Weighted Acyclic Di-Graph Partitioning by Balanced Disjoint Paths

H. Murat AFSAR – Olivier BRIANT

Murat.Afsar@g-scop.inpg.fr
Olivier.Briant@g-scop.inpg.fr

G-SCOP Laboratory
Grenoble Institute of Technology

Presentation Plan

Partitioning Problems - Disjoint Paths

- Covering all the nodes
- Each node appears on just one path

Polynomially solvable - Flow

Problem Context

- Telecommunication networks
  Noninterfering messages
- Transportation
  Aircraft Rotation Problems

Problem Definition

- A directed acyclic graph $G(V, A)$,
- Find K disjoint and weight balanced paths $p_1, p_2, ..., p_K$ such that
  Each node V is covered by exactly one path.

Difficulty

- Objective : $\min \sum_{k=1}^{K} c_{p_k}$

  Where

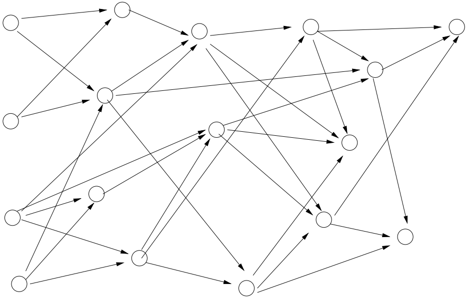
  - $c_{p_k} = g(\sum_{v \in p_k} w_v) = \frac{1}{M}(M - \sum_{v \in p_k} w_v)^2$
  - And M is the ideal path weight

$$M = \frac{1}{K} \sum_{v \in V} w_v$$

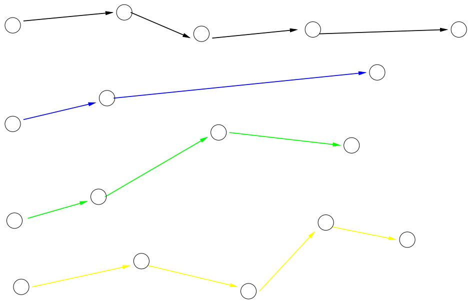
Similar Problems

- Finding 2 node disjoint paths with bounded weight on a *DAG* NP-complete
- Vertex covering by minimum weight paths, on trees NP-Hard (even in unweighted case)
- Partitioning problem NP-hard

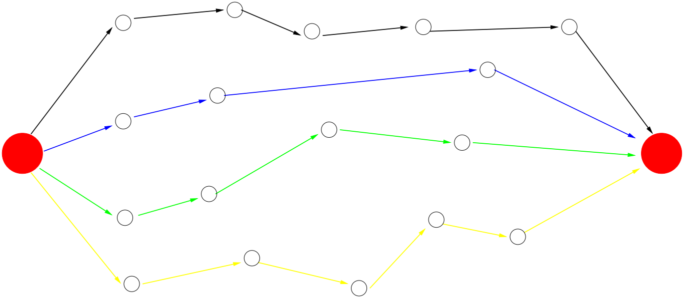
Partitoning Graph by Disjoint Paths



Partitoning Graph by Disjoint Paths



Partitoning Graph by Disjoint Paths



Problem Formulation

$$\min \sum_{v \in V \setminus \{s,t\}} \frac{(My_{(s,v)} - x_{(s,v)})^2}{M}$$

subject to

$$\sum_{(v,v') \in \delta^-(v')} y_{v,v'} - \sum_{(v',v) \in \delta^+(v)} y_{v',v} = 0 \qquad \forall v' \in V \setminus \{s,t\}$$

$$\sum_{(v,v') \in \delta^-(v')} x_{v,v'} - \sum_{(v',v) \in \delta^+(v')} x_{v',v} = w_{v'} \qquad \forall v' \in V \setminus \{s,t\}$$

$$\sum_{v' \in \delta^+(v)} y_{v,v'} = 1 \qquad \forall v \in V \setminus \{s,t\}$$

$$\sum_{(s,v') \in \delta^+(s)} y_{s,v'} = K$$

$$x_{v,v'} \leq W_v y_{v,v'} \qquad \forall (v,v') \in A$$

$$x_{v,v'} \in \mathbb{R} \ \forall (v,v') \in A y_{v,v'} \in \{0,1\} \ \forall (v,v') \in A$$

Decomposition Dantzig-Wolfe

- Master Problem : Combining paths to partition the graph
- Slave Problem : Finding *good* paths

Master Problem

$$\min \sum_{p \in P} c_p \lambda_p$$

subject to

$$\sum_{p \in P} \lambda_p = K$$

$$\sum_{p \in P | v \in p} \lambda_p = 1 \quad \forall v \in V$$

$$\lambda_p \in \{0, 1\} \quad \forall p \in P$$

Reduced Cost of a Path

$$c_p = g(\sum_{v|v\in p} w_v)$$

$$\bar{c}_p(u) = g(\sum_{v|v\in p} w_v) - \sum_{v|v\in p} u_v - u_0$$

Where u_v and u_0 are the dual variables associated to the node v and to the first constraint, respectively

Slave Problem Solution Methods

- Max Flow - Max Cost to cover all the nodes
- 2OPT on flow solution to minimize the minimum reduced cost :
  Finding a feasible solution with promissing paths.
- Optimal dynamic programming :
  A multi label setting dynamic model to find max $\sum_{v \in s \rightarrow v'} u_v$ for each node v' for each total weight $\sum_{v \in s \rightarrow v'} w_v = W_{s \rightarrow v'}$

Max Flow - Max Cost

- Flow 1 on graph $G(V, A)$: $cost(a) = \sum_{p|a \in p} \lambda_p \quad \forall a \in A$
- Flow 2 on graph $G'(V, A')$: With only a limited number of exiting arcs from all node v such that

$$\sum_{p|a \in p} \lambda_p \geq 0.5$$

2 OPT

- For the paths p, p' in the solution S, for every couple of nodes $v \in p$ and $v' \in p'$ compute

$$\begin{aligned}\bar{c}_p^{new} &= g(W_{s\to v} + W_{v'\to t}) - (U_{s\to v} + U_{v'\to t}) \\ \bar{c}_{p'}^{new} &= g(W_{s\to v'} + W_{v\to t}) - (U_{s\to v'} + U_{v\to t})\end{aligned}$$

  Where $U_{s\to v}$ is the sum of dual values of the nodes on the path between s and v

- If $\min\{\bar{c}_p^{new}, \bar{c}_{p'}^{new}\} < \min\{\bar{c}_p, \bar{c}_{p'}\}$ then accept the swap

Dynamic programming

$$f(v, W) = u_v + \max_{v' \in \delta^-(v)} f(v', W - w_v)$$

And naturally

$$f(s, 0) = 0$$

Where W is the total weight of the path between s and v

Dynamic programming

$$f(v, W) = u_v + \max_{v' \in \delta^-(v)} f(v', W - w_v)$$

And naturally

$$f(s, 0) = 0$$

Where W is the total weight of the path between s and v

$$\min_{\substack{p \in P(v) \\ \sum_{v \in p} w_v = W}} \bar{c}_p(u) = g(W) - f(v, W)$$

Where $P(v)$ is the set of paths ending at the node v

Lagrangian Lower Bound

$$\tilde{c}_p(u) = g(\sum_{v|v\in p} w_v) - \sum_{v|v\in p} u_v$$

Lagrangian Lower Bound

$$\tilde{c}_p(u) = g(\sum_{v|v\in p} w_v) - \sum_{v|v\in p} u_v$$

$$\theta_1^*(u) = K \times \min_{p\in P} \tilde{c}_p(u) + \sum_{v\in V} u_v$$

$$\theta_2^*(u) = \sum_{i\in\mathcal{K}} \min_{p\in P(v_i)} \tilde{c}_p(u) + \sum_{v\in V} u_v$$

Where $P(v_i)$ is the set of paths ending with the node v_i
So each path ends at a different node

Obviously $\theta_2^*(u) \geq \theta_1^*(u)$

Instance Characteristics

Table: General DAG with 700 nodes and $K = 80$

A_0.05	5%	$w_{min} = 40$	$w_{max} = 120$
A_0.10	10%	$w_{min} = 40$	$w_{max} = 120$
A_0.15	15%	$w_{min} = 40$	$w_{max} = 120$
A_0.20	20%	$w_{min} = 40$	$w_{max} = 120$
A_0.25	25%	$w_{min} = 40$	$w_{max} = 120$
B_0.05	5%	$w_{min} = 1$	$w_{max} = 1$
B_0.10	10%	$w_{min} = 1$	$w_{max} = 1$
B_0.15	15%	$w_{min} = 1$	$w_{max} = 1$
B_0.20	20%	$w_{min} = 1$	$w_{max} = 1$
B_0.25	25%	$w_{min} = 1$	$w_{max} = 1$

Instance Characteristics

Table: Flight-like nodes with 700 nodes, $w_{min} = 40, w_{max} = 120$, $K = 80$

F_40_80	$wait_{min} = 40$	$wait_{max} = 80$	nb arcs : 14 405
F_40_100	$wait_{min} = 40$	$wait_{max} = 100$	nb arcs : 20 468
F_40_120	$wait_{min} = 40$	$wait_{max} = 120$	nb arcs : 26 887
F_40_150	$wait_{min} = 40$	$wait_{max} = 150$	nb arcs : 36 377
F_40_240	$wait_{min} = 40$	$wait_{max} = 240$	nb arcs : 60 791

RMP Linear Relaxation vs Lagrangean Relaxation

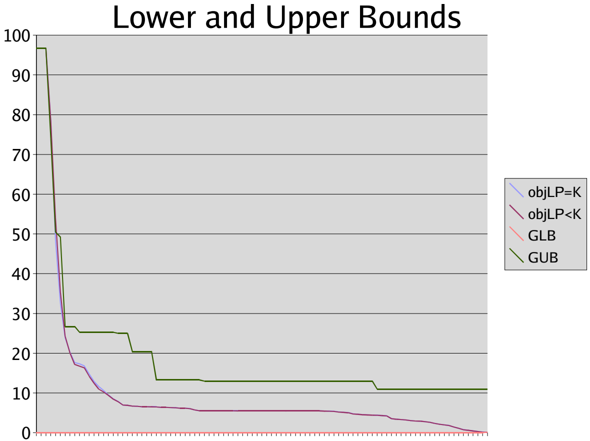
Restricted Master Linear Relaxation 1 :

$$\sum_{p \in P} \lambda_p = K$$

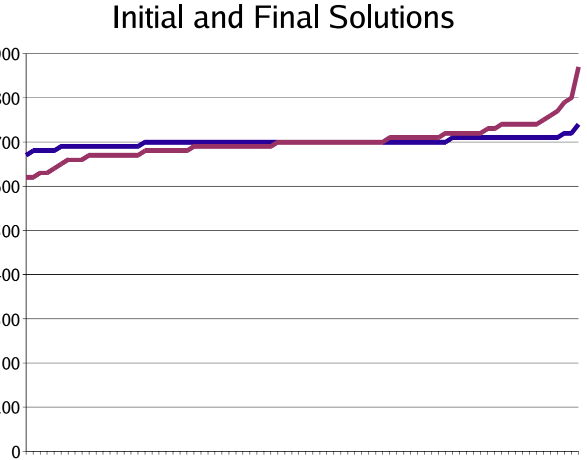
Restricted Master Linear Relaxation 2 :

$$\sum_{p \in P} \lambda_p \leq K$$

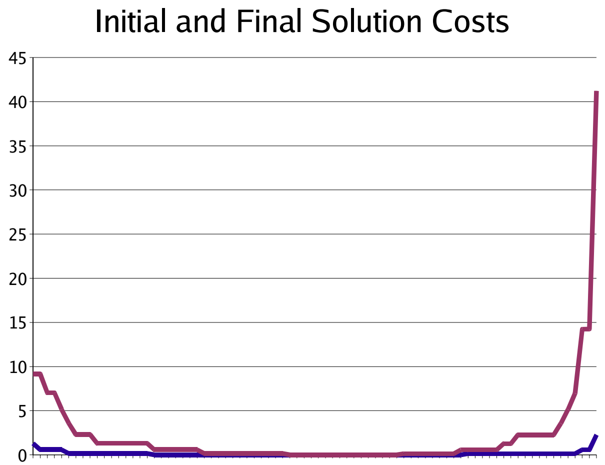
RMP Linear Relaxation vs Lagrangean Relaxation



RMP Linear Relaxation vs Lagrangean Relaxation



RMP Linear Relaxation vs Lagrangean Relaxation



Numerical Results

Instance	iter	$Z^*_{LP\,1}$	$Z^*_{LP\,2}$	GLB	GUB	Columns
A_0.05	95	0,141	0,070	0,141	10,996	9649
A_0.10	18	0,141	0,002	0,141	7,282	4913
A_0.15	11	0,141	0,002	0,141	3,283	3216
A_0.20	7	0,141	0,096	0,141	0,712	1918
A_0.25	6	0,141	0,106	0,141	0,427	1467
B_0.05	34	1,714	0,555	1,714	1,714	4648
B_0.10	6	1,714	0,940	1,714	1,714	1484
B_0.15	6	1,714	0,892	1,714	1,714	1586
B_0.20	6	1,714	0,937	1,714	1,714	1552
B_0.25	5	1,714	1,714	1,714	1,714	1217
F_40_80	6	2,945	2,945	2,945	2,945	1639
F_40_100	6	2,624	1,665	2,624	2,624	1839
F_40_120	5	2,689	2,689	2,689	2,689	1195
F_40_150	5	2,120	2,120	2,120	2,120	1064
F_40_240	5	1,946	1,946	1,946	1,946	1102

Total CPU Time (in sec.) per Method

Instance	Total	Master	Slave	Flow	2OPT
A_0.05	473	265	56	8	133
A_0.10	153	27	19	9	85
A_0.15	176	24	13	4	109
A_0.20	161	13	10	5	96
A_0.25	161	8	9	3	91
B_0.05	63	15	1	2	45
B_0.10	53	5	0	2	46
B_0.15	152	5	0	3	143
B_0.20	184	5	3	4	170
B_0.25	172	2	0	5	162
F_40_80	210	9	0	2	199
F_40_100	253	8	1	1	242
F_40_120	262	4	2	1	255
F_40_150	169	2	2	4	160
F_40_240	290	2	2	3	282

Conclusions

- Gap between lower and upper bounds when the graph is sparse
- Optimality proof of heuristic approach when the graph is dense
- 2-OPT and Master problem are dominating the resolution time

Perspectives

- More tests to come
- Improving the lower bound
- Branch & Price for the sparse graphs to show the optimality

Weighted Acyclic Di-Graph Partitioning by Balanced Disjoint Paths

H. Murat AFSAR – Olivier BRIANT

Murat.Afsar@g-scop.inpg.fr
Olivier.Briant@g-scop.inpg.fr

G-SCOP Laboratory
Grenoble Institute of Technology