaheuristics could be recycled here**


Numerical results
So what?
Conclusion
Open question

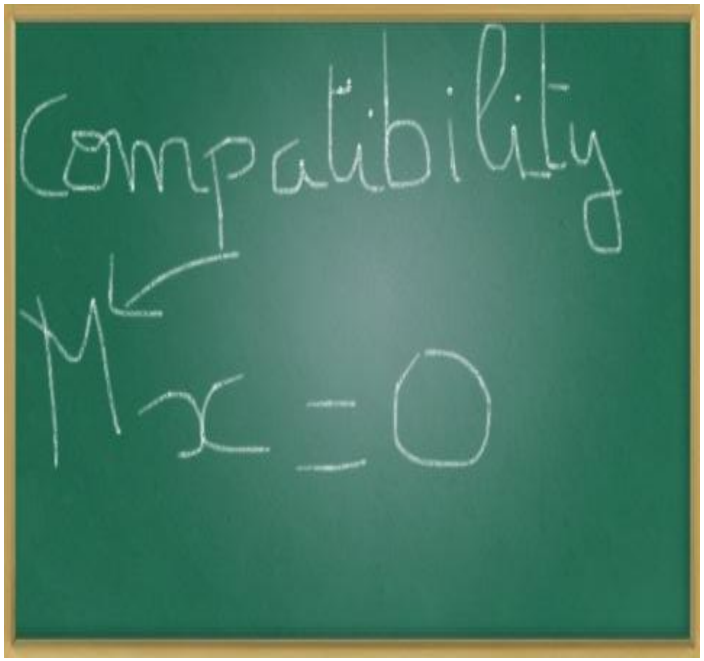
# Numerical results

- Tests on large instances up to **1600** constraints (instead of **163** of Thompson 2002) and **500 000** variables.
- The complementary problem often finds combinations of
  - disjoint columns (**50%-90%**)
  - small size (in average **<= 10** columns/combination)
- ISUD finds optimal solutions in **75%** of cases within **20** minutes.
  - CPLEX takes 10 hours on the easiest large instance (gap 0)
  - CPLEX finds no feasible solutions for the hardest ones

# Open question

**It’s all about…**

# Open question




The degree of incompatibility of a variable depends on M.

**Question:** could we find a compatibility matrix allowing to generate optimal disjoint combinations in polynomial time?

# Conclusion

- Proof of concept showing high potential
- **Ongoing projects:**
  - Extensive experimentation and refinement
  - Local cuts for set partitioning problem
  - Parallel version of ISUD
  - ISUD with cost projection

# Conclusion

**The story continues…**

**Thank you**