Row-Reduced Column Generation

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Column Generation

$\boldsymbol{\lambda} \in \mathbb{R}^n_+$: variables

$A \in \mathbb{R}^m \times \mathbb{R}^n$: constraint matrix

$\boldsymbol{\pi} \in \mathbb{R}^m$: vector of dual variables

$\mathbf{c} \in \mathbb{R}^n$: cost vector

$\mathbf{b} \in \mathbb{R}^m$: right-hand side

$\mathbf{x} \in X$: a combinatorial object

Master problem (MP)

$$\begin{array}{rrcll} \min & \mathbf{c}^\top \boldsymbol{\lambda} & & & \\ \text{st:} & A\boldsymbol{\lambda} & = & \mathbf{b} & [\boldsymbol{\pi}] \\ & \boldsymbol{\lambda} & \geq & \mathbf{0}. & \end{array}$$

Subproblem (SP)

$$\bar{c}^\star_{SP} := \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - \boldsymbol{\pi}^\top \mathbf{a}(\mathbf{x}) \right\}$$

Restricted master problem (RMP)

Defined on a small subset of the variables of MP.

Optimality test

If $\bar{c}^{\star}_{\mathrm{SP}} \geq 0$, no negative reduced cost variables exist.
Solution of RMP (embedded into $\mathbb{R}^n_+$) optimally solves MP.

Otherwise, add to RMP several variables derived from the subproblem's solution and start a new iteration.

Improved Column Generation (ICG)

Notation

- A_{B} : sub-matrix of A indexed by columns in $\mathrm{B} \subset \mathrm{N}$,
- A^{P} : sub-matrix of A indexed by rows in $\mathrm{P} \subset \mathrm{M}$,
- $\bar{A}$: linear transformation of A.

Spot intervention

Given is a basis for MP: $A_{\mathrm{B}}, \quad \mathrm{B} \subset \mathrm{N}.$

$$\bar{\mathbf{b}} = A_{\mathrm{B}}^{-1}\mathbf{b}$$

Simplex-Tableau Transformation by Inverse Basis

MP $\bar{A} = A_{\mathrm{B}}^{-1} A, \quad \bar{\mathbf{b}} = A_{\mathrm{B}}^{-1}\mathbf{b}, \quad \bar{\boldsymbol{\pi}}^{\top} = \boldsymbol{\pi}^{\top} A_{\mathrm{B}}$

$$\begin{array}{llcll} \min & \mathbf{c}^{\top}\boldsymbol{\lambda} & & & \\ \text{st:} & \bar{A}\boldsymbol{\lambda} & = & \bar{\mathbf{b}} & [\bar{\boldsymbol{\pi}}] \\ & \boldsymbol{\lambda} & \geq & \mathbf{0}. & \end{array}$$

SP

$$\bar{c}^{\star}_{\mathrm{SP}} := \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - \bar{\boldsymbol{\pi}}^{\top} \bar{\mathbf{a}}(\mathbf{x}) \right\}$$

Row-partition of MP

Row-partition

Let $\mathrm{P} := \{i \in \mathrm{M} \mid \bar{b}_i > 0\}$ and $\mathrm{Z} := \{i \in \mathrm{M} \mid \bar{b}_i = 0\}$.

$\bar{\mathbf{b}}^{\mathrm{P}} > \mathbf{0}$: values of non-degenerate basic variables $\boldsymbol{\lambda}_{\mathrm{P}}$

$\bar{\mathbf{b}}^{\mathrm{Z}} = \mathbf{0}$: values for degenerate basic variables

$$|\mathrm{P}| = p < m$$

Row-partition of MP

$$
\begin{aligned}
\min \quad & \mathbf{c}^\top \boldsymbol{\lambda} \\
\text{st:} \quad & \bar{A}^{\mathsf{P}} \boldsymbol{\lambda} = \bar{\mathbf{b}}^{\mathsf{P}} \quad [\bar{\pi}^{\mathsf{P}}] \\
& \bar{A}^{\mathsf{Z}} \boldsymbol{\lambda} = \bar{\mathbf{b}}^{\mathsf{Z}} \quad [\bar{\pi}^{\mathsf{Z}}] \\
& \boldsymbol{\lambda} \geq \mathbf{0}
\end{aligned}
$$

SP

$$
\bar{c}^{\star}_{\mathsf{SP}} := \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - (\bar{\pi}^{\mathsf{P}})^\top \bar{\mathbf{a}}^{\mathsf{P}}(\mathbf{x}) - (\bar{\pi}^{\mathsf{Z}})^\top \bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.
$$

Row-partition of MP $\bar{\mathbf{b}}^{\mathsf{Z}} = \mathbf{0}$

$$
\begin{array}{llcll}
\min & \mathbf{c}^{\top}\boldsymbol{\lambda} & & & \\
\text{st:} & \bar{A}^{\mathsf{P}}\boldsymbol{\lambda} & = & \bar{\mathbf{b}}^{\mathsf{P}} & [\bar{\boldsymbol{\pi}}^{\mathsf{P}}] \\
 & \bar{A}^{\mathsf{Z}}\boldsymbol{\lambda} & = & \mathbf{0} & [\bar{\boldsymbol{\pi}}^{\mathsf{Z}}] \\
 & \boldsymbol{\lambda} & \geq & \mathbf{0} &
\end{array}
$$

SP

$$
\bar{c}^{\star}_{\mathsf{SP}} := \min_{\mathbf{x}\in X} \left\{ c(\mathbf{x}) - (\bar{\boldsymbol{\pi}}^{\mathsf{P}})^{\top}\bar{\mathbf{a}}^{\mathsf{P}}(\mathbf{x}) - (\bar{\boldsymbol{\pi}}^{\mathsf{Z}})^{\top}\bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.
$$

Row-partition of MP — Keep original data in row-set P

$$
\begin{array}{llcll}
\min & \mathbf{c}^\top \boldsymbol{\lambda} & & & \\
\text{st:} & A^{\mathsf{P}} \boldsymbol{\lambda} & = & \mathbf{b}^{\mathsf{P}} & [\boldsymbol{\pi}^{\mathsf{P}}] \\
 & \bar{A}^{\mathsf{Z}} \boldsymbol{\lambda} & = & \mathbf{0} & [\bar{\boldsymbol{\pi}}^{\mathsf{Z}}] \\
 & \boldsymbol{\lambda} & \geq & \mathbf{0} &
\end{array}
$$

SP

$$
\bar{c}^{\star}_{\mathsf{SP}} := \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathsf{P}})^\top \mathbf{a}^{\mathsf{P}}(\mathbf{x}) - (\bar{\boldsymbol{\pi}}^{\mathsf{Z}})^\top \bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.
$$

Row-partition of MP

$$\begin{aligned} \min \quad & \mathbf{c}^\top \boldsymbol{\lambda} & & \\ \text{st:} \quad & A^{\mathsf{P}} \boldsymbol{\lambda} & = \; & \mathbf{b}^{\mathsf{P}} \quad [\boldsymbol{\pi}^{\mathsf{P}}] \\ & \bar{A}^{\mathsf{Z}} \boldsymbol{\lambda} & = \; & \mathbf{0} \quad [\bar{\boldsymbol{\pi}}^{\mathsf{Z}}] \\ & \boldsymbol{\lambda} \geq \mathbf{0} & & \end{aligned}$$

SP

$$\bar{c}^{\star}_{\mathsf{SP}} := \min_{\mathbf{x} \in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathsf{P}})^\top \mathbf{a}^{\mathsf{P}}(\mathbf{x}) - (\bar{\boldsymbol{\pi}}^{\mathsf{Z}})^\top \bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.$$

Row-partition of MP

$$
\begin{array}{llll}
\min & \mathbf{c}^\top \boldsymbol{\lambda} & & \\
\text{st:} & A^{\mathrm{P}}\boldsymbol{\lambda} & = \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\
& \bar{A}^{\mathrm{Z}}\boldsymbol{\lambda} & = \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\
& \boldsymbol{\lambda} \geq \mathbf{0} & &
\end{array}
$$

SP?

$$
\bar{c}^{\star}_{\mathrm{SP}} := \min_{\mathbf{x} \in X, \boldsymbol{\zeta}^{Z} \in \mathbb{R}^{m-p}} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathrm{P}})^\top \mathbf{a}^{\mathrm{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\mathrm{Z}})^\top \bar{\mathbf{a}}^{\mathrm{Z}}(\mathbf{x}) \right\}.
$$

Going back to optimality conditions

$\bar{\mathbf{c}}$ on the real axis

Let γ capture the minimal value of $\bar{\mathbf{c}}$:

$$\bar{c}^{\star}_{\mathrm{SP}} := \max\{\gamma \mid \gamma \mathbf{1}^{\top} \leq \bar{\mathbf{c}}^{\top} := \mathbf{c}^{\top} - (\boldsymbol{\pi}^{\mathrm{P}})^{\top} A^{\mathrm{P}} - (\zeta^{\mathrm{Z}})^{\top} \bar{A}^{\mathrm{Z}}\}$$

Rewrite SP

$$\bar{c}^{\star}_{\mathrm{SP}} := \max \ \gamma$$

$$\text{st:} \quad \gamma \mathbf{1}^{\top} + (\zeta^{\mathrm{Z}})^{\top} \bar{A}^{\mathrm{Z}} \leq \ \tilde{\mathbf{c}}^{\top} \qquad [\lambda]$$

SP

$$
\begin{array}{rrcll}
\bar{c}^{\star}_{\mathrm{SP}} := \min & \tilde{\mathbf{c}}^{\top}\boldsymbol{\lambda} & & & \\
\text{st:} & \mathbf{1}^{\top}\boldsymbol{\lambda} & = & 1 & [\gamma] \\
 & \bar{A}^{\mathrm{Z}}\boldsymbol{\lambda} & = & \mathbf{0} & [\zeta^{\mathrm{Z}}] \\
 & \boldsymbol{\lambda} & \geq & \mathbf{0}. &
\end{array}
$$

Solving approach

SP is now a linear program
solved by column generation over $\mathbf{x} \in X$.

Optimal solution $\boldsymbol{\lambda}^*$

An optimal solution to SP satisfies $\bar{A}^{\mathrm{Z}}\boldsymbol{\lambda}^* = \mathbf{0}$.

Hence column vector $[AX^*]$ is compatible
and **strictly** improves the objective function if $\bar{c}^{\star}_{\mathrm{SP}} < 0$.

Compatibility

Definition

Given vector $\boldsymbol{\lambda}_{\mathsf{P}} > \mathbf{0}$ of non-degenerate basic variables, vector $\mathbf{a}$ is *compatible with* $\boldsymbol{\lambda}_{\mathsf{P}}$ if and only if $\bar{\mathbf{a}}^{\mathsf{Z}} = \mathbf{0}$.

Interpretation

Given the solution vector $\boldsymbol{\lambda}_{\mathsf{P}} > \mathbf{0}$ of non-degenerate variables, vector $\mathbf{a}$ is *compatible with* $\boldsymbol{\lambda}_{\mathsf{P}}$ if and only if $\mathbf{a}$ is a linear combination of the columns of A_{P}:

$$\mathbf{a} = A_{\mathsf{P}}\mathbf{y}$$

Observe that right-hand side vector $\mathbf{b}$ is *compatible*.

$$\bar{\mathbf{b}}^{\mathsf{Z}} = \mathbf{0}$$

$$\mathbf{b} = A_{\mathsf{P}}\boldsymbol{\lambda}_{\mathsf{P}}$$

Interest for compatible variables

Consider a non-basic compatible variable $\lambda_j, j \notin B$ with a negative reduced cost that is selected to enter into the basis.

$$\begin{bmatrix} \bar{\mathbf{a}}_j^{\mathrm{P}} \\ \mathbf{0} \end{bmatrix} \text{ vs. } \begin{bmatrix} \bar{\mathbf{b}}^{\mathrm{P}} \\ \mathbf{0} \end{bmatrix}$$

Ratio-test computed on the row-set P only:

$$\min_{i \in \mathrm{P}} \{ \frac{\bar{b}_i}{\bar{a}_{ij}} \mid \bar{a}_{ij} > 0 \}.$$

Since $\bar{b}_i > 0, \forall i \in \mathrm{P}$, the step size for λ_j is greater than zero;
the objective function strictly decreases,
unless $\bar{a}_{ij} \leq 0, \forall i \in \mathrm{P}$, in which case LP is unbounded.

SP

$$\begin{array}{rlrcll} \bar{c}^{\star}_{\mathrm{SP}} := \min & \tilde{\mathbf{c}}^{\top}\boldsymbol{\lambda} & & & \\ \text{st:} & \mathbf{1}^{\top}\boldsymbol{\lambda} & = & 1 & [\gamma] \\ & \bar{A}^{\mathrm{Z}}\boldsymbol{\lambda} & = & \mathbf{0} & [\zeta^{\mathrm{Z}}] \\ & \boldsymbol{\lambda} & \geq & \mathbf{0}. & \end{array}$$

Solving approach

SP is now a linear program

solved by column generation over $\mathbf{x} \in X$.

Optimal solution $\boldsymbol{\lambda}^*$

An optimal solution to SP satisfies $\bar{A}^{\mathrm{Z}}\boldsymbol{\lambda}^* = \mathbf{0}$.

Hence column vector $[A\boldsymbol{\lambda}^*]$ is **compatible**

and **strictly** improves the objective function if $\bar{c}^{\star}_{\mathrm{SP}} < 0$.

Row-partition of MP

$$\begin{aligned} \min \quad & \mathbf{c}^\top \boldsymbol{\lambda} \\ \text{st:} \quad & A^{\mathsf{P}}\boldsymbol{\lambda} = \mathbf{b}^{\mathsf{P}} \quad [\boldsymbol{\pi}^{\mathsf{P}}] \\ & \bar{A}^{\mathsf{Z}}\boldsymbol{\lambda} = \mathbf{0} \quad [\boldsymbol{\zeta}^{\mathsf{Z}}] \\ & \boldsymbol{\lambda} \geq \mathbf{0} \end{aligned}$$

SP

$$\bar{c}^{\star}_{\mathsf{SP}} := \min_{\mathbf{x}\in X, \boldsymbol{\zeta}^{\mathsf{Z}} \in \mathbb{R}^{m-p}} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathsf{P}})^\top \mathbf{a}^{\mathsf{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\mathsf{Z}})^\top \bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.$$

Transformed MP with row and column partitions

Column-partition

- C: index set of compatible variables;
- I: index set of incompatible ones.

Row&Column-Partition of MP

$$\begin{array}{llcllcll}
\text{min} & \mathbf{c}_{\mathrm{C}}^{\top}\boldsymbol{\lambda}_{\mathrm{C}} & + & \mathbf{c}_{\mathrm{I}}^{\top}\boldsymbol{\lambda}_{\mathrm{I}} & & & & \\
\text{st:} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & + & A_{\mathrm{I}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\
 & \bar{A}_{\mathrm{C}}^{\mathrm{Z}}\boldsymbol{\lambda}_{\mathrm{C}} & + & \bar{A}_{\mathrm{I}}^{\mathrm{Z}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\
 & \boldsymbol{\lambda}_{\mathrm{C}}, & & \boldsymbol{\lambda}_{\mathrm{I}} & \geq & \mathbf{0} &
\end{array}$$

SP

$$\bar{c}_{\mathsf{SP}}^{\star} := \min_{\mathbf{x}\in X, \boldsymbol{\zeta}^{\mathsf{Z}}\in\mathbb{R}^{m-p}} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathsf{P}})^{\top}\mathbf{a}^{\mathsf{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\mathsf{Z}})^{\top}\bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.$$

Transformed MP with row and column partitions

Column-partition

- C: index set of compatible variables;
- I: index set of incompatible ones.

Row&Column-Partition of MP

$$\begin{array}{llcllcc l}
\text{min} & \mathbf{c}_{\mathrm{C}}^{\top}\boldsymbol{\lambda}_{\mathrm{C}} & + & \mathbf{c}_{\mathrm{I}}^{\top}\boldsymbol{\lambda}_{\mathrm{I}} & & & \\
 & & & & & & \\
\text{st:} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & + & A_{\mathrm{I}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\
 & \mathbf{0}\boldsymbol{\lambda}_{\mathrm{C}} & + & \bar{A}_{\mathrm{I}}^{\mathrm{Z}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\
 & \boldsymbol{\lambda}_{\mathrm{C}}, & & \boldsymbol{\lambda}_{\mathrm{I}} & \geq & \mathbf{0} &
\end{array}$$

SP

$$\bar{c}_{\mathsf{SP}}^{\star} := \min_{\mathbf{x}\in X, \boldsymbol{\zeta}^{\mathsf{Z}}\in\mathbb{R}^{m-p}} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathsf{P}})^{\top}\mathbf{a}^{\mathsf{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\mathsf{Z}})^{\top}\bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.$$

Transformed MP with row and column partitions

Column-partition

- C: index set of compatible variables;
- I: index set of incompatible ones.

Row&Column-Partition of MP

$$
\begin{array}{lccccccc}
\min & \mathbf{c}_{\mathrm{C}}^{\top}\boldsymbol{\lambda}_{\mathrm{C}} & + & \mathbf{c}_{\mathrm{I}}^{\top}\boldsymbol{\lambda}_{\mathrm{I}} & & & \\
\\
\text{st:} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & + & A_{\mathrm{I}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\
 & & & \bar{A}_{\mathrm{I}}^{\mathrm{Z}}\boldsymbol{\lambda}_{\mathrm{I}} & = & \mathbf{0} & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\
 & \boldsymbol{\lambda}_{\mathrm{C}}, & & \boldsymbol{\lambda}_{\mathrm{I}} & \geq & \mathbf{0} &
\end{array}
$$

SP

$$
\bar{c}_{\mathsf{SP}}^{\star} := \min_{\mathbf{x}\in X,\boldsymbol{\zeta}^{\mathsf{Z}}\in\mathbb{R}^{m-p}} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathsf{P}})^{\top}\mathbf{a}^{\mathsf{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\mathsf{Z}})^{\top}\bar{\mathbf{a}}^{\mathsf{Z}}(\mathbf{x}) \right\}.
$$

A *row-reduced* RMP (rRMP)

Column-partition

- C: index set of compatible variables;
- I: index set of incompatible ones.

Row&Column-Partition of MP — Discard $\boldsymbol{\lambda}_{\mathrm{I}}$

$$\begin{array}{llcll} \min & \mathbf{c}_{\mathrm{C}}^{\top}\boldsymbol{\lambda}_{\mathrm{C}} & & & \\ \text{st:} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} & = & \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\ & & & & [\boldsymbol{\zeta}^{\mathrm{Z}}] \\ & \boldsymbol{\lambda}_{\mathrm{C}} & \geq & \mathbf{0} & \end{array}$$

cSP

$$\bar{c}^{\star}_{\mathrm{cSP}} := \min_{\mathbf{x}\in X, \boldsymbol{\zeta}^{\mathrm{Z}}\in\mathbb{R}^{m-p}} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathrm{P}})^{\top}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) - (\boldsymbol{\zeta}^{\mathrm{Z}})^{\top}\bar{\mathbf{a}}^{\mathrm{Z}}(\mathbf{x}) \mid \bar{\mathbf{a}}^{\mathrm{Z}}(\mathbf{x}) = \mathbf{0} \right\}.$$

A *row-reduced* RMP (rRMP)

Column-partition

- C: index set of compatible variables;
- I: index set of incompatible ones.

Row&Column-Partition of MP

$$
\begin{array}{llll}
\min & \mathbf{c}_{\mathrm{C}}^{\top}\boldsymbol{\lambda}_{\mathrm{C}} & & \\
\text{st:} & A_{\mathrm{C}}^{\mathrm{P}}\boldsymbol{\lambda}_{\mathrm{C}} = \mathbf{b}^{\mathrm{P}} & [\boldsymbol{\pi}^{\mathrm{P}}] \\
 & & [\zeta^{\mathrm{Z}}] \\
 & \boldsymbol{\lambda}_{\mathrm{C}} \geq \mathbf{0} &
\end{array}
$$

cSP

Discard ζ^{Z}

$$\bar{c}_{\text{cSP}}^{\star} := \min_{\mathbf{x}\in X} \left\{ c(\mathbf{x}) - (\boldsymbol{\pi}^{\mathrm{P}})^{\top}\mathbf{a}^{\mathrm{P}}(\mathbf{x}) \;\middle|\; \bar{\mathbf{a}}^{\mathrm{Z}}(\mathbf{x}) = \mathbf{0} \right\}.$$

An ICG Algorithm

Step 0. Initialize RMP, the *Restricted Master Problem*.

Step 1. Solve **RMP**.

Step 2. Identify a basis A_{B} with $p \leq m$ non-degenerate basic variables.

Step 3. If $p < m$ (degenerate solution), Goto Step 7.

Step 4. Otherwise solve **SP**.

Step 5. If $\bar{c}^{\star}_{\mathrm{SP}} < 0$, add columns to RMP; Goto Step 1.

Step 6. Otherwise Stop: RMP and MP are optimal.

An ICG Algorithm

Step 0. Initialize RMP, the *Restricted Master Problem*.

Step 1. Solve **RMP**.

Step 2. Identify a basis A_{B} with $p \leq m$ non-degenerate basic variables.

Step 3. If $p < m$ (degenerate solution), Goto Step 7.

Step 4. Otherwise solve **SP**.

Step 5. If $\bar{c}^{\star}_{\mathrm{SP}} < 0$, add columns to RMP; Goto Step 1.

Step 6. Otherwise Stop: RMP and MP are optimal.

Step 7. $\bar{b} := A_{\mathrm{B}}^{-1} b$. Let $\mathrm{P} := \{i \mid \bar{b}_i > 0\}$ and $\mathrm{Z} := \{i \mid \bar{b}_i = 0\}$.

Step 8. Solve **rRMP**, the *row-reduced* RMP.

Step 9. Solve **cSP**.

Step 10. If $\bar{c}^{\star}_{\mathrm{cSP}} < 0$, add compatible columns to rRMP; Goto Step 8.

Step 11. Otherwise solve **rSP** by *column generation*.

Step 12. If $\bar{c}^{\star}_{\mathrm{rSP}} < 0$, add incompatible columns to RMP; Goto Step 1.

Step 13. Otherwise Stop: rRMP and MP are optimal.

Implementation issues

- Choice of the reduced basis: $A_{\mathrm{P}}^{\mathrm{P}}$.
- Row-partition: when to update?
- Implicit upper bounds: use the smallest rRMP and rSP.
- Priority to cSP but threshold on $\bar{c}^{*}_{\mathrm{cSP}}$ to switch for iSP.
- Compatibility: weaker condition $\bar{\mathbf{a}}_j^{\mathrm{Z}} \leq \mathbf{0}$.
- System with inequalities.

Key to implementation success

Understanding the structure of the problem

Computational experiments

Improved Primal Simplex (IPS) vs. CPLEX-Primal

Simultaneous Vehicle and Crew Scheduling Problems
10 instances: 2000 constraints and 10 000 variables
Degeneracy around 50%
CPU reduced by a factor **4.1**

Aircraft Fleet Assignment
14 instances: 5000 constraints and 25 000 variables
Degeneracy around 65%
CPU reduced by a factor **15.8**

Computational experiments

ubIPS (IPS with upper bounded variables)

Aircraft Fleet Assignment
14 instances: 5000 constraints and 25 000 variables
Degeneracy around 65%
CPU reduced by a factor **32.1**

Statistcal t-test

A t-test on the paired data shows that there is no significant difference between IPS and ubIPS on these instances.

CPLEX considerably slows down when upper bounds are explicitly imposed while ubIPS does not. This observation is at the origin of the research done on ICG with implicit upper bounded variables.

Computational Experiments

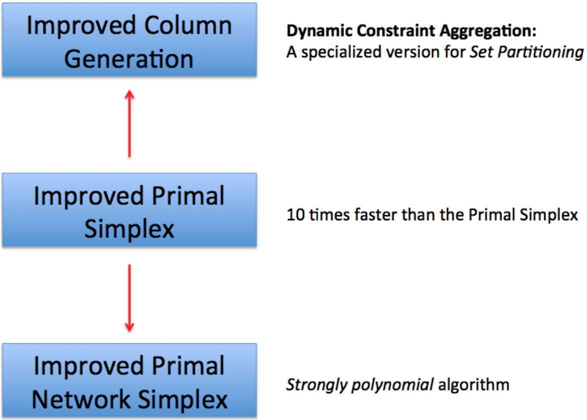
Dynamic Constraints Aggregation (DCA) within GENCOL

Set Partitioning models for routing and scheduling applications
Instances with 2000 constraints and 25 000 variables
Degeneracy around 50% to 60%
CPU reduced by factors **50** to **100**

A combination of many factors

- smaller rRMP dynamically adjusted for the number of rows
- reduced number of degenerate pivots
- smaller cSP (task aggregation also done within networks)
- restricted versions of iSP (low-rank incompatibilities)
- smaller number of column generation iterations
- less fractional linear programming relaxations
- smaller branch-and-bound search trees

The iSeries: ICG, IPS, IPNS



Conclusions

- **CG** exploits the modeling structure of a subproblem.
- Main drawback: tailing-off due to degenerate iterations.

- **ICG** exploits both the modeling structure of a problem and the algebraic structure of its solutions. (dual variables freedom)
- **ICG** takes advantage of degenerate solutions by reducing the row-size of the master problem. (compatible variables priority)

** Future work needed on implementation strategies for generic problems.

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ICG

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