An improved primal simplex algorithm and column generation for degenerate linear programs

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Joint work with

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Linear programming

$$(LP) \qquad z^{\text{LP}} = \min_{x} \quad c^{\top} x \tag{1}$$

$$\text{s.t.} \quad Ax = b \tag{2}$$

$$x \geq 0 \tag{3}$$

where $x \in \mathbb{R}^{n}$, $c \in \mathbb{R}^{n}$, $b \in \mathbb{R}^{m}$, and $A \in \mathbb{R}^{m \times n}$

- **Degeneracy in linear programming**

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Instance	*cp*5	*cp*6	*cp*7
Number of variables	10	15	21
Number of constraints	16	32	64
Number of non-zero elements/column	8	16	32
Number of extreme points	26	158	544
Number of LPF bases	126	48,414	6,450,702
Avg nb of LPF bases/extreme point	4.8	306.4	11,857.9

Table 1: Results of the lrs2 algorithm for three set partitioning instances

- **Set partitioning case:** Dynamic constraint aggregation, Multi-phase dynamic constraint aggregation

1. Elhallaoui, I., A. Metrane, F. Soumis, and G. Desaulniers (2005). Multi-phase Dynamic Constraint Aggregation for Set Partitioning Type Problems. Under minor revisions in *Math Programming*.

2. I. Elhallaoui, G. Desaulniers, A. Metrane, F. Soumis: Bi- Dynamic Constraint Aggregation and Subproblem Reduction. Computer and Operation Research 35 (2008) 1713-1724.

Definition

Let $B = \{A_k \;/\; k \in K \;\subset \{1...n\}\;\}$. A column D is said to be compatible with B if there exists λ such that

$$D = \sum_{k \in K} \lambda_k A_k$$

Definition

A reduced problem w. r. t to a set B is a problem where

- *only columns compatible with B are considered and*
- *the redundant constraints are removed.*

Reduced Problem RP_B

For a degenerate basic solution $\overline{x}$, let $B = \{A_i \;/\; \overline{x}_i > 0\} = \{A_1, \ldots, A_d \;/\; d < m\}$. Denote by RP_B the reduced problem w.r.t to a set B.

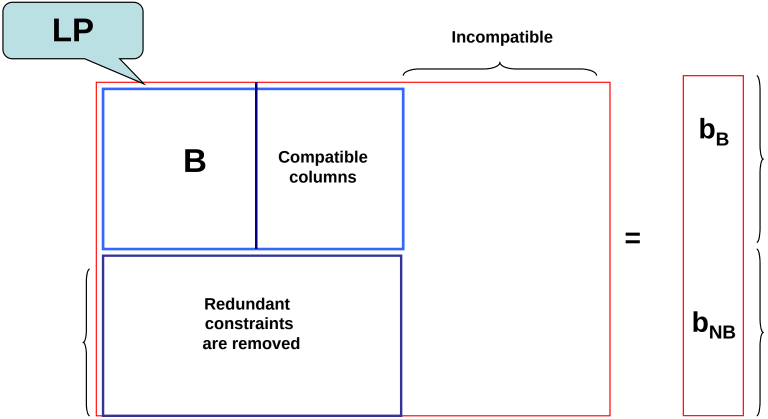
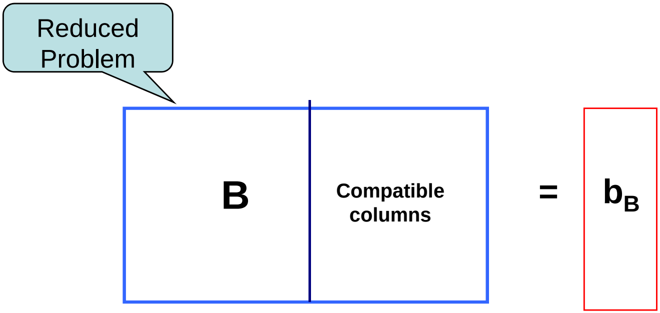


Figure 1: Reduced Problem



Instead of working with LP, it is better to work with RP_B because this problem is smaller than LP.

Figure 2: Reduced Problem

After solving the RP_B

Find one or more columns such that if we add these columns to the reduced problem, the optimal value decreases.

Questions:

- Do these columns exist?
- How can we find these columns?
- Is it easy to find them?

Motivation

We know that a solution x is an optimal solution of LP if and only if there exists a dual solution π to LP such that

$$\bar{c}_j := c_j - \pi^\top A_j = 0, \qquad \forall j \in \{1...d\} \tag{4}$$

$$\bar{c}_j := c_j - \pi^\top A_j \geq 0, \qquad \forall j \in \{d+1...n\} \tag{5}$$

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Complementarity problem

$$\max_{s,\ \pi} \quad s \tag{6}$$

$$\text{s.t.} \quad c_j - \pi^\top A_j = 0, \qquad \forall j \in \{1...d\} \tag{7}$$

$$c_j - \pi^\top A_j \geq s, \qquad \forall j \in I = \{d+1...n\} \tag{8}$$

Duality of complementarity problem

$$(CP_B) \qquad z_B^{CP} = \min_{v} \sum_{j \in I} v_j \bar{c}_j \tag{9}$$

$$\text{s.t.} \quad Mv = 0 \tag{10}$$

$$e^\top v = 1 \tag{11}$$

$$v \geq 0. \tag{12}$$

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Let x_B optimal solution of RP_B

Proposition

- *If $z_B^{CP} \geq 0$, then $(x_B^*, 0)$ is an optimal solution to LP.*
- *If $z_B^{CP} < 0$, then $(x_B^*, 0)$ is not an optimal solution to LP.*

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- *If $z_B^{CP} < 0$, then $(x_B^*, 0)$ is not an optimal solution to LP.*
  $\longrightarrow$ *Add $\{A_j \ / \ v_j^* > 0\} \Rightarrow$ the optimal value of RP_B decreases.*

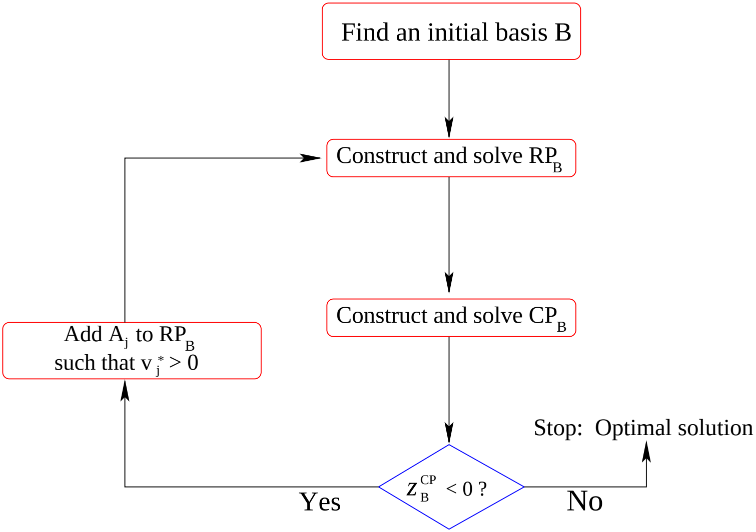

Figure 3: The improved primal simplex algorithm

instance	nb const	nb var	deg (%)
vcs11	1667	14852	63
vcs12	1667	18082	64
vcs13	1878	23683	60
vcs14	1878	27194	62
vcs15	2085	19963	62
vcs16	2085	21687	61
vcs17	2294	15996	59
vcs18	2294	24135	65
vcs19	2498	21431	59
vcs20	2498	32415	60
fa1	3758	13937	41
fa2	3758	14728	40
fa3	3737	11181	39
fa4	3683	14278	47
fa5	3644	13215	52

Table 2: Instance characteristics

vcs: vehicle and crew scheduling problem
fa: fleet assignment and aircraft routing

	TPS (CPLEX)		IPS			
instance	nb piv	time(s)	nb it	nb piv	time(s)	TPS/IPS
vcs11	65864	205	31	77138	133	1.54
vcs12	84194	291	17	113520	159	1.83
vcs13	112396	451	24	86972	225	2.00
vcs14	137056	612	23	91786	258	2.37
vcs15	104698	359	22	70814	164	2.19
vcs16	123209	644	42	105172	299	2.15
vcs17	91443	430	34	72887	211	2.04
vcs18	148769	883	43	121622	403	2.19
vcs19	141125	930	32	139725	466	2.00
vcs20	216296	1500	37	175432	701	2.14
Avg	122505	630.5	30.5	105506.8	50.3	**2.09**
fa1	89173	760	8	39811	240	3.17
fa2	124951	1072	5	51319	307	3.49
fa3	86553	694	4	38243	257	2.70
fa4	94179	609	4	32359	144	4.23
fa5	80693	428	4	27862	114	3.75
Avg	95109.8	712.6	5.0	37918.8	141.2	**3.35**

Table 3: LP results

Remark

Degeneracy can occur while solving RP_B. But, since the reduced problem is small compared to LP, the impact of degeneracy is relatively small in the reduced problem.

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Conclusion

- IPS reduces solution time by an average factor of 2.72 w.r.t CPLEX Elhallaoui, I., A. Metrane, G. Desaulniers and F. Soumis (2007). An improved primal simplex algorithm for degenerate linear programs Submitted to SIAM Journal on Optimization

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- IPS reduces solution time by an average factor of 2.72 w.r.t CPLEX Elhallaoui, I., A. Metrane, G. Desaulniers and F. Soumis (2007). An improved primal simplex algorithm for degenerate linear programs Submitted to SIAM Journal on Optimization
- IPS2: A new version of IPS reduces solution time by an average factor between 4 and 12(developed by Vincent Raymond and François Soumis).

- The same reduced problem RP_B
- The same complementarity problem:

$$(CP_B) \qquad z_B^{CP} = \min_v \sum_{i \in I} v_i \bar{c}_i \tag{13}$$

$$\text{s.t.} \quad Mv = 0 \tag{14}$$

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- The new idea: Add the aggregated column $\omega = \sum_{j \in I} v_j A_j$ in RP_B

Let $I = \{d+1, \dots, n\}$ the index set of incompatible columns, $\omega = \sum_{i \in I} v_i A_i$ with $\sum_{i \in I} v_i = 1$, $\bar{c}_\omega = \sum_{i \in I} v_i \bar{c}_i$.

$$\Omega = \{\omega \ / \ \omega \text{ is compatible with } B \text{ having a negative reduced cost}\}$$

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Theorem

$$x_B \text{ is optimal for } LP$$

$$\Updownarrow$$

$$\Omega = \emptyset$$

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Theorem

$$x_B \text{ is optimal for } LP$$

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$$\Omega = \emptyset \Longleftrightarrow z_B^{CP} \geq 0.$$

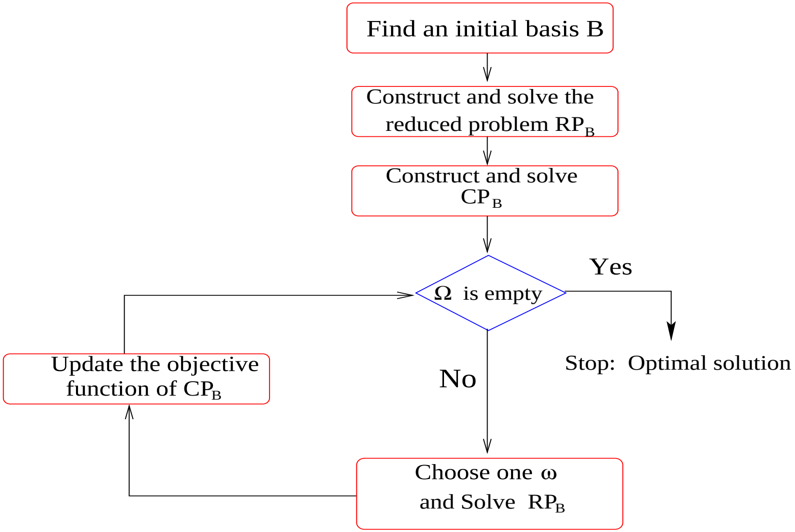

Figure 4: IPS-CG Algorithm

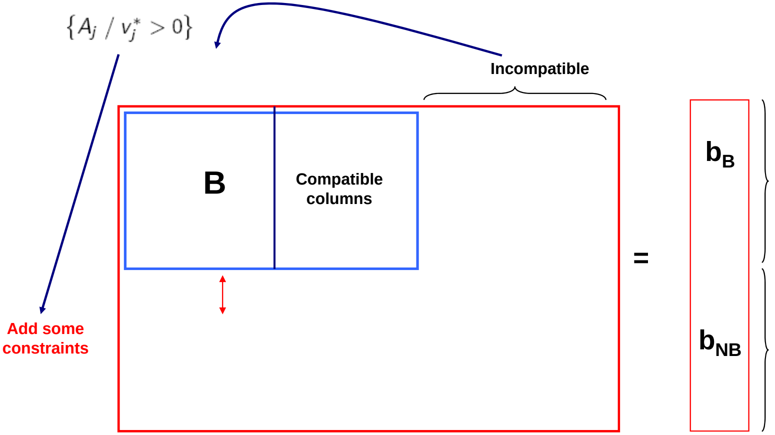

Figure 5: Add $\left\{A_j \ / \ v_j^* > 0\right\}$ in IPS Algorithm

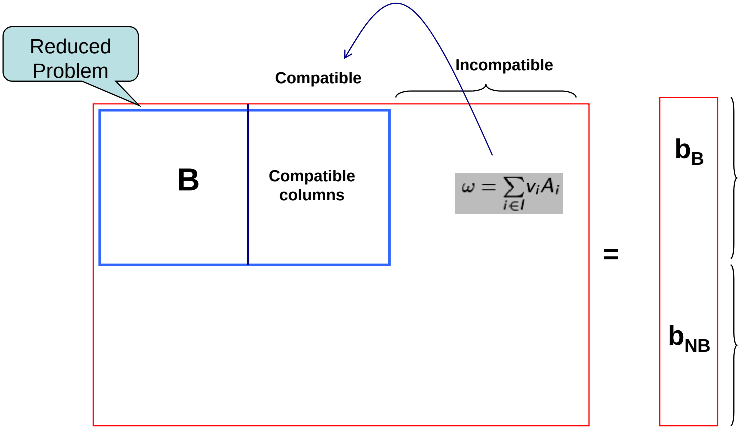

Figure 6: Add $\sum_{i \in I} v_i A_i$ in IPS-GC Algorithm

instance	IPS-CG /IPS2	IPS-CG /Cplex	IPS-Hybrid/IPS2	IPS-Hybrid/Cplex
vcs	0.7	2.45	0.95	3.3
FA	1.3	15	1.4	16

Table 4: Mean of the reduction factor: LP results

- IPS-Hybrid: Start by IPS-CG and finish by IPS2.
- Cplex: Primal simplex

Comments

- Reduction of the solution time by a factor of up to 16 compared to the primal simplex.
- Better understanding of degeneracy.
- It generalizes Dynamic Constraint Aggregation.
- IPS-CG is a new column generation method that doesn't depend on the structure of the problem.

Thank you for your attention