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Optimization problems in planar geometry*

Charles Audet

GERAD and Département de mathématiques et de génie industriel, École Polytechnique de Montréal, C.P. 6079, Succ. Centre-ville, Montréal (Québec), H3C 3A7 Canada, Charles Audet@gerad.ca

Abstract Attributes such as perimeter, area, diameter, sum of distances between vertices and width can be evaluated for every planar convex polygon. Fixing one of these attributes while minimizing of maximizing another defines families of optimization problems. Some of these problems have a trivial solution, and several others have been solved, some since the Greeks, by geometrical reasoning. During the last four decades, this geometrical approach has been complemented by global optimization methods. This combination allowed solution of instances than could be solved by any one of these two approaches alone. This talk surveys research on that topic, and proposes directions for further work.

Keywords: Extremal problems, global optimization, convex polygon, perimeter, diameter, area, sum of distances, width

1. Introduction

Consider a n-sided convex polygon V_n in the Euclidean plane. Let A_n denote its area, P_n its perimeter, D_n its diameter, S_n the sum of distances between all pairs of its vertices and W_n its width. Maximizing or minimizing any of these quantities while setting another to a fixed value defines ten pairs of extremal problems.

These problems were first surveyed in [5], and then some solutions were later updated in [6]. Usually, one problem from each pair has a trivial solution or no solution at all. For example, Zenodorus (200-140 b.c.) showed that the regular polygons have maximal area for a given value of the perimeter, but the question of minimizing the area given a fixed perimeter is trivial, as the area can be made arbitrarily close to zero.

Simple formulations of extremal problems for convex polygons are easily obtained by denoting the consecutive vertices of the n-sided polygon V_n by $v_i = (x_i, y_i)$. Then

- $A_n = \frac{1}{2}\left|\sum_{i=1}^{n}(y_{i+1} - y_i)(x_{i+1} + x_i)\right|,$
- $P_n = \sum_{i=1}^{n} \|v_{i+1} - v_i\|,$
- $D_n = \max_{i<j} \|v_i - v_j\|,$
- $S_n = \sum_{i<j} \|v_i - v_j\|,$

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$$. \; W_n = \min_i \max_{j \neq i, i+1} \frac{|(y_{j+1} - y_j)x_i + (x_j - x_{j+1})y_i + x_{j+1}y_j - x_j y_{j+1}|}{\|v_{j+1} - v_j\|},$$

where the indices $i+1$ and $j+1$ are taken modulo n, $|.|$ denotes absolute value and $\|.\|$ the Euclidean norm. Expressions of the objective function and constraints easily follow.

2. Currently known solutions

Minimizing a first attribute while fixing a second one is equivalent to maximizing the second one while fixing the first one. Therefore, the present work only considers the maximization questions. Table 1 summarizes the current known solutions to these 20 extremal problems for convex polygons, and provides references to where the solutions may be found. Each line of the table corresponds to a fixed attribute, while each column indicates which attribute is maximized.

	max P_n	max A_n	max D_n	max S_n	max W_n
P_n=1	—	Regular $\sim$ 180 BC Zenodorus	Trivial: Segment	Segment 2008 Larcher & Pillichshammer 2008 [16]	Reuleaux for n with odd factor and n=4. 2009 [4]
A_n=1	Trivial: flat	—	Trivial: flat	Trivial: flat	open
D_n=1	Reuleaux for n with odd factor and $n = 4, 8$ Tamvakis 1987 [21] Datta 1997 [11] 2007 [3]	Regular for odd n Reinhardt 1922 [19] and $n \leq 10$. Graham 1975 [13] 2002 [8], Foster & Szabo 2007 [12] Mossinghoff 2006 [18], Henrion & Messine 2010 [15]	—	$n = 3, 4, 5, 6, 7$ 2008 [1]	Reuleaux for n with odd factor and $n = 4$. Bezdek & Fodor [10]
S_n=1	Segment Larcher & Pillichshammer 2008 [16]	open	Trivial: flat	—	open
W_n=1	Trivial: slice	Trivial: slice	Trivial: slice	Trivial: slice	—

Table 1. Convex polygons with maximal attribute

Several of these problems have a trivial solution, and only a few of the non-trivial ones are solved for every value of n. Most of the non-trivial ones have known solutions in the cases where n is very small, or when n is an odd number, or when it has an odd factor. Some of these solutions were obtained numerically, using recent global optimization algorithms. In particular QP [2], a branch and cut algorithm for nonconvex quadratically constrained optimization, IBBA [17], an interval analysis branch and bound algorithm for nonlinear programming, and very recently, GloptiPoly [14], a semidefinite programming approach for polynomial optimization.

Adding the additional constraint that the polygons are equilateral leads to different optimization problems. The cases where the solutions are the regular or clipped-Reuleaux [20] polygons have the equilateral property, and therefore remain optimal. Table 2 details the currently know solutions to these problems. The most recent results are for the maximization of the perimeter, the area and the diameter of unit width equilateral polygon. In the non trivial case (when the number of sides is odd), it is shown that the optimal polygon are arbitrarily close to symmetrical trapezoids.

	max P_n	max A_n	max D_n	max S_n	max W_n
P_n=1	—	Regular Zenodorus $\sim$ 180 BC	Trivial: Segment	open	Reuleaux for n with odd factor. 2009 [4]
A_n=1	Trivial: flat	—	Trivial: flat	open	open
D_n=1	Reuleaux for n with odd factor Vincze 1950 [22] and $n = 4, 8$ 2004 [7]	Regular for odd n Reinhardt 1922 [19] and $n = 4$.	—	$n = 3, 5$ 2008 [1]	Reuleaux for n with odd factor and $n = 4$. 2000 Bezdek & Fodor [10]
S_n=1	open	open	Trivial: flat	—	open
W_n=1	Trivial for even n Trapezoid for odd n 2010 [9]	Trivial for even n Trapezoid for odd n 2010 [9]	Trivial for even n Trapezoid for odd n 2010 [9]	open for odd n	—

Table 2. Equilateral convex polygons with maximal attribute

The presentation will discuss recent progress on some of these problems, and will propose potential research directions.

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