## Integral Simplex Using Decomposition

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## Contents

## What/why?

The set partitioning problem

The story of integral simplex

The decomposition

## How?

The integral simplex using decomposition algorithm

Properties

Numerical results

## So what?

Open question and conclusion


## The set partitioning problen (SPP) Happy example: wedding

Let G be the set of guests. Each guest must be assigned a table. The tables are "feasible" subsets of G. We wish to maximize the total happiness of all of the tables.

## The set partitioning problen (SPP) Mathematical formulation

## Minimize $\mathrm{cx} \longleftarrow$ Unhappiness

Constraint matrix
with binary coefficients

$$
\begin{aligned}
& A x=\mathbb{1} \\
& x_{j} \in\{0,1\}
\end{aligned}
$$

Usually solved by branch and bound and cut and price and ...

## The set partitioning problen (SPP) Applications

- Weddings all over the world
- Transportation


Pairings


Bus driver scheduling


Locomotives


Trucking

- Clustering: data clustering, sensor clustering, ...


## The story

- The story began in 1969 with Trubin when he observed that the polytope $Q$ of SPP is quasiintegral.
Meaning that: every edge of $\operatorname{conv}\left(Q^{\prime}\right)$ is also an edge of $Q$ where $Q^{\prime}$ is the set of its integer points.

conv(Q')


Q

## The story

- Interpretation: there exists a path from $\boldsymbol{x}_{\mathbf{0}}$ to $\boldsymbol{x}^{*}$ where vertices $\boldsymbol{x}_{\boldsymbol{j}}$ are all integer.

- Trubin (1969) proposed a variant of the simplex algorithm to solve this kind of problems.


## Balas and Padberg quoted in 1972 criticizing Trubin:

Trubin, using a completely different line of reasoning, shows that all edges of the convex hull of the feasible integer solutions to $Q$ are also edges of the feasible set of $Q$ '. This property (interesting in itself) then implies the existence of a path containing only integer vertices between any two integer vertices of the feasible set. However, neither the upper bound on the number of pivots required to get from one integer vertex to the other nor the existence of a minimum-length path whose associated objective function values form a monotonic sequence follows directly from Trubin's result.

## The story

- Balas and Padberg (1972) proved the existence of a decreasing sequence of integer solutions leading to the optimal solution with at most $\boldsymbol{m}$ pivots.


## But in practice...



Pivot on $\overline{\mathrm{a}}_{\mathrm{ij}}=1$

- Yemelichev, Kovalev and Kravtsov referred in 1984 (translation) to the algorithm as Integral Simplex for the first time.
- Thompson used in 2002 the integral simplex to solve instances with up to $\mathbf{1 6 3}$ constraints.
- Rönnberg and Larsson developed in 2009 an extension of the integral simplex method to the column generation context.



## The decomposition

$\mathrm{S}_{0}=\{\mathrm{A} 1, \mathrm{~A} 2, \mathrm{~A} 3\}$ of cost 107


Example: $\mathrm{G}=\{\mathrm{g} 1, \mathrm{~g} 2, \mathrm{~g} 3, \mathrm{~g} 4, \mathrm{~g} 5, \mathrm{~g} 6\}$

- $\mathrm{A} 4=\mathrm{A} 2 \mathrm{U} \mathrm{A} 3$
- A4 is compatible of lower cost $4(<2+100)$


## The decomposition

## $S_{0}=\{A 1, A 2, A 3\}$ of cost 107

$\mathrm{S}_{0}=\mathrm{A}-\mathrm{S}_{0}$


Example: $\mathrm{G}=\{\mathrm{g} 1, \mathrm{~g} 2, \mathrm{~g} 3, \mathrm{~g} 4, \mathrm{~g} 5, \mathrm{~g} 6\}$
Definition: if $A k$ is union of some columns of $S, A k$ is said to be compatible with S .

## The decomposition

## The reduced problem:

## contains columns compatible with the partition



Integer pivots until local optimum

The reduced problem improves the current integer solution in polynomial time.

## The decomposition

$$
S_{1}=\{A 1, A 4\} \text { of cost } 8
$$



How to escape the local optimum?

- Balas and Padberg branch or pivot on negative coefficients (degenerate variables) !!!


## The decomposition

## $S_{1}=\{A 1, A 4\}$ of cost 8



- A5 U A6 = A1 U A4
- The combination $C_{1}=\{A 5, A 6\}$ is compatible and minimal.


## The decomposition

$$
S^{*}=\{A 5, A 6\} \text { of cost } 7
$$



## Definition:

- A combination $C_{k}$ is said to be compatible if it is union of some columns of $S$.
- $\mathrm{C}_{\mathrm{k}}$ is minimal if it becomes incompatible by removing any column from it.


## The decomposition

$$
S^{*}=\{A 5, A 6\} \text { of cost } 7
$$



HOW COULD WE FIND SUCH COMBINATIONS?

## The decomposition

## The complementary problem (CP):

 contains columns incompatible with the partition

Compatibility Matrix $\longrightarrow M x=0$
$\mathrm{C}_{\mathrm{k}} \in$ solution subspace

$$
\begin{aligned}
\sum_{j} x_{j} & =1 \longleftarrow \\
x_{j} & \geq 0
\end{aligned} \quad \begin{aligned}
& \text { Normalization constraint } \\
& \text { to close the cone }
\end{aligned}
$$

$A_{j}$ such that $x_{j}>0$ are disjoint

# Integral simplex using decomposition algorithm (ISUD) 



## Properties DW decomposition



Proposition: ISUD is a Dantzig-Wolfe decomposition of the set partitioning problem with $\mathrm{Z}^{*}=\mathrm{Z}_{\mathrm{DW}}$

## Properties Convergence

Theorem: ISUD is exact and guarantees a decreasing sequence of integer solutions leading to the optimal solution.


No degeneracy, no pivoting on negative coefficients

## Properties Minimal combinations

$\boldsymbol{S}_{\mathbf{0}}$ : Set of columns of initial solution
$\boldsymbol{S}^{*}$ : Set of columns of optimal solution
$\boldsymbol{C}^{*}$ : Optimal combination ( $\boldsymbol{C}^{*}=\boldsymbol{S}^{*}-\boldsymbol{S}_{\mathbf{0}}$ )


## Proposition

- $C^{*}=\bigcup_{i=1}^{k} C_{i}$ is union of minimal combinations
- The complementary problem (CP) finds minimal combinations


## ISUD is intrinsically parallelizable

## Properties Integrality of combinations



## ISUD favors integrality

## Properties Local cutting



## Properties Local improvement



## The know how developed in metaheuristics could be

 recycled here

## Numerical results

- Tests on large instances up to 1600 constraints (instead of 163 of Thompson 2002) and 500000 variables.
- The complementary problem often finds combinations of
- disjoint columns (50\%-90\%)
- small size (in average <= 10 columns/combination)
- ISUD finds optimal solutions in $75 \%$ of cases within 20 minutes.
- CPLEX takes 10 hours on the easiest large instance (gap 0)
- CPLEX finds no feasible solutions for the hardest ones

Open question
Compatibility $M^{2} x=0$

## Open question



The degree of incompatibility of a variable depends on $M$.
Question: could we find a compatibility matrix allowing to generate optimal disjoint combinations in polynomial time?

## Conclusion

- Proof of concept showing high potential
- Ongoing projects:
- Extensive experimentation and refinement
- Local cuts for set partitioning problem
- Parallel version of ISUD
- ISUD with cost projection


## Conclusion

The story continues...

Thank you

