The Horizon Decomposition for Capacity Constrained Lot Size Problems with Setup Times

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Capacity Constrained Lot Sizing with Setup Times Notation and Compact Formulation

- P : Set of Products
- T : Set of Time periods
- Problem statement: satisfy demand of each product in each period at minimum cost
- Constraints: machine capacity, Setups

$$
\begin{aligned}
\min \quad & \sum_{i \in P} \sum_{t \in T} (sc_{it} y_{it} + vc_{it} x_{it} + hc_{it} s_{it}) \\
& s_{it} + x_{it} = d_{it} + s_{i,t+1} \quad \forall i \in P, \forall t \in T \\
& x_{it} \leq M y_{it} \quad \forall i \in P, \forall t \in T \\
& \sum_{i \in P} st_{it} y_{it} + \sum_{i \in P} vt_{it} x_{it} \leq cap_t \quad \forall t \in T \\
& y_{it} \in \{0, 1\}, s_{it}, x_{it} \geq 0 \quad \forall i \in P, \forall t \in T
\end{aligned}
$$

Brief Literature Review

Research in CLST is Vast

- Problem introduction: Trigeiro et al. [*Man. Sci.* (1989)]
- Valid inequalities: Barany et al. [*Man. Sci.* (1984)]; Miller et al. [*EJOR* (2000)]
- Reformulations: Eppen & Martin [*Op. Res.* (1987)]; Van Vyve & Wolsey [*Math. Prog.* (2006)]
- DW recomposition – Lagrange relaxation: Thizy et al. [*Dec. Sci.* (1985)]; Sural et al. [*EJOR* (2009)]; Degraeve & Jans [*Op. Res.* (2007)]

The Horizon Decomposition

An illustrative example

$$\min \quad ... + hc_{it}s_{it} + vc_{it}x_{it} + hc_{i,t+1}s_{i,t+1} + vc_{i,t+1}x_{i,t+1} + hc_{i,t+2}x_{i,t+2} + ...$$

$$s_{it} + x_{it} = d_{it} + s_{i,t+1} \qquad \forall i \in P$$

$$s_{i,t+1} + x_{i,t+1} = d_{i,t+1} + s_{i,t+2} \quad \forall i \in P$$

$$\cdots$$

The Horizon Decomposition

An illustrative example

$$\min \quad \ldots + hc_{it}s_{it} + vc_{it}x_{it} + hc_{i,t+1}s_{i,t+1} + vc_{i,t+1}x_{i,t+1} + hc_{i,t+2}x_{i,t+2} + \ldots$$

$$s_{it} + x_{it} = d_{it} + s_{i,t+1} \quad \forall i \in P$$

$$s_{i,t+1} + x_{i,t+1} = d_{i,t+1} + s_{i,t+2} \quad \forall i \in P$$

$$\ldots$$

$$\min \quad \ldots + hc_{it}s_{it} + vc_{it}x_{it} + hc_{i,t+1}[a * s^l_{i,t+1} + (1-a) * s^r_{i,t+1}]$$

$$+ vc_{i,t+1}x_{i,t+1} + hc_{i,t+2}x_{i,t+2} + \ldots$$

$$s_{it} + x_{it} = d_{it} + s^l_{i,t+1} \quad \forall i \in P$$

$$s^r_{i,t+1} + x_{i,t+1} = d_{i,t+1} + s_{i,t+2} \quad \forall i \in P$$

$$s^l_{i,t+1} = s^r_{i,t+1} \quad \forall i \in P$$

$$\ldots$$

$$0 \leq a \leq 1$$

The Horizon Decomposition

Overlapping Horizons

$$\min \quad \ldots + hc_{it}[b * s^r_{it} + (1-b) * s^l_{it}] + vc_{it}x^l_{it} +$$
$$hc_{i,t+1}[a * s^l_{i,t+1} + (1-a) * s^r_{i,t+1})] + vc_{i,t+1}x_{i,t+1} + hc_{i,t+2}x_{i,t+2} + \ldots$$

$$s^l_{it} + x^l_{it} = d_{it} + s^l_{i,t+1} \qquad \forall i \in P$$
$$s^r_{it} + x^r_{it} = d_{it} + s^r_{i,t+1} \qquad \forall i \in P$$
$$s^r_{i,t+1} + x_{i,t+1} = d_{i,t+1} + s_{i,t+2} \qquad \forall i \in P$$
$$s^l_{it} = s^r_{it} \qquad \forall i \in P$$
$$y^l_{it} = y^r_{it} \qquad \forall i \in P$$
$$s^l_{i,t+1} = s^r_{i,t+1} \qquad \forall i \in P$$
$$\ldots$$
$$0 \leq a, b \leq 1$$

The Horizon Decomposition

A MIP Formulation

$$[\mathbf{DW}] \quad \min \quad \sum_{s\in S}\sum_{e\in\mathcal{E}_s} c_{se} z_{se}$$

$$\text{s.t.} \sum_{e\in\mathcal{E}_s} \bar{s}^s_{ite} z_{se} = \sum_{e\in\mathcal{E}_{s+1}} \bar{s}^{s+1}_{ite} z_{s+1,e} \quad \forall i \in P, t \in L_s, s \in S\backslash\{p\}$$

$$\sum_{e\in\mathcal{E}_s} \bar{y}^s_{ite} z_{se} = \sum_{e\in\mathcal{E}_{s+1}} \bar{y}^{s+1}_{ite} z_{s+1,e} \quad \forall i \in P, t \in \bar{L}_s, s \in S\backslash\{p\}$$

$$\sum_{e\in\mathcal{E}_s} z_{se} = 1 \quad \forall s \in S$$

$$y_{it} = \sum_{e\in\mathcal{E}_s} \bar{y}^s_{ite} z_{se} \quad \forall i \in P, \forall (t,s) \in \bar{H}_s \times S : \alpha_{ts} = 1$$

$$z_{se} \geq 0 \quad \forall e \in \mathcal{E}_s, \forall s \in S$$

$$y_{it} \in \{0,1\} \quad \forall i \in P, t \in T$$

Characteristics of the Horizon Decomposition

A Different view of Danzig-Wolfe Decomposition

The Horizon Decomposition has special characteristics

- Each *subproblem is a short-horizon CLST* with as many products as the original problem
- The Danzig-Wolfe *Master* program consists of the *variable splitting* constraints and the usual *convexity constraints*
- The user can control the size of the master and the subproblem (almost) *independently*
- Computational *advantage* stems from the *reduced problem size*, not from special structure of constraints

Computational challenges and strategies

- RMPs very degenerate, column generation tails off
- Two-phase algorithm: subproblem solved to feasibility in phase I
- Add many columns that price out
- Stabilization techniques: extra variables in phase I
- When column generation stalls, update duals with modified subgradient optimization
- Master solved with the sifting algorithm
- Subproblems: find feasible solutions fast in CG, find good lower bounds in LR
- Branch-and-price: RINS diving, best first search

Performance of Horizon Configurations

Criteria: CPU Time, Integrality Gap

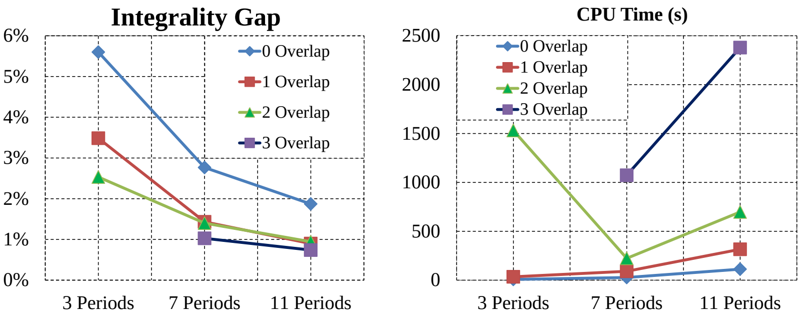
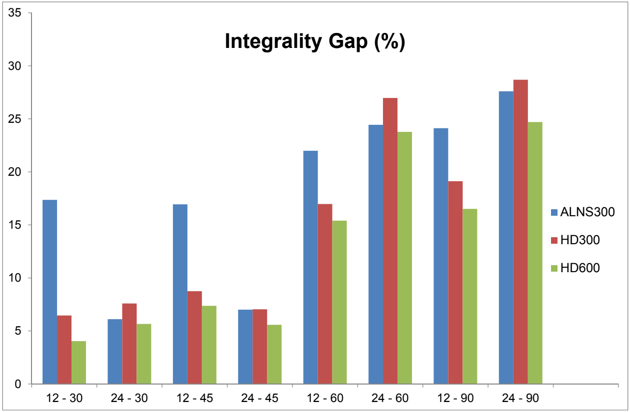


Figure: Performance of Horizon Configurations

Comparison with Adaptive Neighbor Search
Muller et al. (2011)



Comparison with CPLEX

CPU Time: 3600 seconds

	CPLEX		Horizon Decomposition			
Products	Gap (%)	Nodes	(|H|,|L|)	Gap (%)	Gap Closed (%)	Nodes
2	6.73	19,662,721	(12,2)	0.52	93.59	56.7
4	7.77	4,137,254	(12,2)	3.85	58.97	6.2
6	12.60	1,538,035	(10,0)	9.87	22.15	2
8	13.01	644,462	(6,1)	12.33	11.55	0.6
10	9.25	1,903,519	(5,1)	8.47	13.95	0
Total	9.87	5577198		8.63	40.04	16

Table: 100 Period Instances

A Generic MIP

$$
[\mathbf{P}] \qquad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \in X \end{aligned}
$$

- I : Column index set
- R : Row index set
- X denotes integrality and range restrictions

A First Extension: Row Partitioning

$$
\begin{aligned}
[\mathbf{P_1}] \qquad \min \quad & c_{V_1}^T x_{V_1} + c_{V_2}^T x_{V_2} + \lambda c_V^T x_V^1 + (1-\lambda) c_V^T x_V^2 \\
\text{s.t.} \quad & A_{R_1 V_1} x_{V_1} + A_{R_1 V} x_V^1 = b_{R_1} \\
& A_{R_2 V_2} x_{V_2} + A_{R_2 V} x_V^2 = b_{R_2} \\
& x_V^1 - x_V^2 = 0_V \\
& x_{V1} \in X_{V1} \quad x_{V2} \in X_{V2} \quad x_V^1, x_V^2 \in X_V
\end{aligned}
$$

How to Select R_1, R_2 in Practice? (Martin 1999), (Bergner et al. 2011)

A Further Extension: Column Partitioning

Consider $H_1, H_2 \subset I : H_1 \cup H_2 = I, H_1 \cap H_2 = L$
$\bar{c}_{2i} = c_{2i}$ if $i \in H_2 \backslash L$, 0 else

$$
\begin{aligned}
[\mathbf{P_2}] \qquad \min \quad & c_1^T x_1 + \bar{c}_2^T x_2 \\
\text{s.t.} \quad & A_1 x_1 - \lambda A_l x_l^1 + s_1 = b/2 \\
& A_2 x_2 - (1 - \lambda) A_l x_l^2 - s_2 = b/2 \\
& x_l^1 = x_l^2 \\
& s_1 = s_2 \\
& x_1 \in X_1, x_2 \in X_2, x_l^1, x_l^2 \in X_l, s_1, s_2 \in \mathbb{R}^r
\end{aligned}
$$

Potential use: columns with combinatorial description

Conclusions and Future Research

Conclusions

- We introduce the Horizon Decomposition of CLST, a novel approach to solving hard lot sizing problems.
- Branch-and-price gives significant lower bound improvement and has competitive performance against branch-and-cut in challenging instances.
- Our approach is generalizable to any structure

Future research

- Improvement of column generation convergence
- Customized horizon partitioning