An all-integer column generation methodology for set partitioning problems

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What?

- The design of a framework for an unconventional kind of column generation methodology for set partitioning problems.
- Mission: Only integer solutions
- A number of components combined into:
  - A conceptual method with optimising qualities.
  - An outline of a heuristic version.

How?

- Utilise that the set partitioning problem has the *quasi-integrality property*
- Combine:
  - An adaptation of the simplex method.
  - Strategy for finding improved integer solutions.

$$\left.\begin{array}{l}\text{integer solution} = \text{extreme point}\\ \text{associated basis}\end{array}\right\} \Rightarrow \text{simplex multipliers}$$

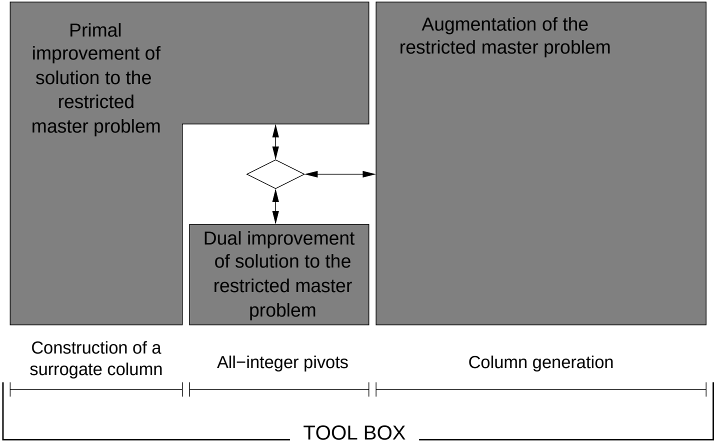
- Column generation tailored for finding improved integer solutions.
- Optimality guarantees: Special case of result in *Larsson and Patriksson, 2006.*

Why?

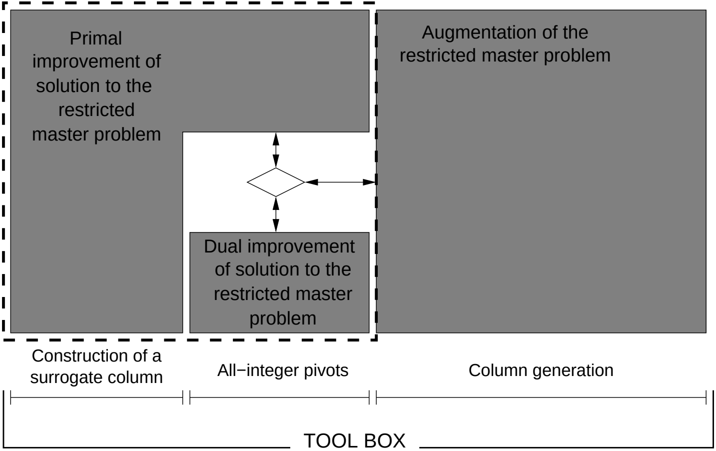
- Quasi-integrality property ⇒ possible to utilise linear programming techniques for an integer program. See *Balas and Padberg, 1972, Trubin, 1969, Thompson, 2002.*

- Not previously exploited in a column generation context.
- Interesting to investigate the possibilities.

Outline — methodology



Outline — methodology



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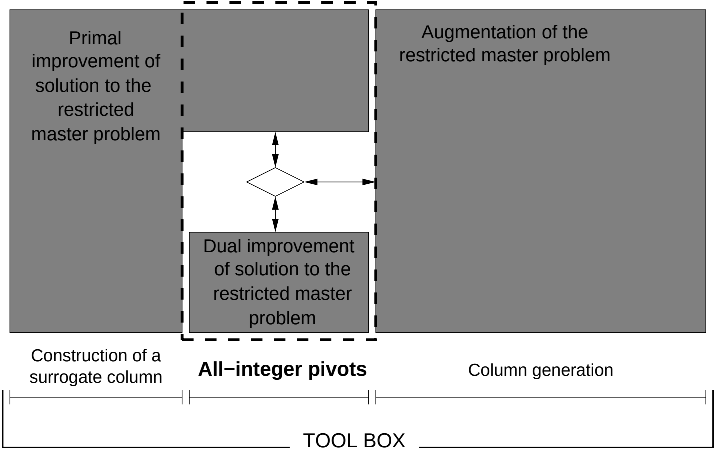

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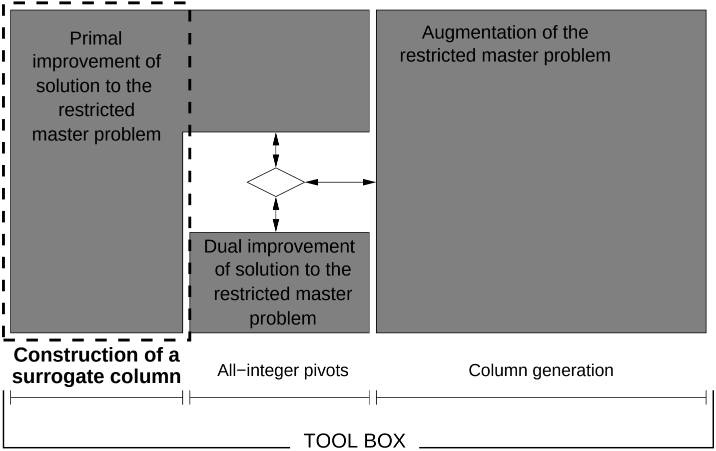

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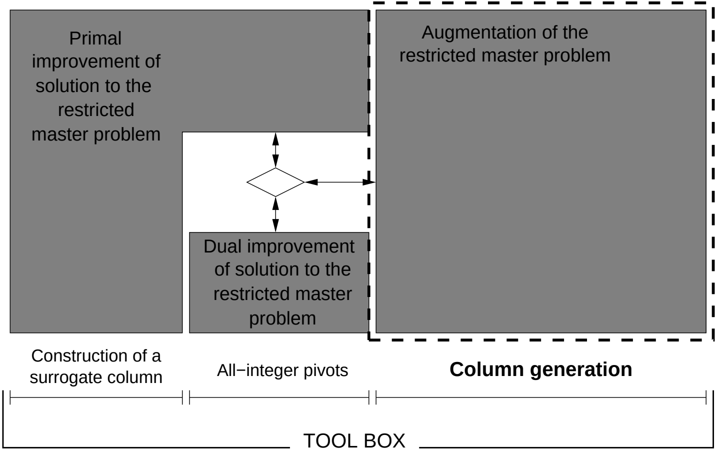


Outline — talk

Notation

- The master problem:

$$[SPP] \qquad z^* = \min \sum_{j \in \mathcal{N}} c_j x_j$$
$$\text{s.t.} \quad \sum_{j \in \mathcal{N}} a_{ij} x_j = e_i, \quad i \in M$$
$$x_j \in \{0,1\}, \quad j \in \mathcal{N}$$

- Set of columns:

$$\mathcal{P} = \{(c_j, a_j) : j \in \mathcal{N}\} \subseteq Z \times \{0,1\}^m$$

- A restricted master problem SPP_N

Notation

- Linear programming relaxation: SPP_N^{LP}
- Basis $\bar{B}$ and dual solution/simplex multipliers $\bar{u}$
- The column generation problem

$$[CG] \qquad \bar{c}_p = \min \; c - \sum_{i \in M} \bar{u}_i a_i \quad \text{s.t.} \quad (c, a) \in \mathcal{P}$$

yields a column (c_p, a_p).

Quasi-integrality — definition

Definition

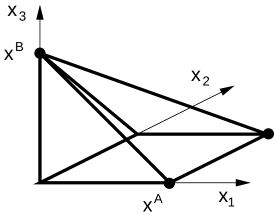
Let X be a polytope and X_I its set of integer points. The polytope X is called quasi-integral if every edge of the convex hull of X_I is also an edge of X.

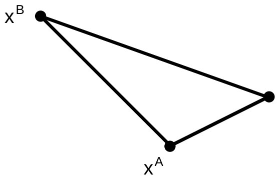
The feasible set of SPP_N^{LP} is a quasi-integral polytope.

Trubin, 1969

Quasi-integrality — an interpretation

'Each pair of integer extreme points belongs to an integral face of the polytope'





The integral simplex method for SPP *Trubin* 1969

- Key observation: A pivot between integer extreme points = a non-degenerate *pivot-on-one* (pivot on a 1-entry)
- The integral simplex method = the simplex method restricted to pivots-on-one only
- Difficulties:
  - Degeneracy
  - Only a few bases enable pivots-on-one
  - Local optima
  - Cycling
- Possible to extend this method

A sufficient condition for integrality

Definition

A basis B is called unimodular if $det(B) = \pm 1$.

- A basis for SPP_N^{LP} is unimodular $\Rightarrow$ the extreme point is integral
- Pivots-on-one preserve unimodularity — utilised in the integral simplex method

All-integer pivots

Observation: Degenerate pivots on *minus* one entries also preserve unimodularity.

Definition

- *Non-degenerate all-integer pivots: On an entry* a_{ij} *such that* $\bar{c}_j < 0$, $\bar{a}_{ij} = 1$, *and* $\bar{e}_i = 1$.
- *Degenerate all-integer pivots: On an entry* a_{ij} *such that* $\bar{c}_j < 0$, $|\bar{a}_{ij}| = 1$, *and* $\bar{e}_i = 0$.

All-integer pivots

- Non-degenerate pivots: *primal* improvement
- Degenerate pivots: might give *dual* progress

- Remaining
    - Degeneracy
    - Only a few bases enable pivots-on-one
    - Local optima
    - Cycling

- Useful when possible!

We need something more than all-integer pivots!

- Use *any* strategy for finding an improved (or optimal) integer solution to SPP_N.
- Example: meta-heuristic

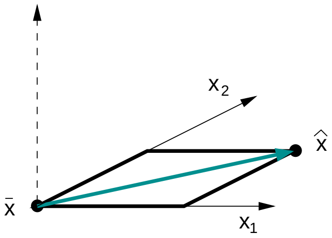
- Drawback: No basis at hand
- Our solution: Construct a *surrogate column*

- Pivot-on-one on the surrogate column $\leftrightarrow$ movement between the current and the improved solution.
- New variable $\leftrightarrow$ increased dimension

What is a surrogate column?

Given:

- A current integer extreme point $\bar{x}$ and a unimodular basis.
- A new integer extreme point $\hat{x}$, but no basis.



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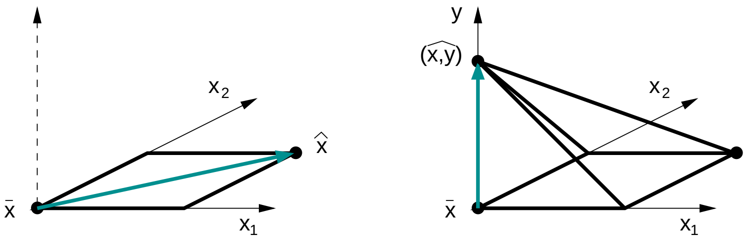


A surrogate column

- Uniquely determined by the current basis and the new, improved extreme point

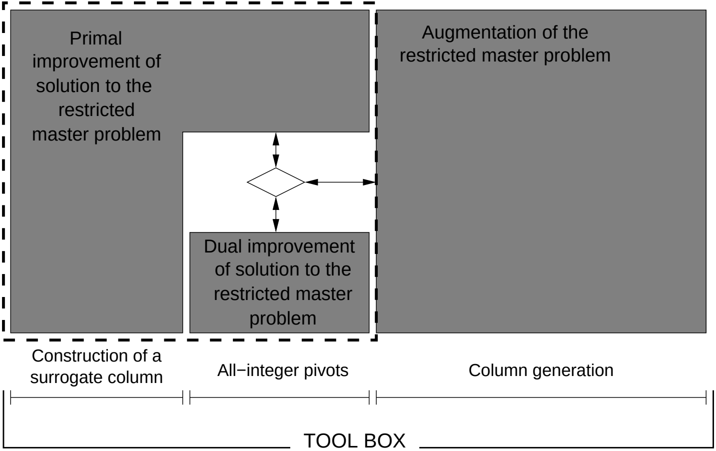
$$(c^y, a^y) = (c_{\bar{N}}^{\mathsf{T}} \hat{x}_{\bar{N}}, \bar{N} \hat{x}_{\bar{N}})$$

- A sum of original columns $\Rightarrow$ of set partitioning type

- Non-degenerate pivot-on-one $\longrightarrow$ unimodular basis

- Drawbacks:
  - Creates redundancy
  - Increases level of degeneracy

Outline — methodology



Linear programming column generation

Three important aspects:

- **Optimality in SPP_N:** $\bar{c}_j \geq 0$, $j \in N$
- **Augmentation:** $CG \Rightarrow$ **new** column p with $\bar{c}_p < 0$
- **Optimality in SPP:** $CG \Rightarrow$ column with $\bar{c}_p = 0$

Our all-integer column generation

The three corresponding aspects:

- **Optimality in SPP_N:** $\bar{c}_j < 0$ possible $\Rightarrow$ need to use integer programming technique
- **Augmentation:** $CG \Rightarrow$ a column p with $\bar{c}_p < 0$ **not** necessarily new
- **Optimality in SPP: ?**

Augmentation of the master problem

- Difficulty: $\bar{c}_j < 0$ in $SPP_N \Rightarrow$ Optimal solution to CG is not necessarily a new column.

- To guarantee the finding of a new column: Apply *over-generation* = generate a number of the best solutions to CG.
- Practically possible in some applications, for example shortest path problems, e.g. *Eppstein 1998*.

Optimality in SPP

To guarantee optimality in SPP — controlled over-generation

Theorem (Sufficient condition for over-generation)
If all columns $j \in \mathcal{N} \setminus N$, such that

$$(i) \quad \bar{c}_j \; < \; -\min \sum_{j \in N} \bar{c}_j x_j \quad s.t. \; \sum_{j \in N} a_{ij} x_j \leq e_i, i \in M; \; x_j \in \{0,1\}, \; j \in N,$$

or

$$(ii) \quad \bar{c}_j \; < \; -\min \sum_{j \in N} \bar{c}_j x_j \quad s.t. \; \sum_{j \in N} a_{ij} x_j \leq e_i, \; i \in M; \; x_j \in [0,1], \; j \in N,$$

or

$$(iii) \quad \bar{c}_j \; < \; -\sum_{j \in N} \min\{0, \bar{c}_j\}$$

are added to SPP_N, then the new restricted master problem will contain an optimal solution to SPP.

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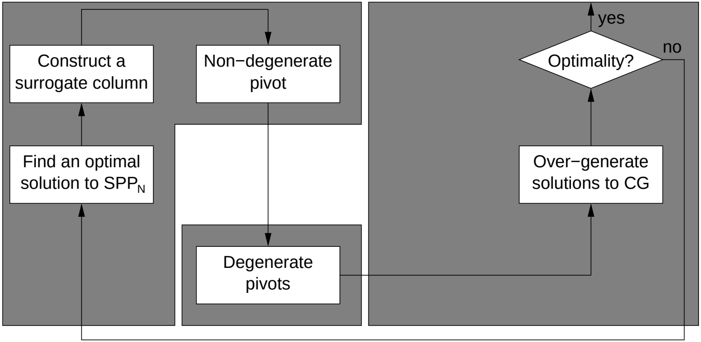
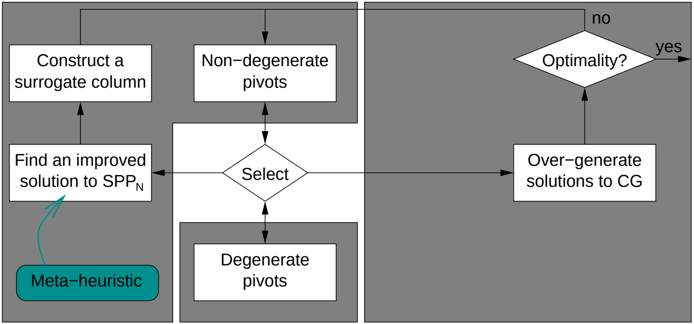
Conceptual exact all-integer column generation



Heuristic all-integer column generation

- Combine the ‘tools’ into a heuristic.
- Optimality conditions still useful.

Heuristic all-integer column generation



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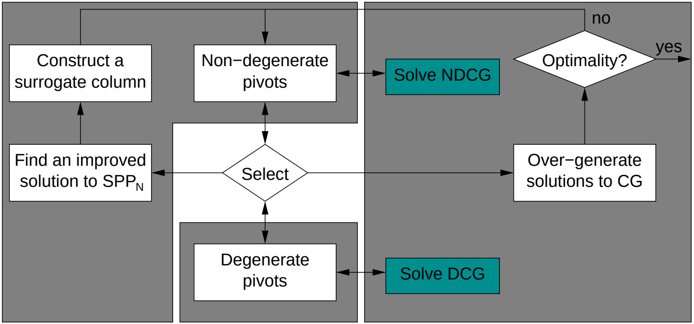
Heuristic all-integer column generation



Tailored column generation

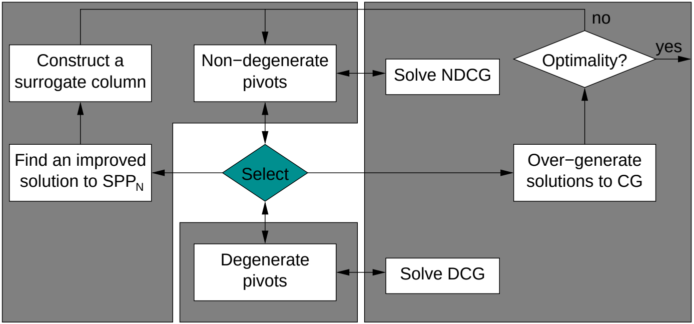
Based on results in *Balas and Padberg 1972*

- $NDCG$: column for non-degenerate all-integer pivot
  $NDCG = CG$ and linear side constraints
- DCG: column for degenerate all-integer pivot
  $DCG = CG$ and linear side constraints

Generates a **new** column if one exists.

Described further in a paper to be published in EJOR.

Heuristic all-integer column generation



Conclusions

- Our tool box:
  - All-integer pivots
  - Surrogate columns
  - Tailored column generation
  - Over-generation based on optimality conditions
- Possible to combine into:
  - A conceptual method with optimising qualities
  - Heuristic versions

Future research

- Develop & implement a heuristic version
- Possible applications: generalised assignment (bin packing), vehicle routing, ...
- Practically useful?

- Add more tools?!

Thanks for listening!

Optimality in $SPP_{\mathcal{N}}$

$$
\begin{aligned}
[AUX] \qquad z_A^* \;=\; & \min\, z_A = \sum_{j \in \mathcal{N}} c_j x_j \\
\text{s.t.} \quad & \sum_{j \in \mathcal{N}} a_{ij} x_j = e_i, \quad i \in M \\
& \sum_{j \in \mathcal{N} \setminus N} x_j \geq 1 \\
& x_j \in \{0, 1\}, \quad j \in \mathcal{N}.
\end{aligned}
$$

- $\tilde{x}$ is an optimal solution to SPP *if and only if* $z_A^* \geq \tilde{z}$
- $\tilde{x}$ is an optimal solution to SPP *if* $LBD_{AUX} \geq \tilde{z}$

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Column for non-degenerate all-integer pivot

$$[NDCG] \qquad \bar{c}_p = \min\ c - \sum_{i \in M} \bar{u}_i a_i$$
$$\text{s.t.} \quad (c, a) \in \mathcal{P},$$
$$\left(B^{-1}a\right)_i \in \begin{cases} [0,1], & i \in M : \bar{e}_i = 1, \\ [-1,0], & i \in M : \bar{e}_i = 0 \end{cases}$$

yields a column (c_p, a_p) that enables a non-degenerate pivot-on-one.

Column for degenerate all-integer pivot

$$[DCG] \qquad \bar{c}_p = \min\ c - \sum_{i \in M} \bar{u}_i a_i$$
$$\text{s.t.} \quad (c, a) \in \mathcal{P},$$
$$\left(B^{-1}a\right)_i = \pm 1 \text{ for some } i \in M : \ \bar{e}_i = 0$$

yields a column (c_p, a_p) enables a degenerate pivot on a plus or minus one entry.