

The Story of Hansen’s Octagon*

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Abstract This paper tells the story of the discovery of *Hansen’s Octagon*, the solution to the problem of the *largest small octagon*—the convex octagon of maximal area under the constraint that the distance between any pair of vertices does not exceed one. Such polygons, called *small polygons*, were introduced by Karl Reinhardt in 1922. While Reinhardt solved the case for odd n , even values remained open. In 1975, Ron Graham solved the hexagon case. Inspired by this, Pierre Hansen proposed using deterministic global optimization to tackle the octagon, leading to a collaborative success more than a decade later. Hansen’s Octagon exemplifies the power of computational methods in modern geometry. Ultimately, Pierre Hansen’s key idea was that when problems, even seemingly simple ones in planar geometry, remained unsolved, computers—and particularly deterministic global optimization methods—could supplement the human mind.

Keywords: Small Polygons, Octagon, Numerical Proof, Reliable Computations

1. Historical Background

The study of small polygons began with Karl Reinhardt in 1922 [8]. He considered the problem of finding, for a given number of vertices n , the convex small polygon that maximizes area. He showed that for odd n , the regular polygon is optimal. For even n , however, the problem remains more nuanced.

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The case $n = 4$ (quadrilateral) was solved by Reinhardt, who showed that the square is one of the optimal solution.

For $n = 6$, the problem is more complex, and remained open until Ron Graham [6] solved it in 1975 through the use of diameter graphs. The nodes of the graphs are the vertices of the polygon, and the edges are the unit-length diagonals. His hexagon possesses an axis of symmetry, and exceeds the regular hexagon's area by nearly 4%.

2. From Hexagon to Octagon

Graham conjectured that when $n \geq 4$ is even, the largest small polygon diameter graph is composed of a cycle with $n - 1$ vertices together with an additional edge attached to a vertex of the cycle. The conjecture was formally shown by Foster and Szabo [5] for arbitrary polygons.

It was during a talk by Ron Graham in the late 1980s, in which he discussed his famous hexagon, that Pierre Hansen got the idea to use deterministic global optimization methods to tackle the more complex case of the octagon ($n = 8$) using deterministic global optimization methods. He shared this idea with Graham, who encouraged the effort despite its difficulty.

With his team [2], Hansen applied Graham strategy to decompose the problem into 31 possible configurations based on the analysis of diameter graphs. Most of the configurations were shown to be suboptimal using elementary geometrical arguments. The remaining ones were modeled as non-convex quadratically constrained optimization problems with a non-convex objective function. A state-of-the-art (at the time) branch-and-cut algorithm [1] found the optimal area up to four decimals through computations that exceeded 100 hours. A few years later, a group of researchers applied an efficient semidefinite programming approach [7] to certify 7 digits of accuracy, in only 5 seconds of computational time. They also pushed the computations to solve the decagon and the dodecagon instances, with computational times of 1 and 25 minutes.

The square (and many other optimal quadrilateral) as well as Graham's octagon have an axis of symmetry, through the pending edge of the diameter graph. The numerical solutions suggest that Hansen's Octagon also has an axis of symmetry. Assuming this to be true, reduces the dimensionality of the search space. Using symbolic computations on a complex-plane representation led to a single-variable root-finding problem, where the area A of the symmetric octagon is the unique root in $[0.7, \frac{\pi}{4}]$ of a degree-42 polynomial with large integer coefficients [3]:

$$p_8(A) = 147573952589676412928 A^{42} - 442721857769029238784 A^{41} + \dots \\ + 3773041038347596515021000956 A - 478425365462547737405343343$$

In 2010, the Global Optimization Workshop TOGO10 [4] brought together researchers to discuss the state-of-the-art in global optimization, covering theoretical, algorithmic, and application-oriented topics. A central theme of the workshop was the 70th birthday celebration of Pierre Hansen. A hand-made whisky glass in the shape of Hansen's Octagon (that maximizes volume) was offered to all participants. The area of this axially symmetrical octagon is within the error bounds proposed in [7], which suggests that Hansen's Octagon has an axis of symmetry. Figure 1 displays the regular octagon together with Hansen's Octagon and the whisky glass.

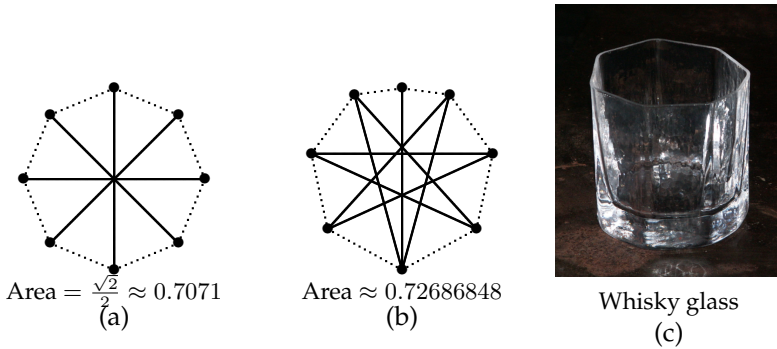


Figure 1: Diameter graph of (a) the regular octagon and (b) Hansen's Octagon, and (c) the whisky glass of TOGO10

3. Discussion

Hansen's Octagon and its associated variants highlights the rich interplay between classical geometry, combinatorial structure, and modern computational optimization. What began as a purely theoretical problem introduced by Reinhardt a century ago has evolved into a fruitful research area where advanced algorithmic tools reveal subtle and surprising optimal shapes.

This journey demonstrates how long-standing questions in planar geometry –such as maximizing area, perimeter, or width under distance or length constraints– can be resolved through global optimization techniques like branch-and-bound, semidefinite programming, and symbolic computation. Ultimately, Hansen’s Octagon serves as a compelling example of how modern optimization can provide rigorous solutions to deep, deceptively simple geometric problems.

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