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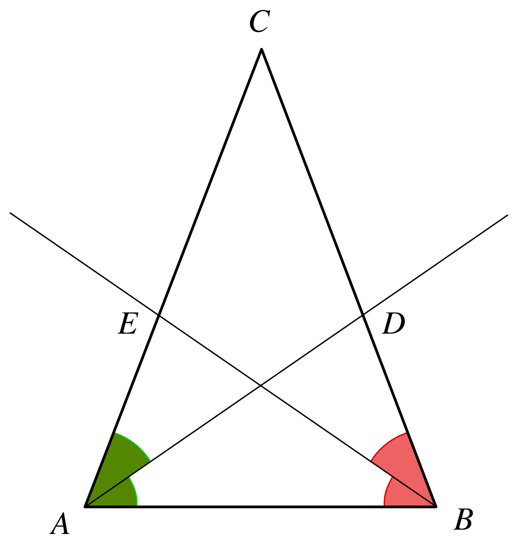

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The Steiner-Lehmus angle bisector theorem

Volume 98
Number 542
July 2014

www.m-a.org.uk

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Ordering 15 marbles with a three-way scale

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À la mémoire de mon frère François.

1. *Introduction*

Many mathematical puzzles involve using a scale to compare the weights of two objects, or two groups of objects. The present work studies a similar but different type of question involving a scale.

We are given a set of marbles, whose weights are all different. The only way to distinguish them is to use a special kind of scale. The scale has three trays and each can accept exactly one marble. The scale then indicates which is the heaviest, the lightest and consequently the middle one of the three marbles.

The paper studies the question of ordering 15 different marbles with as few weighings as possible. The problem is taken from the website *Enigmes, casse-têtes, curiosités et autres bizarreries* [1] and *Toppuzzle* [2] where the best reported strategy requires 23 weighings. The problem can also be found at the website *Trick of Mind* [3] where a strategy requiring 22 weighings is proposed. These websites contain many challenging mathematical puzzles, some are classical problems, others are original creations. The author is not aware of any other work on this problem.

The present paper describes a strategy that improves on the value of 22 weighings, and is structured as follows. Section 2 introduces some useful definitions and provides strategies to sort up to 9 marbles. Section 3 proves our main result, that 20 weighings is always sufficient to sort 15 marbles.

Graphical notation: The paper represents various configurations using graphs, where each node denotes a marble. An edge between two nodes indicates that the highest marble on the figure is known to be lighter than the one below. Edges may be vertical or diagonal, but never horizontal.

2. *Sorting up to nine marbles*

Let us introduce the following notation to manipulate sets of ordered marbles.

Definition: A chain is a set of ordered marbles. A v-chain is a set of marbles composed of a pair of chains with the same heaviest marble.

A chain can contain any number of marbles, including zero. A chain containing a single marble is called a singleton.

Figure 1 represents two chains, one of length three and one of length four, as well as a v-chain containing six marbles. Notice that using the scale once to compare the three shaded marbles will be sufficient to identify the heaviest of the 13 marbles.

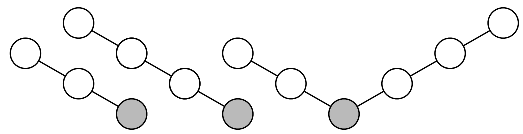

FIGURE 1

Throughout the document, we will use transitivity properties of relative weights, i.e. if marble a is heavier than b and if b is heavier than c, then a is necessarily heavier than c. This implies for example that the chain $a - b - c$ may also be reconfigured as a v-chain composed of the chains $a - b$ and $a - c$. Even though the v-chain contains less information than the chain, the v-chain version will often be used as it makes several subsequent proofs clearer.

The next theorem proposes a strategy to sort any set of marbles partitioned into two chains and one v-chain.

Theorem 1: $p - 2$ weighings suffice to sort two chains together with a v-chain, totalling $p \geqslant 3$ marbles.

Proof: The proof is by induction on p, the total number of marbles. The result is trivial for $p = 3$ since a single weighing will sort them.

Suppose that the result is true for $p \geqslant 3$ marbles. Consider two chains and a v-chain containing a total of $p + 1$ marbles. A single weighing of the heaviest from each of the three groups will yield the overall heaviest (if a group is empty, then pick any marble from another group with at least two marbles).

Observe that the weighing gives the relative weights of the heaviest marbles of the two chains. Consequently, the two chains may be merged into a v-chain. The $p + 1$ marbles are now partitioned into two v-chains.

Remove the overall heaviest marble, breaking one of the two v-chains into two chains and apply the induction hypothesis on the remaining p marbles. It follows that $p - 1$ weighings suffice for $p + 1$ marbles.

An immediate consequence of the previous theorem is that $p - 2$ weighings are sufficient to sort three chains totalling p marbles since a chain can be seen as a v-chain. This observation leads to the following corollaries.

Theorem: 3 weighings suffice to sort 4 marbles.

Proof: Weigh three marbles to form a chain. Theorem 1 ensures that 2 additional weighings suffice.

Note that the numbers of weighings used at each stage are upper bounds. For example, with lucky choices of marbles to weigh, we may get a sorted chain of 4 with only 2 weighings.

The above results suffice to devise a strategy to sort 15 marbles using at most 25 weighings. Indeed, start by using the scale three times to form a chain of length 4. Use the scale three more times to form a chain with the heaviest marble of the first chain with three other marbles. At this point, the seven marbles may be assembled into a *v*-chain, by using transitivity of relative weights if necessary. Now, use the scale six more times (for a total of 12) to form two chains of length 4 with the 8 remaining marbles. Theorem 1 ensures that 13 more weighings suffice.

Theorem 3: 4 weighings suffice to sort 5 marbles.

Proof: Weigh three marbles to form a chain. Theorem 1 ensures that 3 additional weighings suffice.

Armed with this corollary, we can do better than 25 weighings to sort 15 marbles. Use the corollary twice to form a *v*-chain containing nine marbles. This requires 8 weighings. Next, use the scale twice more to form two chains of length three with the 6 remaining marbles. Theorem 1 ensures that 13 more weighings suffice, for a total of 23.

Next we propose strategies to sort up to nine marbles. The idea behind each strategy is to end up with all the marbles being compared to a central one, thereby partitioning them into a group of light and a group of heavy marbles. Then each smaller group is sorted using the previous results.

Theorem 4: 5 weighings suffice to sort 6 marbles.

Proof: Compare three marbles, then weigh the middle one *a* with two of the remaining marbles. There are two cases to consider.

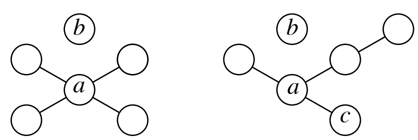

FIGURE 2

Case 1: *a* is the middle marble of the second weighing.

This is illustrated in the left part of Figure 2. Use a third weighing to compare *a* with the last remaining marble *b*. At this point, all marbles have been compared with *a*. A fourth weighing will order the ones heavier than *a* and a fifth weighing will order the lighter ones.

Case 2: *a* is not the middle marble of the second weighing.

By symmetry, and without any loss of generality, suppose that it is the heaviest, as illustrated in the right part of the above figure. Use a third weighing to compare the last remaining marble *b* with *a* and with the heaviest of the first weighing, *c*. If *b* is heavier than *a* then this last weighing has ranked the three overall heaviest marbles *a*, *b* and *c*, and a fourth weighing will be sufficient to rank the three lightest. Otherwise, only *c* is heavier than *a* and Theorem 1 ensures that two more weighings suffice to sort the four marbles lighter than *a*, for a total of 5 weighings.

The following strategy to sort seven marbles is taken from [3].

Theorem 5: 6 weighings suffice to sort 7 marbles.

Proof: Use the scale twice to form two disjoint chains of length three. Let *a* and *b* denote the two middle marbles of the chains and perform a third weighing on *a*, *b* and the last remaining marble *c*. Without any loss in generality, assume that *a* is heavier than *b*. Figure 3 illustrates the three possibilities. In the leftmost graph, *c* is the lightest, in the middle one *c* is the middle marble, and in the rightmost *c* is the heaviest.

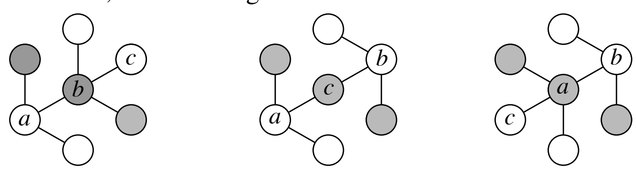

FIGURE 3

In each of these three situations, use the scale for a fourth time to weigh the three shaded marbles. The heaviest marble is therefore heavier than two shaded marbles and two white ones. Similarly, the lightest is lighter than four marbles. Consequently the middle marble will be lighter than one shaded and two white marbles and heavier than the three others. It follows that this middle marble is also the middle of the seven marbles. Therefore two additional weighings will sort the seven marbles: one for three lightest and another for the three heaviest.

With these new results, the strategy to sort 15 marbles can be improved. Use six weighings to form a chain containing seven marbles, and one more to weigh the heaviest with two additional marbles. This yields a *v*-chain of length nine. Next, use the scale twice more to form two disjoint chains of length three with the six remaining marbles. Theorem 1 ensures that 13 more weighings suffice, for a total of 22.

Theorem 6: 8 weighings suffice to sort 8 marbles.

Proof: Perform the same first three weighings on seven of the eight marbles as in Theorem 5, leading to the following graph.

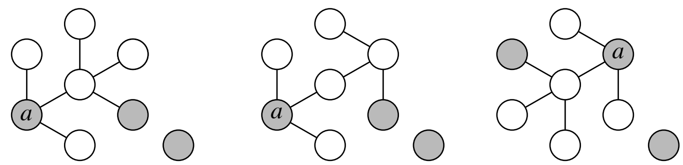

FIGURE 4

With a fourth weighing, compare the three shaded marbles. At this point, the marble a partitions the seven other marbles into a group of lighter and a group of heavier marbles. All possibilities are represented in the figure below (without any loss of generality, symmetrical figures are omitted). The black dot represents marble a. The relative weights of marble a and the other two shaded marbles have been shown, but the relationship between them and the unshaded marbles have been ignored. The cardinality of the two groups are indicated on the left side of each subfigures.

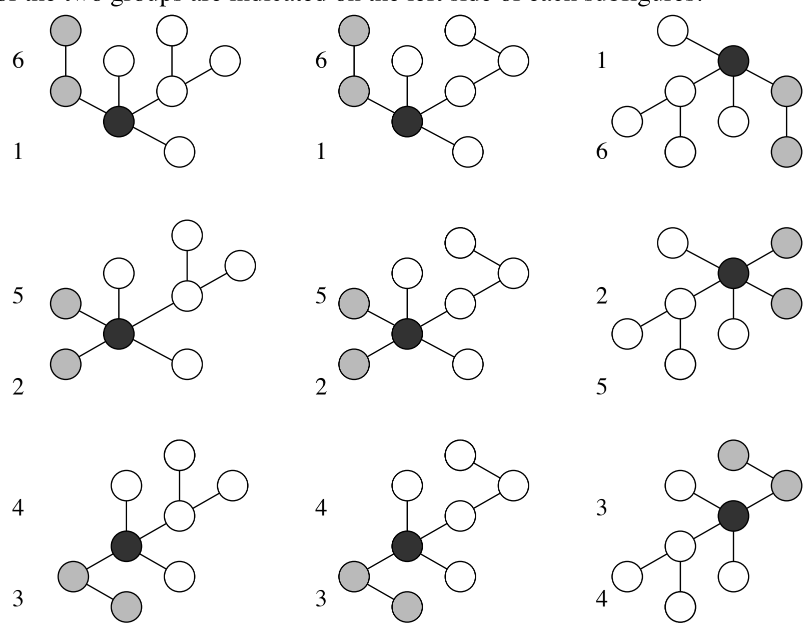

FIGURE 5

If the cardinality of the groups are 1 and 6, then the group of six contains two chains and a v-chain (recall that a chain can be regarded as a special case of a v-chain). Theorem 1 ensures that four additional weighing suffice, for a total of 8.

If the cardinality of the groups are 2 and 5, then the group of five contains two chains and one v-chain, and again, Theorem 1 guarantees that three additional weighings suffice for that group. The group of two requires a single weighing, for a total of 8.

If the cardinality of the groups are 3 and 4, then the group of four may be easily sorted using two additional weighings, and the group of three requires a single weighing, for a total of 7.

Theorem 7: 9 weighings suffice to sort 9 marbles.

Proof: Split the marbles into three groups of three, and weigh each of them. Take the middle one of each chain and weigh them. Let a, b and c denote the heaviest, middle one and lightest of the last weighing. After four weighings, this leads to the following figure:

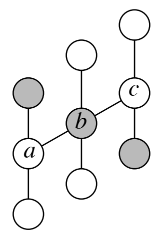

FIGURE 6

With a fifth weighing, compare the three shaded marbles. At this point, every marble has been compared with b, partitioning the marbles into a group of lighter and a group of heavier marbles.

If the cardinality of the groups are three and five, then the group of five is composed of three chains and Theorem 1 ensures that three additional weighing suffice for that group. The group of three requires a single weighing, for a total of 9.

If the both groups have four marbles, then each is composed of three chains and may be sorted using two weighings for a grand total of 9.

Notice that all the strategies to sort six to nine marbles start out by using the scale twice to sort six distinct marbles. This implies for instance that if a set of nine marbles needs to be sorted, and if a chain of length 3 is known, then 8 weighings are sufficient to sort the marbles instead of 9. Similarly, 7 weighings are sufficient when two disjoints chains of length 3 are known.

3. *Twenty weighings to sort* 15 *marbles*

Before presenting our main result, we need to derive preliminary lemmas providing a sufficient number of weighings to rank specific configurations of marbles.

Lemma 1: 4 weighings suffice to sort six marbles containing two disjoint chains of length 2.

Proof: Compare the heaviest marble in each of the two chains with one of the remaining marbles. Let b denote the heaviest, c the lightest and a the middle one. It follows that b is heavier than the lightest marbles of both original chains, and that a is heavier than at least one of them. This produces the following configuration:

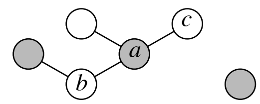

FIGURE 7

Use the scale a second time to weigh the three shaded marbles. The marble a, depicted in black below, partitions the others into a group of light and a group of heavier marbles.

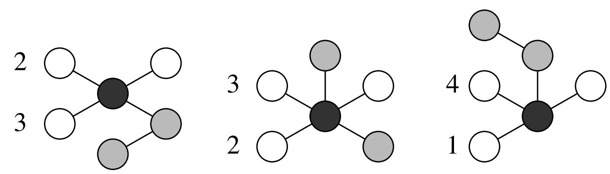

FIGURE 8

If the cardinality of these groups are 2 and 3, then two additional weighings will sort them all. Otherwise, the shaded marbles form a chain with a as the heaviest, and the four lighter marbles may be sorted with two weighings.

In both cases, a grand total of 4 weighings are required.

Lemma 2: 5 weighings suffice to sort seven marbles containing three disjoint chains of length 2.

Proof: Weighing the heaviest marble of each of the three chains yields the following configuration:

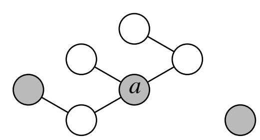

FIGURE 9

Use the scale a second time to weigh the three shaded marbles. The marble a now partitions the others into a group of light and a group of heavier marbles.

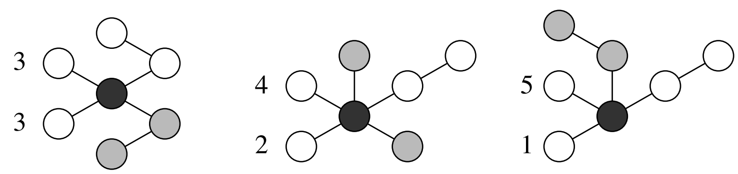

FIGURE 10

If the cardinality of these groups are both equal to 3, then two additional weighings will sort them all. If the cardinality of these groups are 2 and 4, then one additional weighing for the group of two, and 2 weighings for the group of four will sort them all. Otherwise, the shaded marbles form a chain with a as the heaviest, and the five lighter marbles may be sorted with three weighings.

In all three cases, a total of at most 5 weighings are required.

Lemma 3: 13 weighings suffice to sort 15 marbles ordered as in the Figure 11.

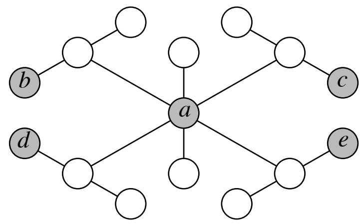

FIGURE 11

Proof: Use the scale twice to weigh a, b, c and a, d, e. The marble a partitions the 14 others into a group of lighter ones, and another of heavier ones. Before these two weighings the unshaded marbles are already partitioned into five lighter and five heavier than a, so after the weighings there are just three possibilities for the cardinalities of the sets of heavy and light marbles: $(7,7)$, $(6,8)$ and $(5,9)$. By symmetry the graphs of $(8,6)$ and $(9,5)$ need not be considered.

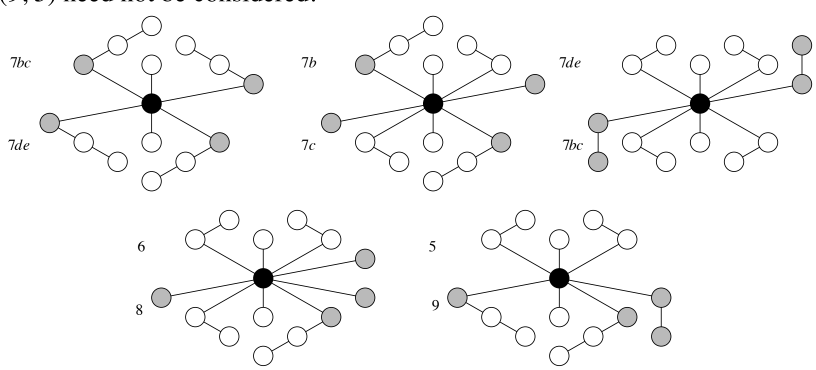

FIGURE 12

Case 1: The cardinality of both groups is 7.

If b and c are both lighter than a (as in the top left figure), then there are two chains of length 3 in the light marbles, and the proof of Theorem 5 ensures that 4 additional weighings suffice to sort the light marbles and 4 for the heavy ones.

If only one of b or c is lighter than a (as in the top middle figure), then the light marbles contain a chain of length three, and Theorem 5 ensures that 5 additional weighings suffice to sort the light marbles and 5 for the heavy ones.

Otherwise, d and e are lighter than a, and both the lighter and heavier groups contain three disjoint chains of length 2, with a fourth chain having a single marble. Lemma 2 ensures that 5 weighings suffice for each group.

Case 2: The cardinality of the groups are 6 and 8.

The group of six contains two disjoint chains of length two (as in the bottom left figure above), thus Lemma 1 guarantees that 4 weighings suffice. The group of eight will necessarily contain a chain of length three, and therefore the proof of Theorem 6 ensures that 7 weighings will sort them.

Case 3: The cardinality of the groups are 5 and 9.

The group of five contains two disjoint chains of length two (as in the bottom right figure in Figure 11), thus Theorem 1 ensures that 3 weighings suffice. The group of nine will necessarily contain two disjoint chains of length 3, and therefore the proof of Theorem 7 ensures that 7 weighings suffice.

The largest number of weighings for these cases is

$$\max\{5 + 5,\ 4 + 7,\ 3 + 7\} = 11,$$

to which we add the 2 initial ones, for a total of 13.

Lemma 4: 6 weighings suffice to sort eight marbles ordered as in Figure 13.

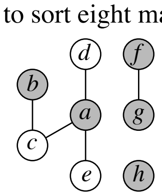

FIGURE 13

Proof: Use the scale twice to compare a, g, h and then a, b, f. Notice that g and h were compared and therefore f, g, h form a v-chain rooted at the heaviest of g and h. The marble a now partitions all other marbles into a group of lighter and a group of heavier marbles. The lighter group contains at least one marble, and up to five. We consider each case appealing to Theorem 1 throughout.

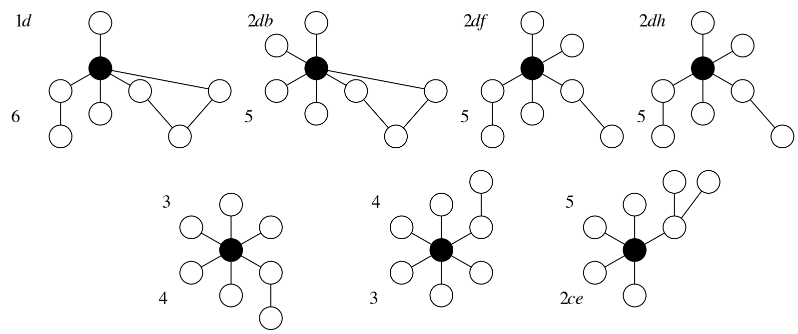

FIGURE 14

Case 1: Only d is lighter than a.

The marbles f, g, h form a v-chain, b, c a chain of length two and e a singleton and therefore four more weighings suffice to sort the six heaviest marbles.

Case 2: Exactly two marbles are lighter than a.

One weighing will sort the two light marbles. If the light marbles are d and b, then the heavy ones contain the v-chain f, g, h and the two singletons c and e. If the light marbles are d and f, then the heavy ones contain two chains of length 2: b, c and g, h and the singleton e. Otherwise, the light marbles are d and h, and the heavy ones contain two chains of length 2: b, c and f, g and the singleton e.

In each of the three subcases, three more weighings suffice to sort the 5 heavy marbles.

Case 3: Three or four marbles are lighter than a.

It follows that there are four or three heavier ones, respectively. The group of three requires a single weighing. The group of four contains a chain of length 2 and two additional weighings suffice for that group.

Case 4: Only c and e are heavier than a.

The two heavy marbles are ordered using a single weighing. The marbles f, g, h form a v-chain, b and d are two singletons and therefore, three more weighings suffice to sort the five lightest marbles.

In all cases, a total of 6 weighings suffices: 2 for splitting the marbles with respect to a and 4 to sort the two partitions.

Lemma 5: 12 weighings suffice to sort 15 marbles ordered as in Figure15.

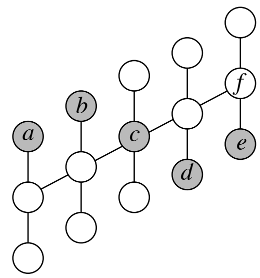

FIGURE 15

Proof: Notice that both d and e are heavier than f. This will be useful in the proof below.

Use the scale twice to weigh a, b, c and c, d, e. We now know if each of the 14 marbles is heavier or lighter than c. Consider the group of marbles heavier than c and the group of marbles lighter than c.

If the cardinality of the groups are 9 and 5, then the group of nine contains two chains of length three and either Theorem 1 or Theorem 7 ensure that 7 additional weighings suffice for that group. The group of five may easily be sorted following Theorem 1, using 3 weighings, for a total of 12.

Consider the situation where the cardinality of the groups are 8 and 6. By symmetry, assume without any loss of generality that the group of six contains the heavy marbles. Two sub-cases need to be considered as illustrated in Figure 16 where c is black and a, b, d and e are shaded.

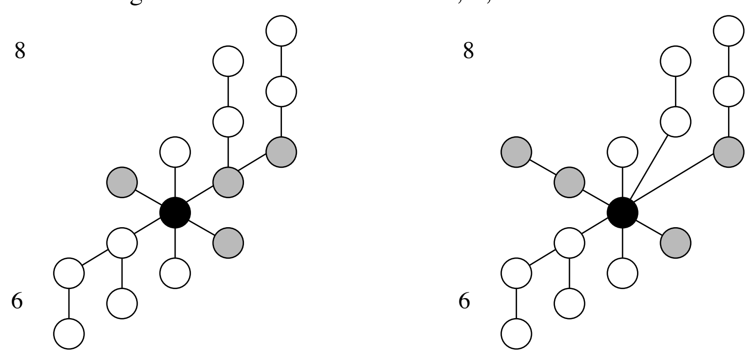

FIGURE 16

Case 1: Only a or b is heavier than c.

The situation is illustrated in the left plot. In both cases the six heavier marbles form contain an inverted v-chain and two singletons (the fact that the v-chain is inverted does not introduce any difficulties). They may be sorted using 4 weighings. The eight lighter marbles contain one v-chain and two singletons, and therefore can be ordered using 6 weighings for a total of 12.

Case 2: Only *d* or *e* is heavier than *c*.

Recall that both *d* and *e* are heavier than *f*, and therefore the situation is illustrated in the right plot. The heavier marbles contain one inverted *v*-chain and two singletons, and therefore 4 weighings suffice to sort them. The eight lighter marbles follow the structure depicted in Lemma 4 and may be sorted using 6 weighings. A total of 12 weighings is used.

Finally, consider the situation where both groups have 7 marbles. Theorem 5 ensures that a total of 12 weighings suffice to sort them all as each group contains a chain of length three.

We now have all the necessary results to state and prove our main result.

Theorem 8: 20 weighings suffice to sort 15 marbles.

Proof: Partition the 15 marbles into five groups of three, and weigh them. Use a sixth weighing to compare the middle marbles of three groups, and let *a* denote the middle one. The following graph illustrates the situation.

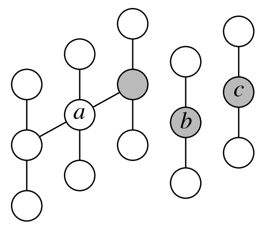

FIGURE 17

Use a seventh weighing to compare *a*, *b*, *c*. Without any loss in generality, assume that *b* is heavier than *c*. Three cases need to be considered.

Case 1: *a* is the middle marble of the seventh weighing.

The marble *b* is heavier than *a*, and *c* is lighter than *a*. The situation is exactly that of Lemma 3, and therefore 13 additional weighings suffice for a total of 20.

Case 2: *a* is the heaviest marble of the seventh weighing.

With an eighth weighing, compare the three shaded marbles. Since these three marbles are lighter than *a*, the resulting graph is exactly that of Lemma 5, and 12 additional weighings suffice, for a grand total of 20.

Case 3: *a* is the lightest marble.

The situation is isomorphic to the previous one, where the roles of heaviest and lightest are reversed.

In summary, 15 marbles can always be sorted using 20 weighings.

4. *Discussion*

We proposed a strategy to sort 15 marbles and proved that it requires at most 20 weighings. Is there a better strategy? Is there a strategy that can guarantee at most 19 weighings to sort the marbles? We do not know the answer to these questions. But we do know that 16 weighings is a valid lower bound on that number, as shown by the following proposition.

Proposition: For the more general problem of sorting p marbles, the best strategy would require at least $\lceil \log_6 (p!) \rceil$ weighings.

Proof: There are $p!$ possible permutations of the marbles. Each weighing compares three marbles, and there are six possible outcomes. So after the first weighing, the $p!$ permutations will be partitioned into six groups, associated with each of the six possible outcomes. At least one of the groups will contain at least $\frac{1}{6}p!$ permutations.

By induction, after an mth weighing, the permutations will be partitioned into six groups, associated with each of the six possible outcomes. At least one of the groups will contain at least $\lceil \frac{p!}{6^m} \rceil$ permutations, where $\lceil \cdot \rceil$ rounds to the next integer.

Therefore, if $\frac{p!}{6^m} > 1$ then m weighings are not sufficient. It follows that the minimal number of weighings m must be such that $6^m \geqslant p!$.

The last proposition gives a lower bound on the minimal number of weighings. For $p = 15$, we have $\log_6 (15!) > 15.57$. The strategy presented in the paper gives an upper bound. Bridging the gap between these two bounds would give the optimal strategy.

Acknowledgements

The author would like to thank Jean Guérin as well as an anonymous referee for meticulous proof reading and constructive suggestions.

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