Strong degree constraints to impose partial elementarity in shortest path problems under resource constraints

Claudio Contardo[1], Guy Desaulniers[2]

[1]École des sciences de la gestion, UQÀM and GERAD

[2]MAGI, École Polytechnique de Montréal and GERAD

Column Generation 2012
Bromont, Canada, June 2012

Outline

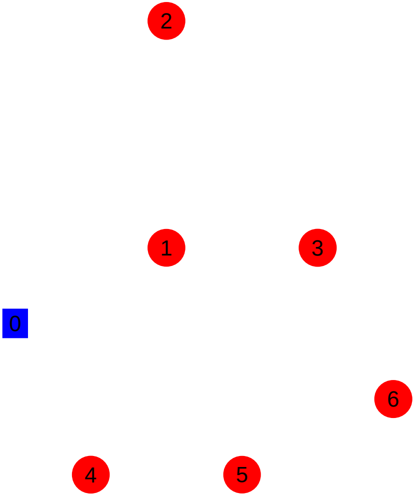
Some notation

- V set of nodes in the graph
  - V_D set of depots
  - V_C set of customers
- $\mathcal{R}$ set of routes. Depending on how $\mathcal{R}$ is defined, routes may contain cycles or not
- For a given customer $i \in V_C$ and route $r \in \mathcal{R}$ we define
  - a_{ir}: number of times that route r visits customer i
  - ξ_{ir}: binary constant equal to 1 iff route r visits customer i

Basic Property

$$a_{ir} \geq \xi_{ir} \qquad \text{for all } i \subseteq V_C, r \in \mathcal{R} \tag{1}$$

Example



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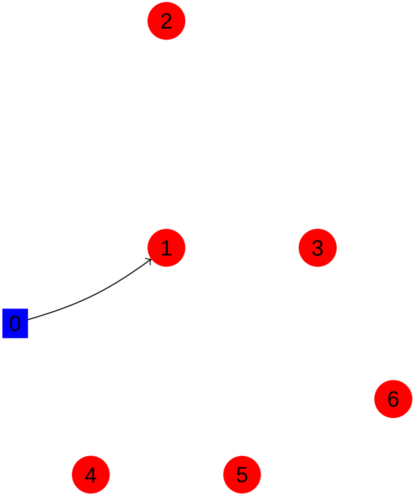

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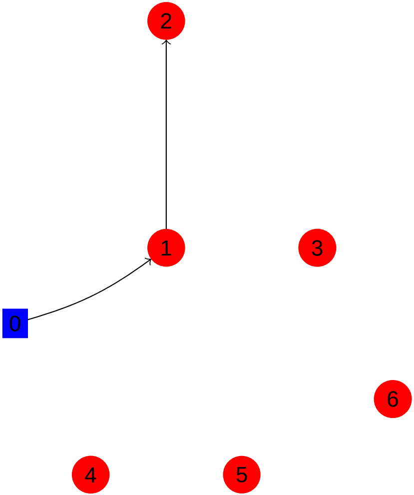

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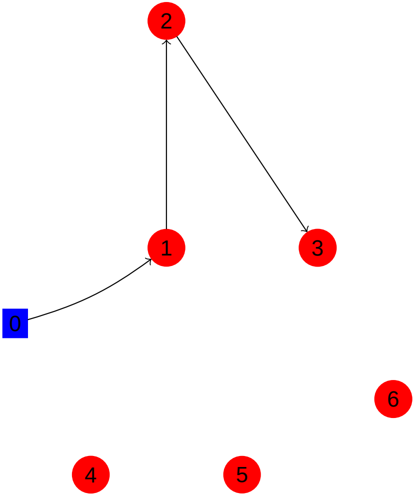

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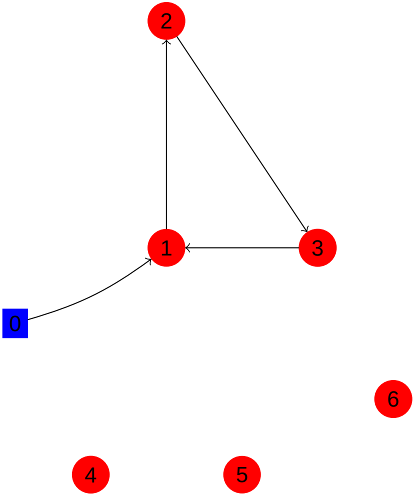

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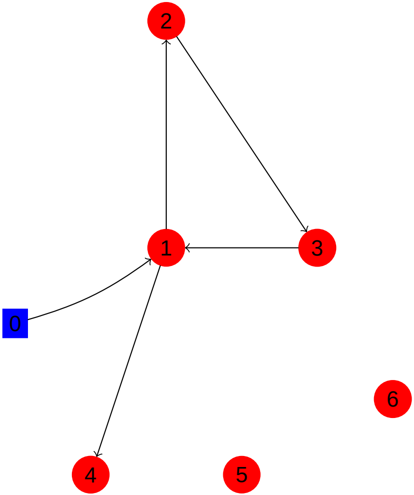

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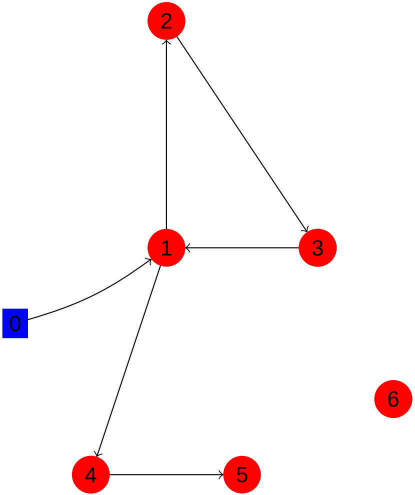

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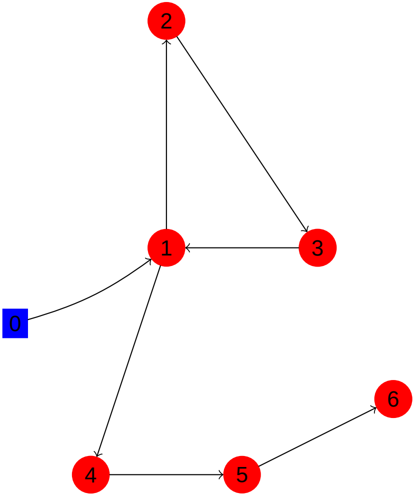

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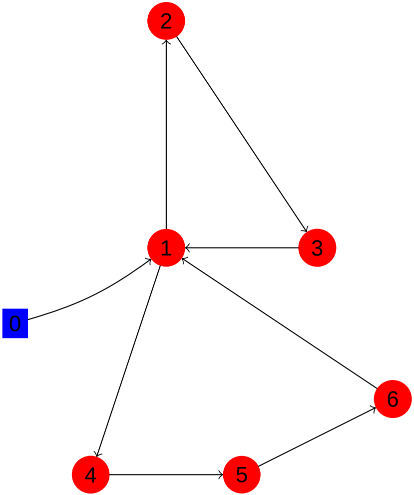

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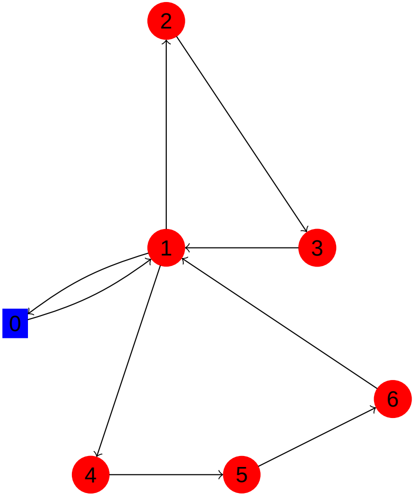

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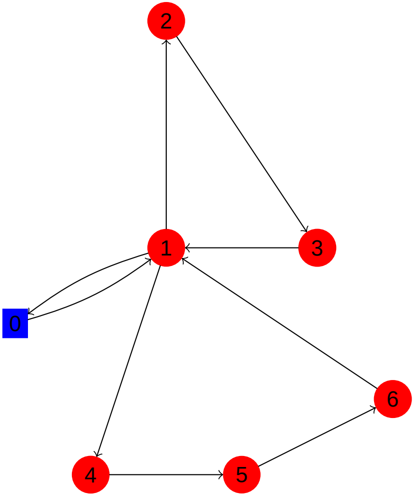
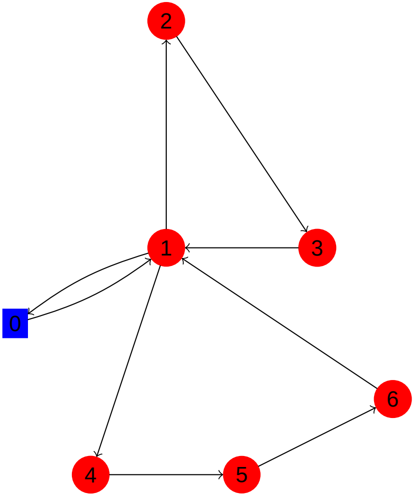


- $r = (0, 1, 2, 3, 1, 4, 5, 6, 1, 0)$
- $a_{1r} = 3$
- $\xi_{1r} = 1$

Example



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General form of the set-partitioning formulation

$$\min \sum_{r \in \mathcal{R}} c_r \theta_r + \text{ other terms} \quad (2)$$

subject to

$$\sum_{r \in \mathcal{R}} a_{ir} \theta_r = 1 \qquad i \in V_C \quad (3)$$

$$\theta_r \in \{0, 1\} \qquad r \in \mathcal{R} \quad (4)$$

$$\text{other variables and constraints} \quad (5)$$

The Capacitated VRP (CVRP)

$$\min \sum_{r \in \mathcal{R}} c_r \theta_r \tag{6}$$

subject to

$$\sum_{r \in \mathcal{R}} a_{ir} \theta_r = 1 \qquad i \in V_C \tag{7}$$

$$\theta_r \in \{0, 1\} \qquad r \in \mathcal{R} \tag{8}$$

Routes in set $\mathcal{R}$ are assumed to respect vehicle capacity Q

The VRP with Time Windows (VRPTW)

$$\min \quad \sum_{r \in \mathcal{R}} c_r \theta_r \tag{9}$$

subject to

$$\sum_{r \in \mathcal{R}} a_{ir} \theta_r = 1 \qquad i \in V_C \tag{10}$$

$$\theta_r \in \{0, 1\} \qquad r \in \mathcal{R} \tag{11}$$

Routes in set $\mathcal{R}$ are assumed to respect vehicle capacity Q and time windows of customers $[e_i, l_i]$

The Multiple Depot VRP (MDVRP)

$$\min \quad \sum_{r \in \mathcal{R}} c_r \theta_r \tag{12}$$

subject to

$$\sum_{r \in \mathcal{R}} a_{ir} \theta_r = 1 \qquad i \in V_C \tag{13}$$

$$\sum_{r \in \mathcal{R}_i} q_r \theta_r \leq b_i \qquad i \in V_D \tag{14}$$

$$\theta_r \in \{0, 1\} \qquad r \in \mathcal{R} \tag{15}$$

Routes in set $\mathcal{R}$ are assumed to respect vehicle capacity Q

The Capacitated Location-Routing Problem (CLRP)

$$\min \quad \sum_{i \in V_D} f_i z_i + \sum_{r \in \mathcal{R}} c_r \theta_r \tag{16}$$

subject to

$$\sum_{r \in \mathcal{R}} a_{ir} \theta_r = 1 \qquad i \in V_C \tag{17}$$

$$\sum_{r \in \mathcal{R}_i} q_r \theta_r \leq b_i z_i \qquad i \in V_D \tag{18}$$

$$\theta_r \in \{0, 1\} \qquad r \in \mathcal{R} \tag{19}$$

$$z_i \in \{0, 1\} \qquad i \in V_D \tag{20}$$

Routes in set $\mathcal{R}$ are assumed to respect vehicle capacity Q

And many others!

- Inventory-Routing Problems
- Periodic Routing Problems
- Prize-collecting Routing Problems
- And many combinations!

Column generation for vehicle routing problems

- Set-partitioning formulations provide stronger bounds than traditional flow formulations
- The strength of set-partitioning formulations depends on the set of feasible routes $\mathcal{R}$
- Pricing subproblem reduces to solve a shortest path problem with additional resource constraints
- Current state-of-the-art methods find a balance between speed and strength of the pricing subproblem
- Bottleneck of column generaton methods is related to the imbalance between strength of the bounds vs. computing times
- Most efficient implementations use bidirectional dynamic programming in the pricing subproblem

Existing pricing algorithms

- Without cycles elimination (SPPRC): very fast (pseudo-polynomial) but weak bounds
- With 2-cycles elimination (2-*cyc*-SPPRC): slower but stronger bounds
- With k-cycles elimination (k-*cyc*-SPPRC): slower but stronger bounds
- *ng*-routes relaxation (*ng*-SPPRC): state-of-the-art, good compromise btw speed and lower bound quality
- Elementary routes (ESPPRC): slowest ($\mathcal{NP}$-hard) but strongest bounds

Strong degree constraints

- It is a family of inequalities that are imposed into the master problem
- As such, they can be embedded into any column generation method
- The goal is to impose partial elementarity in a smart way, taking advantage of master problem information (dual variables)

Strong degree constraints

Strong degree constraints (SDEG)

$$\sum_{r \in \mathcal{R}} \xi_{ir} \theta_r \geq 1 \qquad \text{for all } i \in V_C \tag{21}$$

Separation of SDEG

- For each customer $i \in V_C$ we compute the quantity

$$v(i, \overline{\theta}) = 1 - \sum_{r \in \overline{\mathcal{R}}} \xi_{ir} \overline{\theta}_r$$

- We add to the RMP the SDEG constraints related to the k customers (k being a parameter defined a priori) with the largest value of $v(i, \overline{\theta})$
- This is done if and ony if $v(i, \overline{\theta}) \geq \epsilon_{SDEG}$ with $\epsilon_{SDEG} > 0$ being a parameter defined a priori

Partial Elementarity

- Let us consider a customer $i \in V_C$. Let $\mathcal{R}_{cyc(i)}$ be the set of routes visiting customer i more than once. Assume that a SDEG is added for i.
- Let us subtract from the SDEG, the weak degree constraint associated to customer i

$$\sum_{r \in \mathcal{R}} (\xi_{ir} - a_{ir})\theta_r \geq 0$$

$$\Rightarrow \sum_{r \in \mathcal{R}_{cyc(i)}} (\xi_{ir} - a_{ir})\theta_r \geq 0 \qquad (a_{ir} = \xi_{ir} \text{ for } r \in \mathcal{R} \setminus \mathcal{R}_{cyc(i)})$$

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Theorem (Partial Elementarity)

$$\theta_r = 0 \text{ for all } r \in \mathcal{R}_{cyc(i)} \qquad (a_{ir} > \xi_{ir} \text{ for } r \in \mathcal{R}_{cyc(i)})$$

Effect of SDEG in the pricing subproblem

- Let L, L' be two labels (partial paths) sharing the same terminal node $v(L) = v(L') = v$
- Let $PE \subseteq V_C$ be the subset of customers for which a SDEG constraint has been added
- Let $PE(L), PE(L')$ be the set of customers in PE served by the partial path represented by L and L', respectively

Regular Dominance Rule

$$L \prec L' \Leftrightarrow \overline{c}(L) \leq \overline{c}(L') \text{ and } PE(L) \subseteq PE(L') \text{ and o.c.}$$

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New (SHARPER!) Dominance Rule

$$L \prec L' \Leftrightarrow \overline{c}(L) + \sum_{i \in PE(L) \setminus PE(L')} \sigma_i \leq \overline{c}(L') \text{ and o.c.}$$

We prefer to use SDEG instead of classic PE because...

1. dual variables of SDEG give us an estimate of the importance of imposing PE on a certain node. Small dual $\Rightarrow$ elementarity is not that important $\Rightarrow$ dominance rule becomes much sharper
2. partial and total elementarity are particular cases in which the dual variables of SDEG constraints are "roughly approximated to infinity" in the dominance rule
3. in practice, solving a relaxation of the ESPPRC (2-*cyc*-SPPRC, *ng*-routes, *k*-*cyc*-SPPRC) suffices to obtain near-elementary routes after the addition of a few SDEG

Computational experience

- We consider the VRPTW
- The pricing subproblems are solved using bidirectional dynamic programming
- Five settings are tested
  - ESPPRC [espprc]
  - Incremental *ng*-routes [i-ng]
  - 2-*cyc*-SPPRC + SDEG (ALL) [sdeg-all]
  - 2-*cyc*-SPPRC + SDEG (30) [sdeg-30]
  - *ng*-routes + SDEG [ng+sdeg]
- We compute and report the CPU time in seconds to achieve the elementary bound [LB]

Results on selected instances of the VRPTW

Instance	LB	espprc	i-ng	sdeg-all	sdeg-30	ng+sdeg
R201	1140.30	20.0	10.0	22.1	21.1	10.4
R202	1022.24	40.4	61.9	78.3	60.8	61.6
R203	866.91	117.3	428	299.1	276.9	276.6
R204	724.91	394,001	2,205	2,361.7	4,802.3	1,489.1
R205	938.93	58.9	76.7	145.7	156.2	65.5
R206	866.87	421.4	368.9	310.8	319.8	217.9
R207	790.67	14,117.1	4,522.6	2,219.3	2,286.4	2,160.5
R208	691.98	$>$ 5 DAYS	403,617.4	34,698.3	19,289.3	24,394.4
R209	841.40	117.6	236.1	309.8	356.6	170.9
R210	889.37	340.1	264.1	449.8	369.3	159.2
R211	734.73	$>$ 3 DAYS	2,290.8	683.6	528.2	534.2

Table: Results on some selected VRPTW instances

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Concluding remarks

- State-of-the-art solvers avoid using traditional ESPPRC as pricing subproblem
- Our results suggest that dual information is relevant to derive sharper dominance rules
- SDEG can be efficiently used to derive strong bounds in a fraction of the computational effort
- Because they are cuts applied to the master problem, they can be embedded into any pricing subproblem (2-cyc, ng, k-cyc, etc.)
- We are currently working on some generalizations of the SDEG

Questions? Comments? Suggestions?