Clique Inequalities applied to Vehicle Routing Problem with Time Windows

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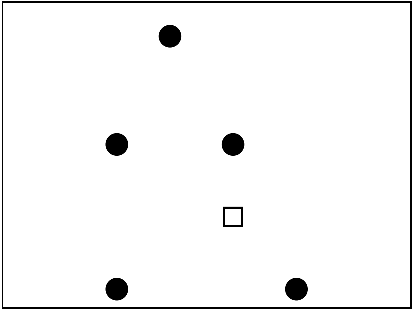
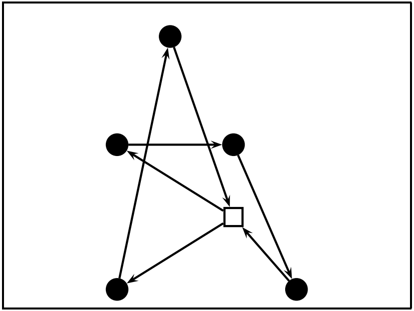
Outline

The Problem

- Route vehicles such that all customers are visited once
- Time windows and capacity must not be violated
- Minimize overall travel cost





Path Formulation

Master Problem

A Set Partitioning Problem:

$$\min \sum_{p \in P} \sum_{(i,j) \in A} c_{ij} \alpha_{ijp} \lambda_p \qquad (1)$$

$$\text{s.t} \sum_{p \in P} \sum_{(i,j) \in \delta^+(i)} \alpha_{ijp} \lambda_p = 1 \qquad \forall i \in C \qquad (2)$$

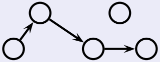
$$\lambda_p \in \{0,1\} \qquad \forall p \in P \qquad (3)$$

where $P = \{\text{valid paths}\}$, $\alpha_{ijp} = \begin{cases} 0 & \text{if } (i,j) \notin p \\ 1 & \text{otherwise} \end{cases}$

Pricing Problem

Elementary Shortest Path Problem with Resource Constraints (ESPPRC)

- $\mathcal{NP}$-hard,



Column Generation

The beginning [Desrochers et al.]:

- Relaxation of ESPPRC into pseudo-polynomial non-elementary problem
- Improved with 2-cycle elimination

Since then:

- Strengthen pricing problem
  - k-cycle elimination [Irnich and Villeneuve]
  - Elementary [Feillet et al.]
  - Bi-directionality [Righini and Salani]
- Apply cuts
  - Arc-flow variables. k-path cuts [Kohl et al.]
  - Path variables. Subset Row inequalities [Jepsen et al.], Chvátal-Gomory rank 1 cuts [Petersen et al.]

Cutting on Arc-Flow Variables

Example

Going from arc to path formulation

$$\sum_{k \in K} \sum_{(i,j) \in A} \beta_{ij} x_{ijk} \leq \beta_0$$

is decomposed to:

$$\sum_{p \in P} \sum_{(i,j) \in A} \beta_{ij} \alpha_{ijp} \lambda_p \leq \beta_0$$

Column Cost:

$$\overline{c}_p = \sum_{(i,j) \in A} (c_{ij} - \pi_j - \beta_{ij}\sigma)\alpha_{ijp}$$

- Dual cost is associated with arc costs in pricing problem

Cutting on Path Variables

Example

Valid inequality for the Set Partitioning:

$$\sum_{p \in P} \beta_p \lambda_p \leq \beta_0$$

Column Cost:

$$\overline{c}_p = \sum_{(i,j) \in A} (c_{ij} - \pi_j) - \beta_p \sigma$$

- To determine β_p when generating p may be difficult.
- Can add complexity to the pricing problem

Cut Representation

Wanted properties:

(i) possible to decide the value of the cut coefficient for a column in reasonable time

(ii) the representation should simplify the handling of the dual value of the cut when solving the pricing problem.

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(i) linear time in the number of arcs ✓

(ii) only modify arc costs ✓

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Cuts on path variables

(i) ?

(ii) ?

Cliques

Definition

The Clique Inequalities.

$$\sum_{p \in \mathcal{Q}} \lambda_p \leq 1$$

Column Cost:

$$\hat{c}_p = \overline{c}_p - \begin{cases} \sigma & \text{if } p \in \mathcal{Q} \\ 0 & \text{otherwise} \end{cases}$$

Separation

Definition

A *conflict graph* is an undirected graph with a node for each column p, and an edge between two nodes if the corresponding columns have at least one common non-zero coefficient.

Exact

The separation problem is the Maximum Weighted Clique Problem which is $\mathcal{NP}$-hard.

Heuristic

Starting from any row greedily build cliques by adding columns that are non-orthogonal.

Representation of Cliques

Cliques on path variables

Assume a submatrix (*key set*) of the master problem that represents a clique

(i) Any column that is a superset of at least one column in the submatrix conflicts with all other columns in the clique, hence the cut coefficient is 1. This can be verified in the number of non-zeroes in the submatrix. ✓

(ii) Given a partial path the cut coefficient can be determined as above. In dynamic programming this can be used to calculate the impact of the dual value of the clique. The step-like function can be exploited when dominance is applied. ✓

Note

A column is part of a clique if it is non-orthogonal to all other columns in the clique. However, (i) is stricter due too a more general setting.

Identifying the Key Set

Definition

The *overlapping* rows of a clique are the rows represented by the edges (possibly more than one row per edge) that connect the clique in the conflict graph.

Definition

The *minimum overlapping* rows of a clique are a minimum subset of the overlapping rows such that the edges represented by the minimum overlapping rows still form a clique in the conflict graph.

Example

Discard either c or d to get minimum overlapping rows.

row			
a	1	1	
b	1		1
c		1	1
d		1	1

Dominance of Rows in the Key Set

Proposition

Given two overlapping rows r and r' for a clique and their coefficient sub-vectors v_r and $v_{r'}$ containing the coefficients of the columns in the clique then row r dominates r' if $v_r \geq v_{r'}$.

Proof.

If $v_r \geq v_{r'}$ then all conflicts due to row r' will also be conflicts due to r as the inequality implies that column with a non-zero coefficient in r' is also has a non-zero coefficient in r. □

Example

a dominates c

row			
a	1	1	
b	1		1
c		1	

Dominance of Columns in the Key Set

Proposition

Given to columns c and c' for a clique and coefficient sub-vectors v_c and $v_{c'}$ containing the coefficients of the overlapping rows in the clique then column c dominates c' if $v_c \leq v_{c'}$.

Proof.

If $v_c \leq v_{c'}$ then column c conflicts with all other columns in the clique on a subset of the overlapping rows that column c' uses. Hence, the overlapping rows used by c is sufficient and the denser column c' is dominated. □

Example

c dominates a

column	*a*	*b*	*c*
	1	1	
	1		1
		1	

The Key Set

Definition

The *key set* of a clique is a minimum set of overlapping rows and a set of column vectors given by the coefficients of the overlapping rows in each of the column vectors in the clique, i.e., a set of sub-vectors for each column in the clique based on coefficients entries at the overlapping rows.

The Key Set - Example

Example

A clique and the key set

$$
\begin{bmatrix}
1 & 1 &   &   &   & 1 \\
  &   & 1 &   &   &   \\
1 &   & 1 &   &   & 1 \\
1 &   &   & 1 & 1 & 1 \\
  & 1 & 1 & 1 &   &   \\
  &   &   & 1 & 1 & 1 \\
  & 1 &   &   & 1 &   \\
  &   & 1 &   & 1 &   \\
1 &   &   &   &   &
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
1 & 1 &   &   &   \\
1 &   & 1 &   &   \\
1 &   &   & 1 & 1 \\
  & 1 & 1 & 1 &   \\
  & 1 &   &   & 1 \\
  &   & 1 &   & 1
\end{bmatrix}
$$

Determine the Cut Coefficient

Proposition

Given a clique with a key set and a column c with a coefficient sub-vector v_c based on the set of overlapping rows in the key set then c is part of the clique, i.e., the cut coefficient is 1, if $v_c \geq v_{c'}$ for some sub-vector $v_{c'}$ for a column c' in the key set of the clique.

Proof.

The proof follows from the Proposition on dominated columns. Adding c in this case corresponds to adding a dominated column in the key set. Hence, c uses at least the same overlapping rows as c' and the column is part of the clique. □

Determine the Cut Coefficient - Example

Example

Comparing three columns a, b and c with a key set $p = 1, \ldots, 5$ for a clique.

$p =$	1	2	3	4	5	a	b	c
	1	1				1		
	1		1			1		1
	1			1	1	1	1	1
		1	1	1			1	
		1			1			1
			1		1		1	

- a is in the clique since $a =$ column 1
- b is in the clique since $b \geq$ column 4
- c is not in the clique although it conflicts with all columns

Implementation in Pricing Problem

Observations on paths

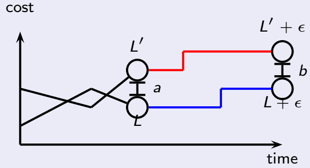
- Negative dual σ means a penalty on a path if it is in the clique
- Objective function is a step function with one step of size $-\sigma$
- The cliques do not constrain solution space

Adding Clique Resources

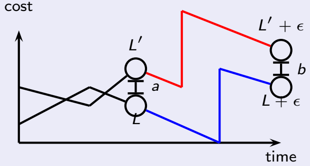
- One resource per clique
- Contains information on the visited customers by a label with regard to the columns in the key set of the clique
- If the label sub-vector is in the clique (Proposition 3) the dual cost is subtracted from the label cost

Exploiting The Step Function when Dominating

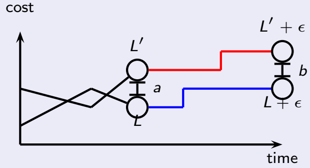
(1) $\mathcal{Q}(L) \leq \mathcal{Q}(L'), a \geq -\sigma$



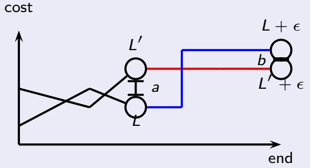
(2) $\mathcal{Q}(L) \leq \mathcal{Q}(L'), a < -\sigma$



(3) $\mathcal{Q}(L) > \mathcal{Q}(L'), a \geq -\sigma$



(4) $\mathcal{Q}(L) > \mathcal{Q}(L'), a < -\sigma$



Dominance Criteria

Theorem

In cases (1) and (2) label L dominates label L' with regard to cost if:

$$
\begin{aligned}
\textit{Cost:}\quad & \hat{c}(L) \leq \hat{c}(L') & \\
\textit{Resources:}\quad & r(L) \leq r(L') & \forall r \in R \\
\textit{Clique Resource:}\quad & \mathcal{Q}(L) \leq \mathcal{Q}(L') &
\end{aligned}
$$

Dominance Criteria

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$$\begin{aligned} \text{Cost: } & \hat{c}(L) \leq \hat{c}(L') \\ \text{Resources: } & r(L) \leq r(L') \qquad \forall r \in R \\ \text{Clique Resource: } & \mathcal{Q}(L) \leq \mathcal{Q}(L') \end{aligned}$$

Theorem

In case (3) label L dominates label L' if

$$\begin{aligned} \text{Cost: } & \hat{c}(L) - \sigma \leq \hat{c}(L') \\ \text{Resources: } & r(L) \leq r(L') \qquad \forall r \in R \\ \text{Clique Resource: } & \mathcal{Q}(L) > \mathcal{Q}(L') \end{aligned}$$

Experimental Results

Instances

- 39 instances from [Solomon 1987] with 100 customers
- Distribution is random (R) and random/clustered (RC)
- Time windows are narrow (1) and wide (2)

		solved			gap closed		bb reduced	
Instance	No.	none	SR	C	SR	C	SR	C
R1	12	50.0	100.0	100.0	87.3	89.9	91.4	92.4
RC1	8	37.5	100.0	100.0	97.1	97.5	99.3	99.1
R2	11	9.0	72.7	72.7	94.3	94.8	95.6	95.6
RC2	8	37.5	75.0	75.0	99.7	99.7	95.6	95.6
	39	33.3	87.2	87.2	90.9	91.9	94.5	94.9
			0.0		10.9		7.2	

Running times increased by approx. 25 % in current implementation.

Final Remarks

- Description of data structures to represent cliques in a BCP algorithm
- Presented a modified dominance criteria used in the pricing problem
- Improvement of lower bounds reduces number of nodes in search tree
  - minimum violation was 0.1, i.e., bounds may be improved
- Slower dominance criteria.
  - 32 bit implementation limits key set size
  - Prototype implementation
- The pricing problem remains the current bottleneck. With or without clique inequalities.