Classification of Dantzig-Wolfe Reformulations for MIP's

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Outline and Motivation

- Dantzig-Wolfe reformulation for lotsizing
  - CLSP (big bucket) : we cannot put binary conditions on the new columns.
  - CSLP (small bucket): we can put binary conditions on the new columns
- Question: In which cases is it (not) allowed to put binary conditions on the new master variables
- Classification for Binary Mixed Integer Programs

Lot Sizing Problems

- Lot Sizing = determine the timing and level of production
- Production planning in a deterministic environment with a finite time horizon
- A classic OR problem (Wagner and Whitin 1958)
- Core substructure in many industrial planning problems
- A small example:

		Period					
		1	2	3	4	5	6
Item 1	**Demand**	**10**	**0**	**35**	**15**	**15**	**10**
	Production	**20**	**20**	**20**	**-**	**15**	**10**
	Inventory	**10**	**30**	**15**	**0**	**0**	**0**
Item 2	**…**	**…**					
Item 3	**…**	**…**					

The Uncapacitated Lot Sizing Problem

$$Min \quad \sum_{t=1}^{m} \left(vc_t x_t + sc_t y_t + hc_t s_t \right)$$

$$s.t. \qquad s_{t-1} + x_t = d_t + s_t \qquad \forall t \in T$$

$$x_t \leq sd_{tm} y_t \qquad \forall t \in T$$

$$x_t, s_t \geq 0; y_t \in \{0,1\} \qquad \forall t \in T$$

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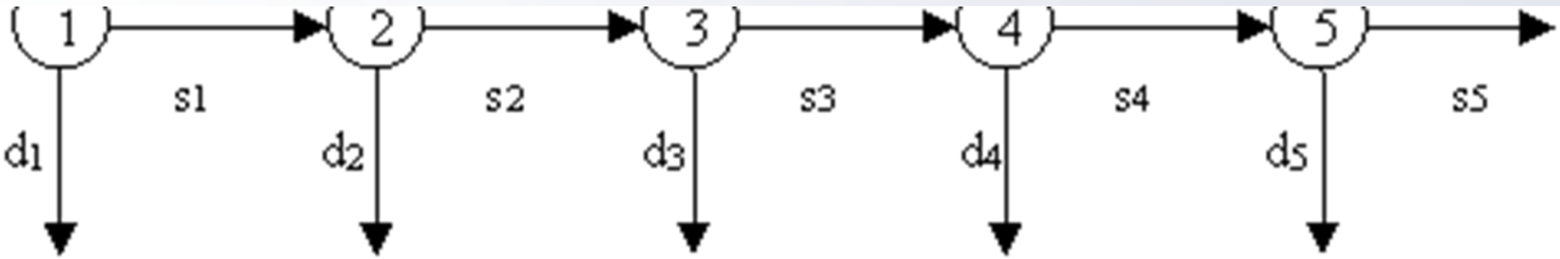
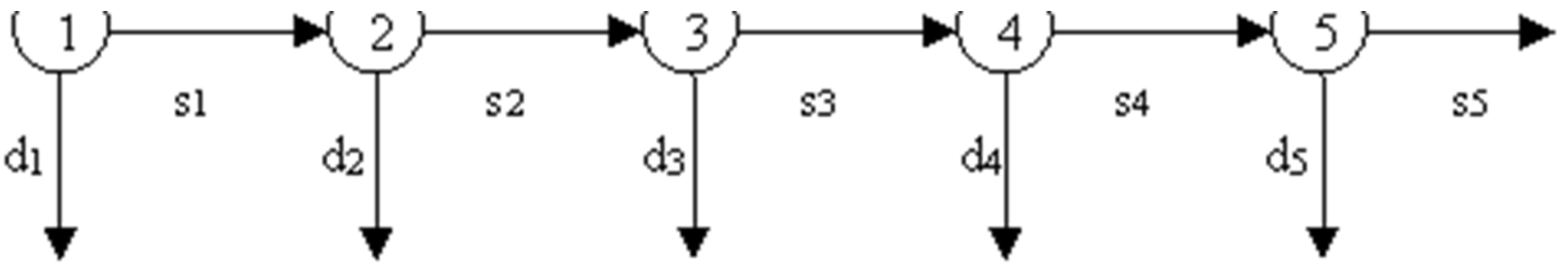
The Uncapacitated Lot Sizing Problem: A fixed charge network problem





In an uncapacitated network with one source, the extreme flow can have at most one positive input in each node. (Zangwill 1969)

This is exactly the Wagner-Whitin property (1958):

$$s_{t-1} x_t = 0$$

Capacitated Lotsizing with Setup Times

- Minimize total cost

$$Min \sum_{i \in P} \sum_{t \in T} \left(sc_i y_{it} + vc_i x_{it} + hc_i s_{it} \right)$$

- Satisfy demand

$$s_{i,t-1} + x_{it} = d_{it} + s_{it} \qquad \forall i \in P, \forall t \in T$$

- Set up forcing

$$x_{it} \leq \min\{cap_t - st_i, sd_{it}\} y_{it} \qquad \forall i \in P, \forall t \in T$$

- Limited capacity

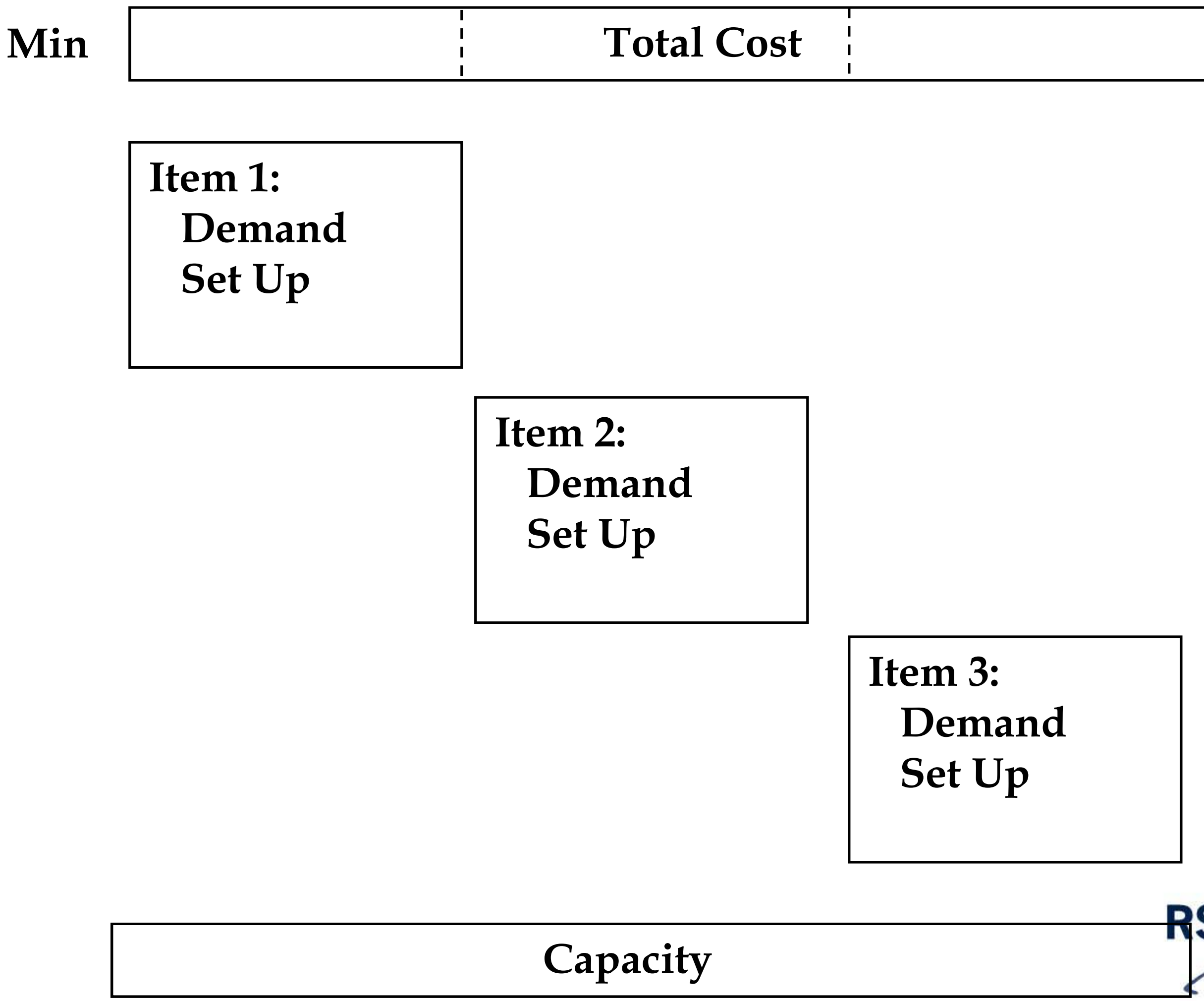
$$\sum_{i \in P} \left(st_i y_{it} + vt_i x_{it} \right) \leq cap_t \qquad \forall t \in T$$

- Integrality $\quad s_{i,0} = 0; s_{i,m} = 0;\ x_{it}, s_{it} \geq 0; y_{it} \in \{0,1\}$

Capacitated Lot Sizing: Literature Review

	DP	Cutting Planes	Variable Re definition	Lagrange Relaxation	DW Decomposition
1958	Wagner & Whitin				Manne
1965					Dzielinski & Gomory
1971	Florian & Klein				
1975				Kleindorfer & Newson	
1977			Krarup & Bilde		
1984		Barany et al			
1985				Thizy & Van Wassenhove	
1987			Eppen & Marti		
1989		Leung et al		Trigeiro et al	
1990					Cattrysse et al.
1991	Wagelmans et al.	Pochet & Wolsey			
2000		Belvaux & Wolsey			
2004			Jans and Degraeve		

Dantzig-Wolfe Decomposition



CLST : Decomposition and CG approach

Master

- Minimize total cost

$$Min \sum_{i \in P} \sum_{q \in Q_i} c_{iq} \, z_{iq}$$

- Convexity constraint

$$\sum_{q \in Q_i} z_{iq} = 1 \qquad \mu_i$$

- Capacity constraint

$$\sum_{i \in P} \sum_{q \in Q_i} r_{iqt} \, z_{iq} \le cap_t \qquad \pi_t$$

- Non-negativity

$$z_{iq} \ge 0$$

Structural Problem with textbook decomposition

Subproblem

- Minimize reduced cost

$$Min \sum_{t \in T} \left(sc_i y_{it} + vc_i x_{it} + hc_i s_{it} \right) - \mu_i + \sum_{t \in T} (st_i \, y_{it} + vt_i \, x_{it}) \pi_t$$

- Satisfy demand
- Set up forcing
- Integrality

- **Wagner Whitin (1958):** $s_{t-1} x_t = 0$
- **A solution (= column) consists of both a set up and production quantity decision**

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CLST : Key Observation

Every feasible production schedule for a specific setup schedule can be written as a convex combination of the Wagner-Whitin production schedules, associated with the subset of the setup.

	Set up Schedule	Set up Subset				Production Subset			
		S1	S2	S3	S4	P1	P2	P3	P4
Period 0	1	1	1	1	1	$d_1+d_2+d_3$	0	d_1+d_2	0
Period 1	1	0	1	0	1	0	$d_1+d_2+d_3$	0	d_1+d_2
Period 2	0	0	0	0	0	0	0	0	0
Period 3	1	0	0	1	1	0	0	d_3	d_3

$$\begin{bmatrix} si \\ x1 \\ 0 \\ x3 \end{bmatrix} = \lambda1 \begin{bmatrix} d1+d2+d3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda2 \begin{bmatrix} 0 \\ d1+d2+d3 \\ 0 \\ 0 \end{bmatrix} + \lambda3 \begin{bmatrix} d1+d2 \\ 0 \\ 0 \\ d3 \end{bmatrix} + \lambda4 \begin{bmatrix} 0 \\ d1+d2 \\ 0 \\ d3 \end{bmatrix}$$

$$\lambda1+\lambda2+\lambda3+\lambda4=1$$

$$\lambda1, \lambda2, \lambda3, \lambda4 \geq 0$$

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CLST : New Extreme Point Formulation

- Minimize total cost

$$Min \sum_{i \in P} \sum_{j \in S} cs_{ij} zs_{ij} + \sum_{i \in P} \sum_{j \in S} \sum_{k \in S_j} cp_{ik} zp_{ijk}$$

- Select one setup proposal

$$\sum_{j \in S} zs_{ij} = 1 \qquad \forall i \in P$$

- Relationship setup and production

$$zs_{ij} = \sum_{k \in S_j} zp_{ijk} \qquad \forall i \in P, \forall j \in S$$

- Capacity constraint

$$\sum_{i \in P} \sum_{j \in S} st_i ss_{ijt} zs_{ij} + \sum_{i \in P} \sum_{j \in S} \sum_{k \in S_j} vt_i ps_{ikt} zp_{ijk} \le cap_t \qquad \forall t \in T$$

- Integrality

$$zs_{ij} \in \{0,1\} \qquad \forall i \in P, \forall j \in S$$

Another lotsizing problem

- Continuous Setup Lotsizing Problem
  - Small bucket problem
  - Single mode constraint
- Decomposition: single mode constraint is the complicating constraint (Vanderbeck 1998)
- We can put binary conditions on the new column variables to obtain an equivalent reformulation

Central question

In which cases can we (not) put integrality constraints on the new master variables?

Generalizability: DW reformulations for MIP's

$$\begin{aligned} & Min\ gx + hy \\ \textbf{s.t.}\quad & Ax + By \geq c \\ & Dx + Ey \geq f \\ & x \geq 0 \\ & y \in \{0,1\} \end{aligned}$$

Some Notation

X^{BMIP}	the set of feasible solutions to the overall BMIP problem;
$conv(X^{BMIP})$	the convex hull of X^{BMIP};
Q^{BMIP}	the set of extreme points of $conv(X^{BMIP})$;
X^S	the feasible MIP region for the special structure constraints: $X^S = \{(x, y) \in R^m \times N^n : Dx + Ey \geq f, x \geq 0, y \in \{0,1\}\}$;
$conv(X^S)$	the convex hull of X^S;
Q^S	the finite set of extreme points of $conv(X^S)$;
$q \in Q^S$	an extreme point of Q^S.

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Convexification approach for MIP's

- In the *convexification* approach we keep the binary restrictions on the original variables (Barnhart et al. 1998, Vanderbeck 2000).
- The feasible MIP region can then be rewritten as follows:

$$X^S = \left\{ (x,y) \in R^m \times N^n : x = \sum_{q \in Q^S} x^q \lambda_q, y = \sum_{q \in Q^S} y^q \lambda_q, \sum_{q \in Q^S} \lambda_q = 1, \lambda_q \geq 0, y \in \{0,1\} \right\}$$

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Convexification approach

(DW-BMIP):

$$Min \sum_{q \in Q^S} Gx^q \lambda_q + \sum_{q \in Q^S} Hy^q \lambda_q$$

$$s.t. \quad \sum_{q \in Q^S} Ax^q \lambda_q + \sum_{q \in Q^S} By^q \lambda_q \geq c$$

$$\sum_{q \in Q^S} \lambda_q = 1$$

$$x = \sum_{q \in Q^S} x^q \lambda_q$$

$$y = \sum_{q \in Q^S} y^q \lambda_q$$

$$\lambda_q \geq 0$$

$$y \in \{0,1\}$$

Generalizability: DW reformulations for MIP's

- **The set of feasible solutions to BMIP is a subset of the set of feasible solutions to the special structure constraints:** $X^{BMIP} \subseteq X^{S}$.
- **We cannot impose such a relationship in general between the extreme points of** $conv(X^{BMIP})$ **and the extreme points of** $conv(X^{S})$.

Overview of different BMIP models

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	<u>Case 1:</u> $Ax \geq c$ $Dx \geq f$	<u>Case 2:</u> $Ax \geq c$ $Ey \geq f$	<u>Case 3:</u> $Ax \geq c$ $Dx + Ey \geq f$
	$By \geq c$	<u>Case 4:</u> $By \geq c$ $Dx \geq f$	<u>Case 5:</u> $By \geq c$ $Ey \geq f$	<u>Case 6:</u> $By \geq c$ $Dx + Ey \geq f$
	$Ax + By \geq c$	<u>Case 7:</u> $Ax + By \geq c$ $Dx \geq f$	<u>Case 8:</u> $Ax + By \geq c$ $Ey \geq f$	<u>Case 9:</u> $Ax + By \geq c$ $Dx + Ey \geq f$

Case 1: Generalized Linear Programming

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	Case 1: $Ax \geq c$ $Dx \geq f$	Case 2: $Ax \geq c$ $Ey \geq f$	Case 3: $Ax \geq c$ $Dx + Ey \geq f$
	$By \geq c$	Case 4: $By \geq c$ $Dx \geq f$	Case 5: $By \geq c$ $Ey \geq f$	Case 6: $By \geq c$ $Dx + Ey \geq f$
	$Ax + By \geq c$	Case 7: $Ax + By \geq c$ $Dx \geq f$	Case 8: $Ax + By \geq c$ $Ey \geq f$	Case 9: $Ax + By \geq c$ $Dx + Ey \geq f$

Cases 2 and 4: independent subproblems

		Subproblem structure		
		$Dx \ge f$	$Ey \ge f$	$Dx + Ey \ge f$
Linking constraints	$Ax \ge c$	<u>Case 1:</u> $Ax \ge c$ $Dx \ge f$	<u>Case 2:</u> $Ax \ge c$ $Ey \ge f$	<u>Case 3:</u> $Ax \ge c$ $Dx + Ey \ge f$
	$By \ge c$	<u>Case 4:</u> $By \ge c$ $Dx \ge f$	<u>Case 5:</u> $By \ge c$ $Ey \ge f$	<u>Case 6:</u> $By \ge c$ $Dx + Ey \ge f$
	$Ax + By \ge c$	<u>Case 7:</u> $Ax + By \ge c$ $Dx \ge f$	<u>Case 8:</u> $Ax + By \ge c$ $Ey \ge f$	<u>Case 9:</u> $Ax + By \ge c$ $Dx + Ey \ge f$

Case 7: No binary variables in the subproblem

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	Case 1: $Ax \geq c$ $Dx \geq f$	Case 2: $Ax \geq c$ $Ey \geq f$	Case 3: $Ax \geq c$ $Dx + Ey \geq f$
	$By \geq c$	Case 4: $By \geq c$ $Dx \geq f$	Case 5: $By \geq c$ $Ey \geq f$	Case 6: $By \geq c$ $Dx + Ey \geq f$
	$Ax + By \geq c$	Case 7: $Ax + By \geq c$ $Dx \geq f$	Case 8: $Ax + By \geq c$ $Ey \geq f$	Case 9: $Ax + By \geq c$ $Dx + Ey \geq f$

Case 5: The pure binary problem

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	Case 1: $Ax \geq c$ $Dx \geq f$	Case 2: $Ax \geq c$ $Ey \geq f$	Case 3: $Ax \geq c$ $Dx + Ey \geq f$
	$By \geq c$	Case 4: $By \geq c$ $Dx \geq f$	Case 5: $By \geq c$ $Ey \geq f$	Case 6: $By \geq c$ $Dx + Ey \geq f$
	$Ax + By \geq c$	Case 7: $Ax + By \geq c$ $Dx \geq f$	Case 8: $Ax + By \geq c$ $Ey \geq f$	Case 9: $Ax + By \geq c$ $Dx + Ey \geq f$

Case 5: The pure binary problem

- The binary constraint on the original variables can be replaced by binary conditions on the new master variables:

$$\lambda_q \in \{0,1\} \iff y \in \{0,1\}$$

- There … ence:

$$X^{BMIP} = Q^{BMIP} \subseteq X^{S} = Q^{S}$$

Case 8: Johnson (1989)

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	Case 1: $Ax \geq c$ $Dx \geq f$	Case 2: $Ax \geq c$ $Ey \geq f$	Case 3: $Ax \geq c$ $Dx + Ey \geq f$
	$By \geq c$	Case 4: $By \geq c$ $Dx \geq f$	Case 5: $By \geq c$ $Ey \geq f$	Case 6: $By \geq c$ $Dx + Ey \geq f$
	$Ax + By \geq c$	Case 7: $Ax + By \geq c$ $Dx \geq f$	Case 8: $Ax + By \geq c$ $Ey \geq f$	Case 9: $Ax + By \geq c$ $Dx + Ey \geq f$

The interesting cases: BMIP subproblem

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	Case 1: $Ax \geq c$, $Dx \geq f$	Case 2: $Ax \geq c$, $Ey \geq f$	Case 3: $Ax \geq c$, $Dx + Ey \geq f$
	$By \geq c$	Case 4: $By \geq c$, $Dx \geq f$	Case 5: $By \geq c$, $Ey \geq f$	Case 6: $By \geq c$, $Dx + Ey \geq f$
	$Ax + By \geq c$	Case 7: $Ax + By \geq c$, $Dx \geq f$	Case 8: $Ax + By \geq c$, $Ey \geq f$	Case 9: $Ax + By \geq c$, $Dx + Ey \geq f$

Case 3 and 9: The capacitated Lotsizing Problem (with setup times)

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	<u>Case 1:</u> $Ax \geq c$ $Dx \geq f$	<u>Case 2:</u> $Ax \geq c$ $Ey \geq f$	<u>Case 3:</u> $Ax \geq c$ $Dx + Ey \geq f$
	$By \geq c$	<u>Case 4:</u> $By \geq c$ $Dx \geq f$	<u>Case 5:</u> $By \geq c$ $Ey \geq f$	<u>Case 6:</u> $By \geq c$ $Dx + Ey \geq f$
	$Ax + By \geq c$	<u>Case 7:</u> $Ax + By \geq c$ $Dx \geq f$	<u>Case 8:</u> $Ax + By \geq c$ $Ey \geq f$	<u>Case 9:</u> $Ax + By \geq c$ $Dx + Ey \geq f$

The capacitated Plant Location Problem

The capacitated plant location problem

$$Min \sum_{i \in N} \sum_{j \in M} c_{ij} x_{ij} + \sum_{j \in M} f_j y_j$$

s.t.

$$\sum_{j \in M} x_{ij} = 1 \qquad \forall i \in N$$

$$\sum_{i \in N} d_i x_{ij} \le s_j y_j \qquad \forall j \in M$$

$$x_{ij} \le y_j \qquad \forall i \in N, \forall j \in M$$

$$y_i \in \{0,1\}, x_{ij} \ge 0 \qquad \forall i \in N, \forall j \in M$$

The capacitated Plant Location Problem

- One possible decomposition is to leave the capacity constraints in the master.
- The resulting subproblem is the Simple Plant Location Problem, which is NP-hard (Krarup and Pruzan 1983).
- An optimal solution to this simple Plant Location Problem has the ‘Single Assignment Property’.
- In the optimal solution for the capacitated case, however, it is highly likely that demand for some customers will be supplied from more than one open plant.
- By putting integrality constraints on the columns generated by the subproblem, we can never attain such a split supply.
- In this case, we will need to apply a similar approach as with the CLST, namely a separation of the binary location decision and the fractional supply decision. (Klose and Götz, 2007).

Case 6: CSLP
Vehicle Routing with Time Windows

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	Case 1: $Ax \geq c$ $Dx \geq f$	Case 2: $Ax \geq c$ $Ey \geq f$	Case 3: $Ax \geq c$ $Dx + Ey \geq f$
	$By \geq c$	Case 4: $By \geq c$ $Dx \geq f$	Case 5: $By \geq c$ $Ey \geq f$	Case 6: $By \geq c$ $Dx + Ey \geq f$
	$Ax + By \geq c$	Case 7: $Ax + By \geq c$ $Dx \geq f$	Case 8: $Ax + By \geq c$ $Ey \geq f$	Case 9: $Ax + By \geq c$ $Dx + Ey \geq f$

Case 6:

Proposition 1:

For case 6, if (x^p, y^p) is an extreme point of $conv(X^{BMIP})$, then it is an extreme point of $conv(X^S)$.

- **Hence, we can impose the binary conditions directly on the new master variables.**

Overview of different BMIP models

		Subproblem structure		
		$Dx \geq f$	$Ey \geq f$	$Dx + Ey \geq f$
Linking constraints	$Ax \geq c$	Case 1: $Ax \geq c$, $Dx \geq f$	Case 2: $Ax \geq c$, $Ey \geq f$	Case 3: $Ax \geq c$, $Dx + Ey \geq f$
	$By \geq c$	Case 4: $By \geq c$, $Dx \geq f$	Case 5: $By \geq c$, $Ey \geq f$	Case 6: $By \geq c$, $Dx + Ey \geq f$
	$Ax + By \geq c$	Case 7: $Ax + By \geq c$, $Dx \geq f$	Case 8: $Ax + By \geq c$, $Ey \geq f$	Case 9: $Ax + By \geq c$, $Dx + Ey \geq f$

Remarks

1. If general integer variables are present, the optimum solution to the original MIP may be an interior point of conv(X^s).

   - Imposing integrality constraints on the new master variables will hence not give an equivalent formulation.

2. Model formulation versus algorithmic solution approach:

   - Even if you can impose integrality on the new master variables, this is probably not an efficient branching strategy because of the unbalanced tree and the possible difficulties in the pricing problem.

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Conclusions

- Focus on correct reformulations, not on algorithms.
- Systematic analysis of the effect of DW-decomposition on the binary variables.
- General overview of when it is allowed to put integrality constraints on the new master variables and when not.