The Thin Green Line: The Transboundary Pollution Problem in a Coupled System of Shallow Lakes

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Abstract

This paper provides a contribution to the dynamic games literature on the transboundary pollution problem. The results from the dynamic game characterize the impact from the flow of pollution from the neighboring upstream lake to the downstream lake in terms of a closed-loop Stackleberg equilibrium. We show that when both systems are at steady state, even a small flow of pollution from the upstream lake can radically change the optimal pollution policy of the downstream lake from one that converges to a lower (oligotrophic) state to one that leads to an upper (eutrophic) steady state. The convergent paths for the downstream lake illustrate the impact of the upstream lakes convergence to a low steady state given low or high accumulations of pollution. As a consequence, efficiency requires that upstream agents limit their own consumption so as to increase that of downstream agents whose marginal benefits are higher.

Key Words: Transboundary Pollution, Coupled Lake Systems, Dynamic Games, Closed-Loop Stackleberg Equilibrium, Dynamic Lake Game

JEL Classification: C73, Q52, Q53

1 Introduction

This paper concentrates on the problem of pollution in system of shallow lakes. In the model presented in this paper there are two lakes, an upstream and a downstream lake, and two communities situated around each of the lakes. The upstream lake feeds into the downstream lake via a connecting waterway. As a byproduct of their day to day economic activities, the communities situated around each of the shallow lakes will input phosphorus into their local lake system, contributing to its phosphorus loading. It is well known that shallow lakes present a hysteresis in their response to this phosphorous loading (Carpenter, Brock and Hansen 1999, Carpenter Ludwig and Brock 1999). Hence both lakes will remain in an oligotrophic (or healthy) state over long periods of time with gradual increases in phosphorous loading to a point at which it suddenly flips to an alternate, eutrophic state. Once the flip has occurred, the lake then remains eutrophic despite decreases in phosphorous loading, and is reversible to a healthy state only if the pollution loading could be reduced well below the oligotrophic steady state level (Brock et al. 1999). The threshold point at which the lake changes state, from oligotrophic to eutrophic, is known as a Skiba point (Skiba 1978, Dechert and Nishimura 1983).

One natural question concerns what happens in systems of shallow lakes when there are multiple communities sharing its use? Dechert and Brock (2000) and Mäler, Xepapadeas and De Zeeuw (1999) were the first papers to pose this problem as a dynamic game between multiple communities sharing a single lake. Both of these papers show that runoff from pollution creates asymmetry in terms of the ability of each community to support local economic activity. Mäler, Xepapadeas and De Zeeuw (1999, 2003) demonstrate that it is possible to address the problem by imposing of Pigouvian taxes on the level of economic activity for this problem leading back to a steady state Pareto efficiency. However one problem with this outcome is that the tax policies constructed in Mäler, Xepapadeas and De Zeeuw (1999, 2003) are for open-loop Nash equilibrium. Salerno, Beard and McDonald (2008) show that because these tax policy are not time consistent, they will be subject to manipulation leading to potentially catastrophic outcomes.

However, the cumulative impact of this localized pollution is compounded in the model presented in our paper. Here the runoff from upstream pollution creates asymmetry in terms of the ability of each lake to support local economic activity. We show in the cooperative model that Pareto efficiency requires that upstream agents limit their own consumption so as to increase that of downstream agents whose marginal benefits are higher. The results from the non-cooperative dynamic game characterize the impact from the flow of pollution from the neighboring upstream lake to the downstream lake when each of the communities adopt a separate management strategy. They show that when both systems are at steady state, even a small flow of pollution from the upstream lake can radically change the optimal pollution policy of the downstream lake from one that converges to a lower (oligotrophic) state to one that leads to an upper (eutrophic) steady state.

The modeling approach is based on Dechert and Brock (2000), which use a potential as a means of finding the open-loop Nash equilibrium of the dynamic game as a control problem. As shown in Dechert (2000) one benefit from employing this procedure is that because the equilibrium emerges as a solution to an optimal control, when the optimal control problem is solved via dynamic programming its will automatically solve the closed-loop problem. Hence the open-loop Nash equilibrium will also be closed-loop Nash equilibrium. One potential difficult with results obtained using the non-cooperative model is that the outcomes are dynamic Stackleberg equilibrium in which the leader is the upstream community. However, closed-loop Stackleberg equilibria are notoriously difficult to solve (see Baszar and Tamer (1999) for a discussion).

Hence, we adapt the approach of Dechert and Brock, by using backward induction methods from the steady state point to solve the potential function iteratively, to generate the closed-loop equilibrium. The numerical results show that an upstream lake converging to its steady state with low accumulation increases the speed of convergence of the downstream lake but does not change its path overall. If the upstream lake is converging to its steady state from high accumulation of pollution, the downstream lake eventually converges to its lower steady state but initially, the lake undergoes a transient state with the accumulation of pollution in the downstream lake converging toward its upper steady state. Hence the convergent paths for the downstream lake illustrate the impact of the upstream lakes convergence to a low steady state given low or high accumulations of pollution.

The paper is organized as follows. Section 2 describes the couple lake system and structure and preferences of the communities situated around each of the lake. Section 3 provides the cooperative model for the coupled lake game. Section 4 provides a description of the non-cooperative model. Section 5 examines the Hamiltonian derived from the cooperative game and the non-cooperative game. For the non-cooperative game the dynamics are constructed for both stationary and non-stationary transition paths. This section shows that for both games all dynamics will converge to a steady state equilibrium path. Section 6 provides numerical simulations of the coupled lake system and a discussion of their implications for each of the local communities.

2 Model

The model in this paper introduces a coupled lake system and is an extension of work that was first introduced in Dechert and Brock (2000, 2008). In this model P_i is the quantity of phosphorus in lake i, where we will denote the upstream lake by $i = 1$ and the downstream lake by $i = 2$. L_i is the loading of phosphorus per unit time into lake i. We assume the dynamics of $P_i(t)$, $i = 1, 2$ are given by

$$\dot{P}_1 = L_1 - \delta_1 P_1 + r_1 \frac{(P_1/m_1)^{q_1}}{1 + (P_1/m_1)^{q_1}}, \qquad P_1(0) = P_1^0, \tag{2.1}$$

$$\dot{P}_2 = L_2 - \delta_2 P_2 + r_2 \frac{(P_2/m_2)^{q_2}}{1 + (P_2/m_2)^{q_2}} + \gamma P_1, \quad P_2(0) = P_2^0. \tag{2.2}$$

Here δP_i is the rate (per unit time) at which phosphorus flows out of lake water[1], r is the quantity of phosphorus that is reintroduced from the mud per unit time and

[1]Some of the outflow is sequestered into the mud at the bottom of the lake which can be stirred up and reintroduced into the lake water, while the rest of the outflow is passed along down stream.

m is the anoxic level (which is one half of the quantity of phosphorus that will cause the lake water to be saturated with phosphorus). The amount of phosphorus from the upstream lake that is introduced into the downstream lake is denoted by γP_1. Since this cannot be more than the outflow from the upstream lake, the restriction that $\gamma \leq \delta_1$ is imposed on the model.

In order to reduce the number of independent parameters in equations (2.1) and (2.2) as well as to establish a "standard" lake we use the following transformation of time

$$\tau = \frac{hr_1}{m_1}t,$$

where h is the small increment of time that will be used in the discrete time version of the model. This leads to the following transformed system of differential equations:

$$h\dot{x}_1 = a_1 + (b_1 - 1)x_1 + \frac{x_1^{q_1}}{1 + x_1^{q_1}}, \qquad x_1(0) = x_{1,0}, \tag{2.3}$$

$$h\dot{x}_2 = a_2 + (b_2 - 1)x_2 + \nu\frac{x_2^{q_2}}{1 + x_2^{q_2}} + \mu x_1, \;\; x_1(0) = x_{2,0}, \tag{2.4}$$

where the constants in this system of differential equations are defined as follows:

$$b_1 = 1 - m_1\delta_1/r_1, \; b_2 = 1 - \frac{m_1\delta_2}{r_1}, \; \nu = \frac{r_2 m_1}{r_1 m_2} \text{ and } \mu = \frac{\gamma m_1^2}{r_1 m_2}.$$

and the restriction $m_2\mu/m_1 \leq 1/b_1$ must hold[2]

There is a farming community situated around each lake that loads phosphorus (mostly as runoff from fertilization) into their own lake, and each community derives benets from their lake. In this paper we will examine the nature of the solution to two variations of this model. In the rst the two communities act non cooperatively. In the second we examine the social optimum for the two communities together. The preferences of each community are modeled using a representative agent, whose

[2] This follows from the restriction that $\gamma \leq \delta_1$.

preferences are given by

$$\int_0^\infty e^{-\rho t} v_i(x_i(t), a_i(t))dt, \quad i = 1, 2, \tag{2.5}$$

where J_i denotes the utility that community i derives from the loading and the pollution level, ρ is the rate of discounting the future and v_i is the utility for community i at time t.

Increases in the loading are assumed to bring increased economic production which in turn implies higher consumption levels. Increases in the level of pollution are assumed to reduce the ecosystem service quality. Thus the loading is an economic good and the level of pollution is an economic bad in utility terms. We will assume that benefits and costs are separable, so that

$$v_i(x_i, a_i) = u_i(a_i) - c_i(x_i), \quad i = 1, 2, \tag{2.6}$$

where u_i denotes the benefits that are derived from increased economic production and c_i denotes the social costs (measured in terms of disutility) that are derived the pollution which emerges as a by-product of this activity[3].

In this paper we focus on outcomes generated by two types of games: a cooperative game, where the two communities jointly manage the coupled lake system, and a non-cooperative game, where each community's focus is on an independent management strategy for their respective lake. In the cooperative game the objective function is a weighted average of utilities for the upstream and downstream communities:

$$W(x_1, x_2) = \max_{a_1, a_2} \int_0^\infty v_1(x_1(t), a_1(t)) + \lambda \left(v_2(x_1(t), a_1(t))\right), \tag{2.7}$$

subject to equations (2.3) and (2.4). This corresponds to a social welfare function with different weights $\lambda > 0$ on the upstream and downstream utilities, as would

[3]This assumption is standard in much of ecological economics, see for example Maler, Xepapadeas, and de Zeeuw (2003), Brock and Starrett (2003), Dasgupta and Maler (2003) and Xepapadeas (2006), and the references cited in those articles.

be the case is there were differences in population in the two regions. The outcome of this game delivers a benchmark – the Pareto optimal outcome. However, in the non-cooperative setting of this paper, the core problem that we are attempting to analyze is the possibility of a non-binding agreement being formed between both parties for joint management of the lake that implement this outcome.

The problem with such an agreement is that the upstream polluter has minimal incentive to commitment its agreed level of economic activity given the structure of the coupled lake environment. This is modeled as a dynamic Stackleberg game, where the follower is the downstream community by virtue that its decisions have no direct impact on the quality of the upstream community's lake. Here is what happens in the open-loop problem: At time 0 the upstream polluter announces a control path a_1. The downstream polluter then takes this as given and makes his decision regarding the choice of a_2:

$$J_2(x_2, a_1) = \max_{a_2} \int_0^\infty e^{-r_2 t} v_2(x_2(t), a_1(t), a_2(t)) dt$$

$$\text{s.t. } \dot{x}_2 = f_2(x_2(t), a_1(t), a_2(t)), \quad x_2(0) \text{ given}, \tag{2.8}$$

where $f_2(x_2(t), a_1(t), a_2(t))$ is as defined in equation (2.4). The first order conditions for the downstream polluter are then given as follows:

$$\frac{\partial v_2(x(t), a_2(t))}{\partial a_2(t)} - \frac{\partial \lambda_2(t) f_2(x_2(t), a_1(t), a_2(t))}{\partial a_2(t)} = 0 \tag{2.9}$$

and

$$\dot{\lambda}_2 = r_2 \lambda_2(t) - \frac{\partial v_2(x_2(t), a_2(t))}{\partial x_2(t)} - \frac{\partial \lambda_2(t) f_2(x_2(t), a_1(t), a_2(t))}{\partial x_2(t)} \tag{2.10}$$

with the transversality condition given by

$$\lim_{t \to \infty} e^{-r_2 t} \lambda_2(t) x_2(t) = 0. \tag{2.11}$$

These first order conditions, when taken together with state equation and the transversality condition (i.e., equations (2.8) – (2.11)), characterize the follower's best response control path $a_2(t) := \varphi_2(x_2(t), a_1(t))$.

Given that there is complete information here and the downstream community's control path is known by every agent, the upstream community can now choose a path $a_1(t)$ by applying the following maximization problem:

$$J_1 = \max_{a_1} \int_0^\infty e^{-r_1 t} v_2(x_1(t), a_1(t)) dt,$$

$$\text{s.t. } \dot{x}_1 = f_1(x_1(t), a_1(t)), \quad x_1(0) \text{ given.}$$

Hence, for each optimal time path $a_1^*(t)$ chosen by the upstream community their will be a corresponding best response path $a_2^*(t)$ for the downstream community. The state and co-state variables corresponding to this optimal path are given by $x_2^*(t)$ and $\lambda_2^*(t)$, respectively and satisfy $x_2^*(0)$ and the transversality condition $\lim_{t\to\infty} e^{-r_2 t} \lambda_2^*(t) x_2^*(t) = 0$. If $\lambda_2^*(0)$ depends on the optimal control path of the leader, $a_1^*(t)$, then the open-loop equilibrium will not be time consistent. Hence at any point in time in the future the control path for the follower at time t will depend on future behavior of the leader and because of this, the open-loop Stackleberg equilibria are usually not the appropriate equilibrium for this dynamic setting.

3 Dynamic Cooperative Game

Because we are looking at a numerical solution for this problem, we begin by breaking time into short durations by letting $\tau = nh$ for $n = 0, 1, \cdots$, and let $x_{1,n} = x1(n\tau)$ and $x_{2,n} = x_2(n\tau)$. Similarly, $a_{1,n} = a_1(n\tau)$ and $a_{2,n} = a2(n\tau)$. Using the Caucy scheme

$$\dot{x} \approx \frac{x(\tau + h) - x(\tau)}{h}$$

the differential equations for x_1 and x_2 can be approximated by

$$x_{1,n+1} = b_1 x_{1,n} + \frac{x_{1,n}^{q_1}}{1 + x_{1,n}^{q_1}} + a_{1,n} \tag{3.1}$$

$$x_{2,n+1} = \mu x_{1,n} + b_2 x_{2,n} + \nu \frac{x_{2,n}^{q_2}}{1 + x_{2,n}^{q} 2} + a_{2,n} \tag{3.2}$$

Notice that the differences in equations (3.1) and (3.2) are the term $\mu x_{1,n}$ in the latter as well as the constant ν in the non-linear term. The units of m_i/r_i are the same as t (i.e., units of time) so when

$$\frac{m_1}{r_1} > \frac{m_2}{r_2}$$

Using the discrete time steps of $\tau = nh$, the objective function for the upstream community becomes

$$\begin{aligned}
\int_0^\infty e^{-\rho t} v_1(x_1(t), a_1(t))dt &= \int_0^\infty e^{-r_2 t} u_1(x_1(t)) - c_1(a_1(t))dt \\
&= \sum_{n=0}^{\infty} \int_{nh}^{(n+1)h} e^{-\frac{\rho m_1 \tau}{h r_1}} u_1(x_1(t)) - c_1(a_1(t))dt \\
&\approx \sum_{n=0}^{\infty} \beta^n \left[u_1(a_{1,n}) - c_1(x_{1,n})\right] \qquad (3.3)
\end{aligned}$$

where $\beta^n = \exp \rho m_1/r_1$. In similar fashion, we can approximate the objective function for the downstream community as follows:

$$\begin{aligned}
\int_0^\infty e^{-\rho t} v_2(x_2(t), a_2(t))dt &= \int_0^\infty e^{-r_2 t} u_2(x_2(t)) - c_2(a_2(t))dt \\
&\approx \sum_{n=0}^{\infty} \beta^n \left[u_2(a_{2,n}) - c_2(x_{2,n})\right]. \qquad (3.4)
\end{aligned}$$

In the literature on the lake game it is common to use the values of $q_1 = q_2 = 2$ and the functional forms of $u_i(a_i) = ln(a_i)$ and $c_i(x_i) = k_i x_i$ for the community utility and cost functions[4]

For the cooperative we get the dynamic optimization, which we solve to get the socially efficient outcome:

$$\max \quad \sum_{n=0}^{\infty} \beta^n \left[u_1(a_{1,n}) - c_1(x_{1,n}) + \lambda \Big(u_2(a_{2,n}) - c_2(x_{2,n}) \Big) \right] \tag{3.5}$$

subject to:

$$x_{1,n+1} = b_1 x_{1,n} + \frac{x_{1,n}^{q_1}}{1 + x_{1,n}^{q_1}} + a_{1,n} \tag{3.6}$$

$$x_{2,n+1} = \mu x_{1,n} + b_2 x_{2,n} + \nu \frac{x_{2,n}^{q_2}}{1 + x_{2,n}^{q_2}} + a_{2,n} \tag{3.7}$$

For convenience of notation, we define $G_i(x, b, q) = b x_i + x_i^{q_i}/(1 + x_i^{q_i})$. The value function for the cooperative game is:

$$\begin{aligned} W(x_1, x_2) \quad = \quad & \max_{a_1, a_2} \{ u_1(a_1) - c_1(x_1) + \lambda (u_2(a_2) - c_2(x_2)) \\ & + \beta W(G_1(x_1, b_1, q_1) + a_1, \mu x_1 + \nu G_2(x_2, b_2/\nu, q_2) + a_2) \} . \end{aligned} \tag{3.8}$$

The parameter λ is the relative weight of the two communities in the social costbenet optimization. Now let $u_1(a) = u_2(a) = \ln(a)$ and $c_i(x) = k_i x^2$.

The Euler equations for the cooperative game are as follows:

$$0 \quad = \quad W_1(n) + \beta^{-1} F_3(n-1) \tag{3.9}$$

$$0 \quad = \quad F_2(n) + \beta^{-1} F_4(n-1) \tag{3.10}$$

where the W_i denote the partial derivatives of the value function for the cooperative

[4] See Maler, Xepapadeas, and de Zeeuw (2003) for the specification of the functional form of the model.

game. These partial derivatives of W are given as follows:

$$W_1(n) = -\left\{ \frac{G'(x_n, b_1)}{x_{n+1} - G(x_n, b_1)} + 2k_1 x_n + \frac{\lambda\mu}{y_{n+1} - \nu G(y_n, b_2/\nu) - \mu x_n} \right\} \tag{3.11}$$

$$W_2(n) = -\frac{\lambda\nu G'(y_n, b_2/\nu)}{y_{n+1} - \nu G(y_n, b_2/\nu) - \mu x_n} - 2\lambda k_2 y_n \tag{3.12}$$

$$W_3(n-1) = \frac{1}{x_n - G(x_{n-1}, b_1)} \tag{3.13}$$

(3.14)

$$W_4(n-1) = \frac{\lambda}{y_n - \nu G(y_{n-1}, b_2/\nu) - \mu x_{n-1}}, \tag{3.15}$$

where for shallow lakes ($q = 2$):

$$G_i(x_i, b) = bx + \frac{x_i^2}{1 + x_i^2} \text{ and } G_i'(x_i, b) = b + \frac{2x_i}{(1 + x_i^2)^2}.$$

The steady state, $(\bar{x}, \bar{y})$, which satisfies the Euler equations with $x_n = x_{n+1} = \bar{x}$ and $y_n = y_{n+1} = \bar{y}$ is given below:

$$\frac{\beta^{-1}}{\bar{x} - G(\bar{x}, b_1)} = \frac{G_1'(\bar{x}, b_1)}{\bar{x} - G_1(\bar{x}, b_1)} + 2k_1\bar{x} + \frac{\lambda\mu}{\bar{y} - \nu G_2(\bar{y}, b_2/\nu) - \mu\bar{x}} \tag{3.16}$$

and

$$\frac{\beta^{-1}}{\bar{y} - \nu G(\bar{y}, b_2/\nu) - \mu\bar{x}} = \frac{\nu G_2'(\bar{y}, b_2/\nu)}{\bar{y} - \nu G_2(\bar{y}, b_2/\nu) - \mu\bar{x}} + 2k_2\bar{y} \tag{3.17}$$

The interesting term is the last one in the first steady state equation. Without it the upper lake steady state would be the same as in the non cooperative solution.

4 The Non-Cooperative Game

Under the discrete time formulation of this problem the upstream community solves the following control problem

$$J_1 = \max_{a_{1,n}} \sum_{n=0}^{\infty} \beta^n \left[u_1(a_{1,n}) - c_1(x_{1,n}) \right] \tag{4.1}$$

$$\text{s.t. } x_{1,n+1} = b_1 x_{1,n} + \frac{x_{1,n}^{q_1}}{1 + x_{1,n}^{q_1}} + a_{1,n}$$

The downstream community solves the following optimization problem

$$J_2 = \max_{a_{2,n}} \sum_{n=0}^{\infty} \beta^n \left[u_2(a_{2,n}) - c_2(x_{2,n}) \right] \tag{4.2}$$

$$\text{s.t. } x_{2,n+1} = \mu x_{1,n} + b_2 x_{2,n} + \nu \frac{x_{2,n}^{q_2}}{1 + x_{2,n}^{q_2}} + a_{2,n}$$

As in the last section, the analytics (as well as the programming) can be simplified by defining the auxiliary function

$$G_i(x_i, b_i, q_i) = b_i x_i + \frac{x_i^{q_i}}{1 + x_i^{q_i}}, \quad i = 1, 2$$

The dynamics for the two lakes can then be written as

$$x_{1,n+1} = G_1(x_{1,n}, b_1, q_1) + a_{1,n}$$

$$x_{2,n+1} = \mu x_{1,n} + G_2(x_{2,n}, b_2, q_2) + a_{2,n}$$

We will use the notation $G_i'(x_i, b_i, q_i)$ to denote the derivative of G_i with respect to x_i, and similarly $G_i''(x_i, b_i, q_i)$ for the second derivative.

Note that the optimization problem for the downstream community depends also on the outcome generated from the upstream community's optimization problem. Hence this is a dynamic Stackleberg game. For the upstream community, we know from Brock and Dechert (2000) that the upstream community will be converging to one of the two steady states: x_1^o, which denotes the oligotrophic steady state, or x_1^e, which we use to denote the eutrophic steady state for that lake. When the upstream community is at one of these two steady states, the solution of the downstream communitys optimization is stationary. In this case the technique in Brock and Dechert can be used to get the noncooperative solution for the problem. For a general initial condition, $(x_{1,0}, x_{2,0})$, it is the case that the upstream lake converges to a steady state in a relatively few number of periods, typically between 30 and 50. We can solve the optimal control problem for the downstream lake by using the solution sequence for the upstream lake and iterating the Euler equation for the down stream lake backwards from the steady state.

For this stationary case, the optimization problem for the downstream community as

$$J_2 = \max_{x_{1,n}} \sum_{n=0}^{\infty} \beta^n \left[u_2 \left(\mu x_{1,n} + x_{2,n+1} + \nu G_2(x_{2,n}, b_2/\nu, q_2) \right) - c_2(x_{2,n}) \right]$$

$$\text{s.t. } x_{2,n+1} \geq \max \left\{ 0, \nu G_2 \left(x_{2,n}, b_2/\nu, q_2 \right) - \mu x_{1,n} \right\},$$

and so the Euler equation for the downstream lake is

$$0 = \beta^{-1} u_2' \left(\mu x_{1,n} + x_{2,n+1} + \nu G_2(x_{2,n}, b_2/\nu, q_2) \right.$$

$$\left. - u_2' \left(\mu x_{1,n} + x_{2,n+1} + \nu G_2(x_{2,n}, b_2/\nu, q_2 \right) \nu G' \left(x_{2,n}, b_2/\nu, q_2 \right) - c_2' \left(x_{2,n} \right)$$

To iterate this backwards, we need to be able to solve for $x_{2,n-1}$ given values for $x_{2,n}$, $x_{2,n+1}$, $x_{1,n}$ and $x_{1,n+1}$.

For the non-stationary case, we note that the upper lake in the non cooperative game converges to the steady state fairly quickly We can program the non time stationary dynamic programing problem as follows so that it exploits this observation. Let $\bar{x}$ be the steady state that the upper lake is converging towards. (Note that this may depend on the initial state if it is a Skiba lake.) Let $x_0, x_1, \cdots$ be the optimal sequence statrting from x_0, and suppose that by time T this sequence has for all practical purposes converged to $\bar{x}$ (i.e., we assume that $x_t = \bar{x}$ for all $t \geq T$). The dynamic programming recursion for the lower lake is now given as follows

$$V_{t-1}(y) = -k_2 y^2 + \max_u \{\ln(u) + \beta V_t(\nu G(y, b_2/\nu) + \mu x_{t-1} + u)\} \tag{4.3}$$

This is solved recursively as follows: By our assumption that $x_t = \bar{x}$ for $t \geq T$, it is the case that $V_T(y) = V(y)$ where V is the solution to the dynamic programming problem when the upper lake is at steady state:

$$V(y) = -k_2 y^2 + \max_u \{\ln(u) + \beta V(\nu G(y, b_2/\nu) + \mu \bar{x} + u)\}$$

for which we already have a program. So, the way to solve for V_0 in equation (4.3) is to solve it backwards from $V_T = V$. Read in V and then compute V_{T-1}. Now iterate, for $V_{T-2}, \cdots, V_1, V_0$. To simulate the model (once you have the solution) then you will need to store the optimal policy function for each stage. That is, if $h_{t-1}(y)$ is the maximizing value of u in equation (4.3), then you will have to write to disk the value of h_{T-1}, $h_{T-2}, \cdots, h_1, h_0$.

5 Hamiltonian Dynamics

5.0.1 Stationary Case

Let's start with the stationary dynamics (when the upper lake is at a steady state, $\bar{x}$, so that the downstream lake gets the constant amount of loading $\mu\bar{x}$ from the upper lake). The Hamiltonian is

$$H(y, p, u) = (\nu G(y, b_2/\nu) + \mu \bar{x} + u)\, p - \ln(u) + k y^2. \tag{5.1}$$

Maximizing the Hamiltonian with respect to u implies that $p - \frac{1}{u} = 0$ or that $u = 1/p$. The dynamics of the state–costate equations are:

$$y_{t+1} = \nu G(y_t, b_2/\nu) + \mu\bar{x} + \frac{1}{p_t} \tag{5.2}$$

$$\beta^{-1} p_{t-1} = \nu G'(y_t, b_2/\nu) p_t + 2ky_t \tag{5.3}$$

Notice that to iterate these equations forward (in t) given values of y_0 and p_0 equation (5.2) can be solved for y_1. Then with y_1 and p_0 equation (5.3) can be solved for p_1. To iterate these equations backwards given values of y_T and p_T, solve equation (5.3) for p_{T-1}. Then with the values of y_T and p_{T-1} solve equation (5.2) for y_{T-1}. Notice that this step requires solving the non-linear equation

$$\nu G(y_{T-1}, b_2/\nu) = y_T - \mu\bar{x} - \frac{1}{p_{T-1}} \tag{5.4}$$

for y_{T-1}. Then continue in this fashion. If either $y_t < 0$ or $y_t - \mu\bar{x} - 1/p_{t-1}$ at some iteration, then stop.

The problem in this approach is that we cannot iterate the equations forward, since we do not know the initial value to use for p_0. However, what we do know is that the sequence of state–costate variables is converging to a steady state which is the solution to

$$\bar{y} = \nu G(\bar{y}, b_2/\nu) + \mu\bar{x} + \frac{1}{\bar{p}}$$

$$\beta^{-1}\bar{p} = \nu G'(\bar{y}, b_2/\nu)\bar{p} + 2k\bar{y}.$$

The next thing to do is to linearize the dynamics at the steady state and to solve for the stable and unstable manifolds. To that end, subtract equations (5.5) and

(5.5) from equations (5.2) and (5.3) to get:

$$y_{t+1} - \bar{y} = \nu \left(G(y_t, b_2/\nu) - G(\bar{y}, b_2/\nu) \right) + \frac{1}{p_t} - \frac{1}{\bar{p}} \tag{5.5}$$

$$\beta^{-1}(p_{t-1} - \bar{p}) = \nu \left(G'(y_t, b_2/\nu) p_t - G'(\bar{y}, b_2/\nu) \bar{p} \right) + 2k(y_t - \bar{y}) \tag{5.6}$$

Next, expand the right hand sides of equations (5.5) and (5.6) in a Taylor series at $(\bar{y}, \bar{p})$ *keeping only* the first order terms:

$$y_{t+1} - \bar{y} = \nu G'(\bar{y}, b_2/\nu)(y_t - \bar{y}) - \frac{1}{\bar{p}^2}(p_t - \bar{p}) \tag{5.7}$$

$$\beta^{-1}(p_{t-1} - \bar{p}) = \left(\nu G''(\bar{y}, b_2/\nu)\bar{p} + 2k \right)(y_t - \bar{y}) + \nu G'(\bar{y}, b_2/\nu)(p_t - \bar{p}) \tag{5.8}$$

Finally, step up the dates from $(t-1, t)$ in equation (5.8) to $(t, t+1)$ and collect the terms in $t+1$ onto the left hand side and the terms in t onto the right hand side to get the matrix equation:

$$A \begin{bmatrix} y_{t+1} - \bar{y} \\ p_{t+1} - \bar{p} \end{bmatrix} = B \begin{bmatrix} y_t - \bar{y} \\ p_t - \bar{p} \end{bmatrix}$$

The matrices A and B are

$$A = \begin{bmatrix} 1 & 0 \\ \nu G''(\bar{y}, b_2/\nu)\bar{p} + 2k & \nu G'(\bar{y}, b_2/\nu) \end{bmatrix} \quad B = \begin{bmatrix} \nu G'(\bar{y}, b_2/\nu) & \bar{p}^{-2} \\ 0 & \beta^{-1} \end{bmatrix}$$

Note that this linear system is of the form

$$z_{t+1} = C z_t \tag{5.9}$$

where $C = A^{-1}B$. Now, get the eigenvalues and eigenvectors of equation (5.9). Call them (λ_1, λ_2) and $(\boldsymbol{v}_1, \boldsymbol{v}_2)$, respectively. One of the two eigenvalues will be less than one in magnitude, say $|\lambda_1| < 1$. Then $\boldsymbol{v}_1$ will be the linear approximation to the stable manifold. If we let ϵ be small, we can start the backwards iteration scheme at

$$\begin{bmatrix} y_T \\ \\ p_T \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \\ \bar{p} \end{bmatrix} \pm \epsilon \boldsymbol{v}_1 \tag{5.10}$$

where the sign on ϵ determines which side of the steady state to start the dynamics.

5.1 Time Non-Stationary Case

Now, we turn to the time non stationary case, when the loading from the upper lake into the lower lake varies over time, μx_t. The Hamiltonian for this case is time dependent:

$$H_t(y, p, u) = \Big(\nu G(y, b_2/\nu) + \mu x_t + u\Big)\, p - \ln(u) + k y^2$$

and the time dependence does not affect the solution of maximizing the Hamiltonian, $p - \frac{1}{u} = 0$.

Basically the rest of the solution is the same as above, but with $\bar{x}$ replaced by x_t. The dynamics of the state–costate equations are:

$$y_{t+1} = \nu G(y_t, b_2/\nu) + \mu x_t + \frac{1}{p_t} \tag{5.11}$$

$$\beta^{-1} p_{t-1} = \nu G'(y_t, b_2/\nu) p_t + 2k y_t \tag{5.12}$$

The steady state equations (5.5) and (5.5) are the same since the time dependent case has the optimal sequence $\{x_t\}$ converging to $\bar{x}$. To start the backwards iteration of equations (5.11) and (5.12) simulate the upper lake for $x_0, x_1, \cdots, x_T$, where x_T is close (within ϵ) to $\bar{x}$. Pick (y_T, p_T) according to equation (5.10) and iterate backwards to get $\{(y_{T-1}, p_{T-1}), \cdots, (y_0, p_0)\}$. Along this solution path, the optimal loading for the lower lake at (x_t, y_t) is $u_t = 1/p_t$.

5.2 Cooperative Game

Equations (3.5) – (3.7) are the optimal control problem for the social optimum of the coupled lake system. Using the function G_i, $i = 1, 2$, and letting $\mathbf{x_t} = (\mathbf{x_{1,t}}, \mathbf{x_{2,t}})$, $\boldsymbol{u}_t = (u_{1,t}, u_{2,t})$ and $\boldsymbol{p}_t = (p_{1,t}, p_{2,t})$. The following notation will be used when there is no time subscript: $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{p} = (p_1, p_2)$). The Hamiltonian for the cooperative game is

$$\begin{aligned} H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}) &= -\ln(u_1) + k_1 x_1^2 - \lambda \left(\ln(u_2) - k_2 x_2^2\right) \\ &\quad + \left(G(x_1, b_1) + u_1\right) p_1 + \left(\nu G(x_2, b_2/\nu) + \mu x_1 + u_2\right) p_2 \end{aligned}$$

Maximizing the Hamiltonian with respect to $\boldsymbol{u}$ implies that

$$u_1 = \frac{1}{p_1} \qquad u_2 = \frac{\lambda}{p_2} \tag{5.13}$$

and the resulting dynamics are

$$x_{1,t+1} = G(x_{1,t}, b_1) + \frac{1}{p_{1,t}} \tag{5.14}$$

$$x_{2,t+1} = \nu G(x_{2,t}, b_2/\nu) + \mu x_{1,t} + \frac{\lambda}{p_{2,t}} \tag{5.15}$$

$$\beta^{-1} p_{1,t-1} = G'(x_{1,t}, b_1) p_{1,t} + \mu p_{2,t} + 2k_1 x_{1,t} \tag{5.16}$$

$$\beta^{-1} p_{2,t-1} = \nu G'(x_{2,t}, b_2/\nu) p_{2,t} + 2\lambda k_2 x_{2,t} \tag{5.17}$$

The steady states[5] satisfy the equations:

$$\bar{x}_1 = G(\bar{x}_1, b_1) + \frac{1}{\bar{p}_1} \tag{5.18}$$

$$\bar{x}_2 = \nu G(\bar{x}_2, b_2/\nu) + \mu\bar{x}_1 + \frac{\lambda}{\bar{p}_2} \tag{5.19}$$

$$\beta^{-1}\bar{p}_1 = G'(\bar{x}_1, b_1)\bar{p}_1 + \mu\bar{p}_2 + 2k_1\bar{x}_1 \tag{5.20}$$

$$\beta^{-1}\bar{p}_2 = \nu G'(\bar{x}_2, b_2/\nu)\bar{p}_2 + 2\lambda k_2\bar{x}_2 \tag{5.21}$$

The linearized system around the steady states is:

$$x_{1,t+1} - \bar{x}_1 = G'(\bar{x}_1, b_1)(x_{1,t} - \bar{x}_1) - \frac{1}{\bar{p}_1^2}(p_{1,t} - \bar{p}_1) \tag{5.22}$$

$$\begin{aligned} x_{2,t+1} - \bar{x}_2 &= \nu G'(\bar{x}_2, b_2/\nu)(x_{2,t} - \bar{x}_2) + \mu(x_{1,t} - \bar{x}_1) \\ &\quad - \frac{\lambda}{\bar{p}_2^2}(p_{2,t} - \bar{p}_2) \end{aligned} \tag{5.23}$$

$$\begin{aligned} \beta^{-1}(p_{1,t} - \bar{p}_1) &= G'(\bar{x}_1, b_1)(p_{1,t+1} - \bar{p}_1) + G''(\bar{x}_1, b_1)\bar{p}_1(x_{1,t+1} - \bar{x}_1) \\ &\quad + \mu(p_{2,t+1} - \bar{p}_2) + 2k_1(x_{1,t+1} - \bar{x}_1) \end{aligned} \tag{5.24}$$

$$\begin{aligned} \beta^{-1}(p_{2,t} - \bar{p}_2) &= \nu G'(\bar{x}_2, b_2/\nu)(p_{2,t+1} - \bar{p}_2) + \nu G''(\bar{x}_2, b_2/\nu)\bar{p}_2(x_{2,t+1} - \bar{x}_2) \\ &\quad + 2\lambda k_2(x_{2,t+1} - \bar{x}_2) \end{aligned} \tag{5.25}$$

[5] Note that It is probably easier to calculate the steady states in two steps. First, use equations (3.16) and (3.17) and solve for $\bar{x}_1$ and $\bar{x}_2$. Then use equations (5.20) and (5.21) to solve for $\bar{p}_1$ and $\bar{p}_2$. These latter two equations are linear in $\bar{p}_1$ and $\bar{p}_2$ and so are trivial to solve.

This time let $\boldsymbol{z}_t$ be

$$\boldsymbol{z}_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ p_{1.t} \\ p_{2,t} \end{bmatrix},$$

then the matrix equation is again of the form $\boldsymbol{z}_{t+1} = C\boldsymbol{z}_t$ where $C = A^{-1}B$ and A and B are

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ G''(\bar{x}_1, b_1)\bar{p}_1 + 2k_1 & 0 & G'(\bar{x}_1, b_1) & \mu \\ 0 & \nu G''(\bar{x}_2, b_2/\nu)\bar{p}_2 + 2\lambda k_2 & 0 & \nu G'(\bar{x}_2, b_2/\nu) \end{bmatrix}$$

and

$$B = \begin{bmatrix} G'(\bar{x}_1, b_1) & 0 & -\bar{p}_1^{-2} & 0 \\ \mu & \nu g'(\bar{x}_2, b_2/\nu) & 0 & -\lambda\bar{p}_2^{-2} \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{bmatrix}$$

Let the eigenvalue–eigenvector pairs be $(\lambda_i, \boldsymbol{v}_i)$ for $i = 1, \cdots, 4$ and assume that they are ordered in such a way that the first two have $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Then the linear approximation to the stable manifold at a steady state is the span of

$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. So for the backwards iteration of equations (5.14) – (5.17) start at

$$\begin{bmatrix} x_{1,T} \\ x_{2,T} \\ p_{1,T} \\ p_{2,T} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{p}_1 \\ \bar{p}_2 \end{bmatrix} \pm \epsilon_1 \boldsymbol{v}_1 \pm \epsilon_2 \boldsymbol{v}_2$$

for small values of ϵ_1 and ϵ_2. Again, the signs will determine what direction the dynamics are coming from. Along the dynamics the optimal level of loadings will be

$$u_{1.t} = \frac{1}{p_{1,t}} \text{ and } u_{2,t} = \frac{1}{p_{2,t}}. \tag{5.26}$$

6 Simulation Results

In this section we will focus on the equilibrium loadings generated from the non-cooperative game. The results are not included here at present. We state that the total loadings for the cooperative are always less than the total loadings of the non-cooperative (except for the case where the welfare weight on the lower lake community is zero, where the cooperative outcome coincides with the non-cooperative game. We also state that for both cooperative and non-cooperative games, there are four different "phases" that the system can be in with respect to the parameters. They are

1. **LSS**: only the lower of the two steady states is an optimal steady state;
2. **USS**: only the upper of the two steady states is an optimal steady state;
3. **SSS**: there is only one steady state, which is therefore the optimal steady state; and
4. **Skiba**: both steady states are optimal steady states and there is a Skiba point in between them.

In Figure 1 both plots describe non-cooperative choices made for the downstream lake as the upstream lake converges to its lower steady state. In the upper diagram, there are two policy functions for the downstream lake. The red line is the policy function for the downstream lake when the upstream lake is at the lower steady state. At steady state, the upstream lake is sending down 0.003347 units of phosphorus (when $\mu = 0.01$). Both lakes are oligotrophic. The blue line is the policy function when the upstream lake is 40 periods away from steady state. At period 40, the upstream lake is sending down 0.018237 units. Policy function characterizes a Skiba lake.

Notice that the blue line is below the red line for low accumulations of phosphorus in the downstream lake and above the red line for high accumulations. This influences the convergent path of the downstream lake when the upstream lake experiences a shock. We define the flow from the upstream that changes the downstream lake policy functions as μx. Figure 2 maps out the convergence paths given the condition of the upstream lake. To produce this map, a program was written that computed the policy function for the downstream lake when the upstream lake was m periods away from its steady state. For this example, the upstream lake took up to 41 periods to converge. The program generate 42 policy functions $\mu_{t,41}$ to $\mu_{t,0}$ (one for each period and steady state).

The lower diagram in Figure 1 illustrates the paths when $y_0 = 1.0$. When the policy function is Skiba, this initial value is to the left of the Skiba point. The steady state value for the downstream lake is indicated with the dashed line. The path when the upstream lake is at a steady state is represented with the solid black line. If the upstream lake has a low accumulation of phosphorus, the initial runoff to the downstream lake is less than μx. The downstream lakes policy function lies below the policy function when the upstream lake is at a steady state. This implies that convergence to the downstream steady state is faster. In the graph, this is illustrated with the red line. The line was generate in the upstream lake was 10 periods away from steady state.

If the upstream has a high accumulation ($\mu x_{0,j} > \mu x$) and the initial accumulation of downstream phosphorus is high, the system eventually convergences to the lower steady state but the time is longer and the system in the early stages accumulates more phosphorus. This is illustrated by the blue line. The upstream lake was 40 periods away from steady state.

7 Conclusion

The model presented in this paper examined the situation where there are two lakes, an upstream and a downstream lake, and two communities situated around each of the lakes. The upstream lake feeds into the downstream lake via a connecting waterway. As a byproduct of their day to day economic activities, the communities situated around each of the shallow lakes will input phosphorus into their local lake system, contributing to its phosphorus loading. However, the cumulative impact of this localized pollution is compounded in the model presented in our paper. Here the runoff from upstream pollution creates asymmetry in terms of the ability of each lake to support local economic activity.

We show in the cooperative model that Pareto efficiency requires that upstream agents limit their own consumption so as to increase that of downstream agents whose marginal benefits are higher. The results from the non-cooperative dynamic game characterize the impact from the flow of pollution from the neighboring upstream lake to the downstream lake when each of the communities adopt a separate management strategy. We show that when both systems are at steady state, even a small flow of pollution from the upstream lake can radically change the optimal pollution policy of the downstream lake from one that converges to a lower (oligotrophic) state to one that leads to an upper (eutrophic) steady state.

In the dynamic game the modeling approach used in this paper uses a potential as a means of finding the open-loop non-cooperative equilibrium. As shown in Dechert (2000) one benefit from employing this procedure is that because the equilibrium emerges as a solution to an optimal control, when the optimal control problem is solved via dynamic programming its will automatically solve the closed-loop problem. Hence the open-loop equilibrium will also be closed-loop Nash equilibrium. One potential difficult with results obtained using the non-cooperative model is that the outcome of the game is dynamic Stackleberg equilibrium in which the leader is the upstream community. Closed-loop Stackleberg equilibria are notoriously difficult to solve.

We adapt the approach of Dechert and Brock, by using backward induction methods from the steady state point to solve the potential function iteratively, to generate the closed-loop equilibrium. The numerical results show that an upstream lake converging to its steady state with low accumulation increases the speed of convergence of the downstream lake but does not change its path overall. If the upstream lake is converging to its steady state from high accumulation of pollution, the downstream lake eventually converges to its lower steady state but initially, the lake undergoes a transient state with the accumulation of pollution in the downstream lake converging toward its upper steady state. Hence the convergent paths for the downstream lake illustrate the impact of the upstream lakes convergence to a low steady state given low or high accumulations of pollution.

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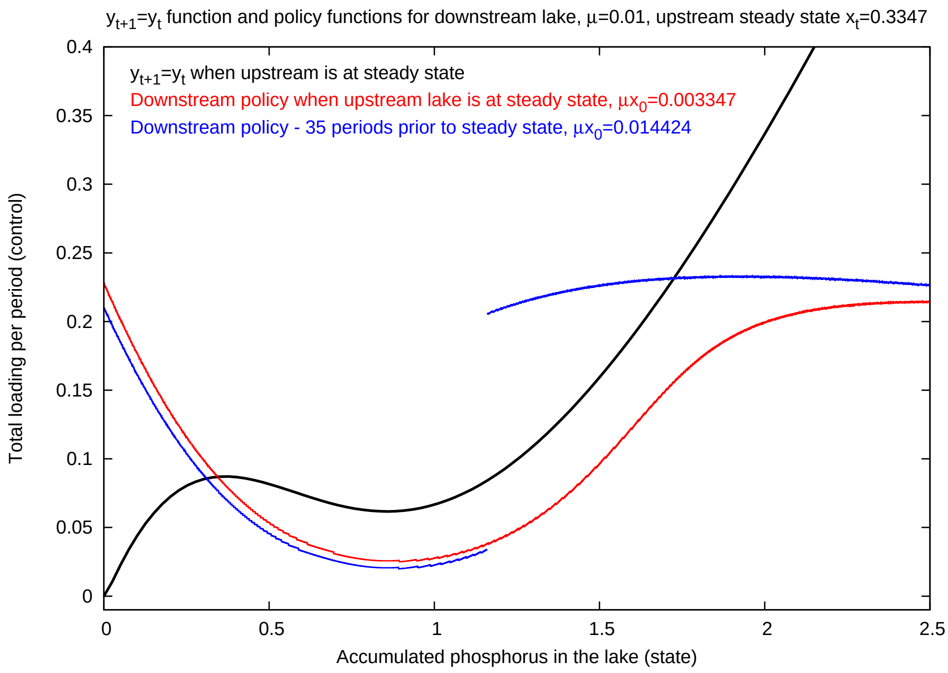

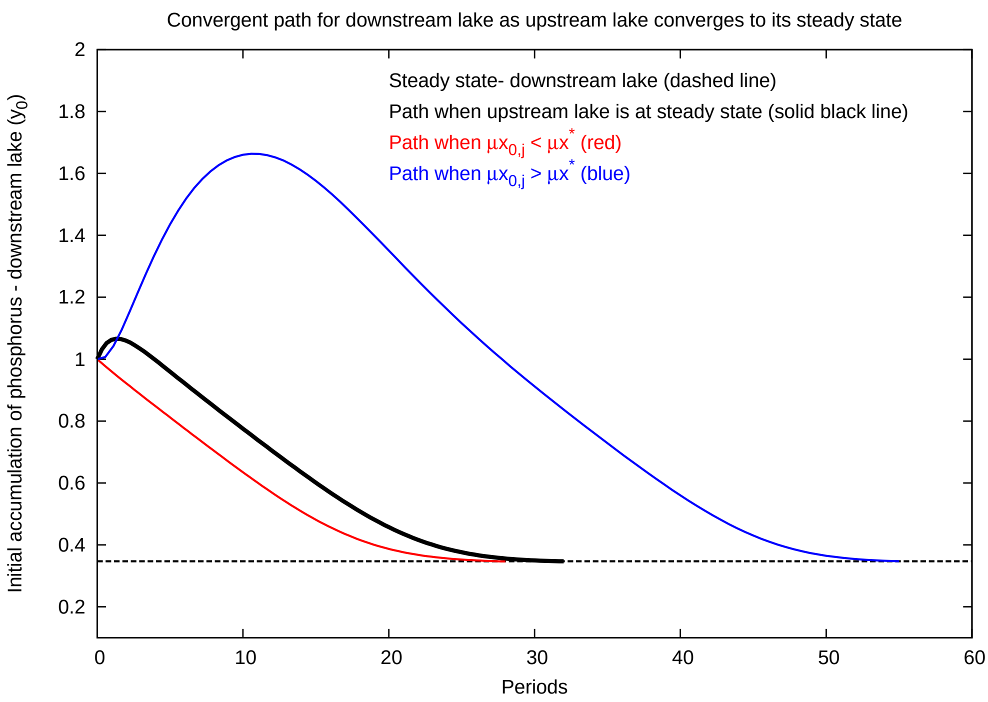

Figure 1: Policy functions and convergences for the downstream lake community

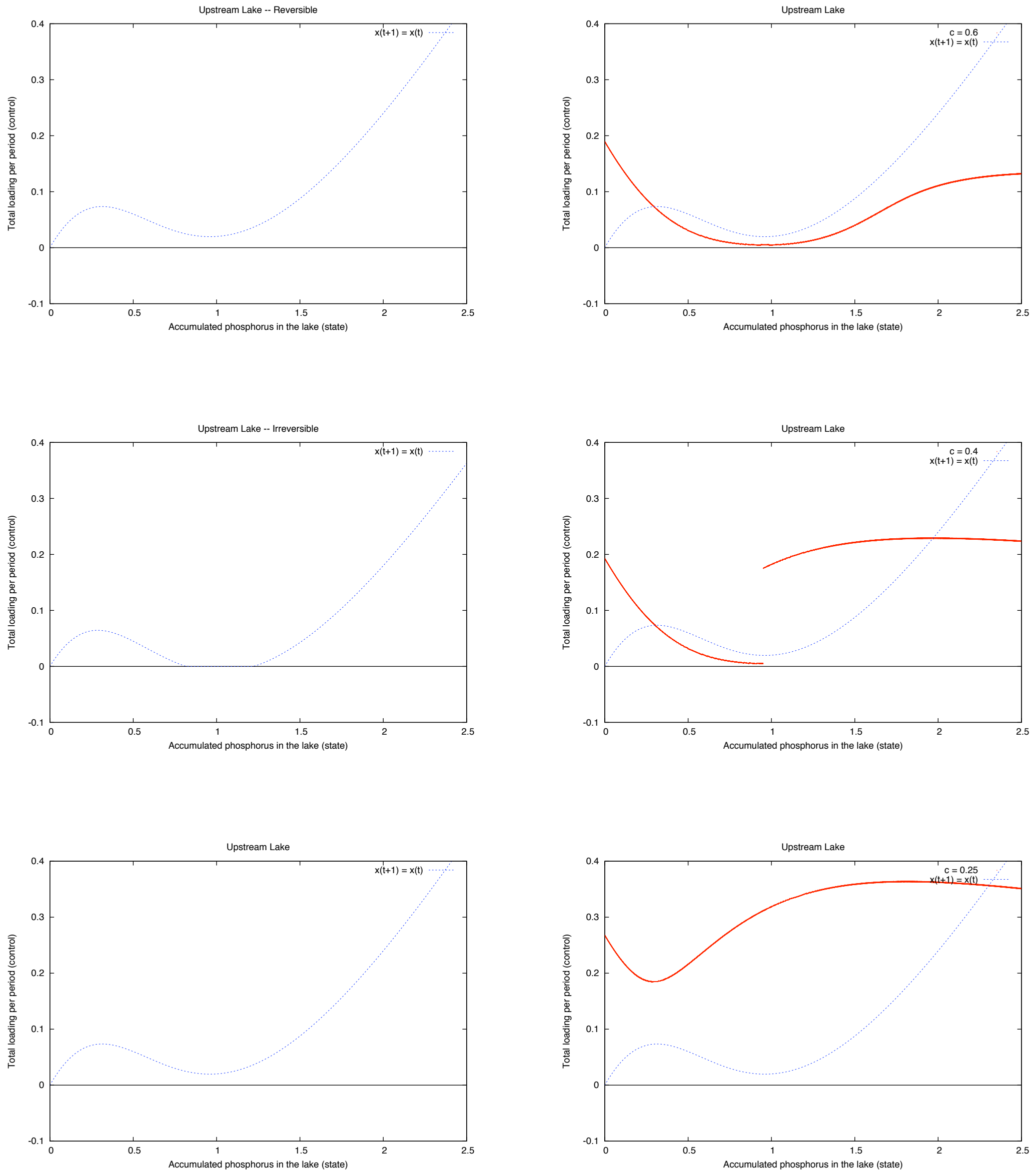

Figure 2: Stability analysis of steady states and the policy functions for the upstream lake community changing c

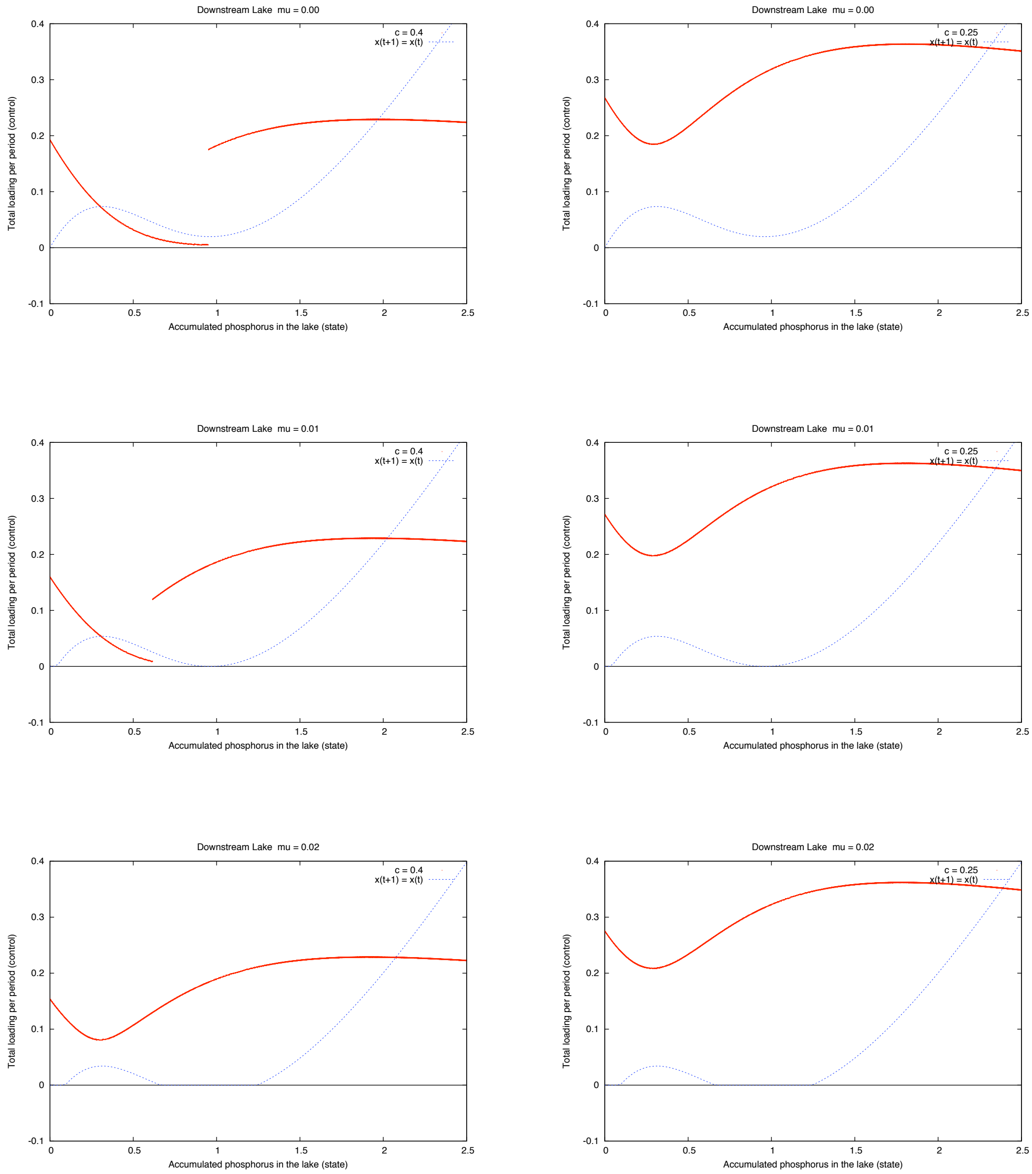

Figure 3: Stability analysis of the policy functions for the downstream lake community changing μ