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On the use of Boolean methods for the computation of the stability number

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Abstract

We study a transformation of pseudo-Boolean functions which, when applicable, amounts to constructing from a graph $G = (V, E)$ a new graph $G' = (V', E')$ with the same stability number and such that $|V'| = |V| - 1$. This transformation provides a polynomial time algorithm for the computation of the stability number of graphs which contain neither an induced chordless cycle with four vertices, nor its complement. The transformation might also be useful for reducing the size of a claw-free graph while preserving its stability number.

Keywords: Boolean methods; Stability number

1. Introduction

In a simple graph $G = (V, E)$, a set S of vertices is *stable* (or *independent*) if no two vertices in S are linked by an edge. A stable set S is *maximum* if its cardinality $|S|$ is maximum. The maximum cardinality of a stable set in G is denoted $\alpha(G)$ and is called the *stability number* of G. For a weighted graph G, the maximum weight of a stable set is denoted $\alpha_w(G)$.

Given a positive integer k, finding whether an arbitrary graph contains a stable S set with $|S| \geqslant k$ is NP-complete [9]. However, there are special classes of graphs for which $\alpha(G)$ can be computed in polynomial time (e.g. [1, 5, 6, 11, 12, 14, 15, 18–22, 24, 25]).

Ebenegger et al. [7] have described the relation between the maximization of a pseudo-Boolean function and the determination of a stable set having maximum weight in a graph. In the same paper, the authors consider the computation of the stability number $\alpha(G)$ of a graph G (unweighted case) and describe a transformation of an associated pseudo-Boolean function which amounts to constructing another graph

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G' with $\alpha(G') = \alpha(G) - 1$. By repeatedly applying this construction, one may compute $\alpha(G)$ in at most $\alpha(G) \leqslant |V|$ steps. Unfortunately, the number of vertices is generally increasing when the transformation is applied. However, specialized versions of this construction have provided polynomial time algorithms for some classes of graphs [11, 14, 15, 18].

More recently, Hammer and Hertz [13] have studied a simplification of pseudo-Boolean functions which, when applicable, amounts to constructing from a graph $G = (V, E)$ another graph $G' = (V', E')$ with $|V'| = |V| - 1$ and $\alpha(G') = \alpha(G)$. This construction provides a polynomial time algorithm for the computation of the stability number in some classes of graphs.

We study, in this paper, another simplification of pseudo-Boolean functions and it is shown that the proposed transformation, when applicable, also amounts to constructing from a graph $G = (V, E)$ a new graph $G' = (V', E')$ with $|V'| = |V| - 1$ and $\alpha(G') = \alpha(G)$.

In the next section, we briefly recall the relation described in [7] between the maximization of a pseudo-Boolean function and the determination of the stability number. Previous studies based on the use of Boolean methods for the computation of the stability number are summarized in Section 3. The new graph transformation is described in Section 4 and applied to classes of graphs in Section 5.

Let $G = (V, E)$ be a graph and $W \subseteq V$ a subset of its vertices. We denote by $E(W)$ the subset of edges in E having both endpoints in W. The subgraph *induced* by W is the graph $G = (W, E(W))$ and is denoted by $G[W]$. A graph $H = (V', E')$ is a *partial* subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E(V')$. The *complement* of G which is denoted $\bar{G}$ has the same vertex set V as G, while two vertices are linked in $\bar{G}$ if and only if they are not linked in G. A *clique* is a set of pairwise adjacent vertices. Hence, a set of vertices is a clique in a graph G if and only if it is stable in $\bar{G}$. The maximum cardinality of a clique in G is denoted $\omega(G)$ and is called the *clique number* of G.

A graph $G = (V, E)$ is *bipartite* if its vertex set V can be partitioned into two sets V_1 and V_2 such that each edge of E has one endpoint in V_1 and the other in V_2; such a graph will be denoted $G = ((V_1, V_2), E)$. A bipartite graph $G = ((V_1, V_2), E)$ is *complete* if each vertex in V_1 is adjacent to each vertex in V_2. A *star* is a complete bipartite graph $G = ((V_1, V_2), E)$ with $|V_1| = 1$ or $|V_2| = 1$.

For two sets A and B, $A \backslash B$ denotes the set of elements which are in A, but not in B. For a graph $G = (V, E)$ and vertices a, b and c in V, we denote

$[a, b]$	an edge linking the vertices a and b;
$N_G(a)$	the set of vertices which are adjacent to a in G (i.e., $\{v \mid [a, v] \in E\}$);
$N_G[a]$	the set $N_G(a) \cup \{a\}$;
$\mathscr{C}_G(a, b, c)$	the set $N_G(a) \cap N_G(b) \cap N_G(c)$;
$\mathscr{I}_G(^{a}_{bc})$	the set $N_G(a) \backslash (N_G[b] \cup N_G[c])$;
$\mathscr{O}_G(^{a}_{bc})$	the set $(N_G(b) \cap N_G(c)) \backslash N_G[a]$.

We denote $P_q(v_1, \ldots, v_q)$ the chordless chain on q vertices with edges $[v_i, v_{i+1}]$ $(1 \leqslant i < q)$, while $C_q(v_1, \ldots, v_q)$ denotes the chordless cycle on q vertices with edges $[v_i, v_{i+1}]$ $(1 \leqslant i < q)$ and $[v_q, v_1]$.

All graph-theoretical terms not defined here are borrowed from [2] while for pseudo-Boolean definitions, the reader is referred to [16].

2. Posiforms and conflict graphs

It is known that a pseudo-Boolean function f can always be written in a polynomial form i.e.,

$$f(x_1, \ldots, x_n) = K + \sum_{i=1}^{p} w_i T_i,$$

where $T_i = \prod_{j \in A_i} x_j \prod_{k \in B_i} \bar{x}_k$ with $A_i, B_i \subseteq \{1, \ldots, n\}$ and $A_i \cap B_i = \emptyset$.

If all w_i $(1 \leqslant i \leqslant p)$ are strictly positive and $K = 0$, we say that f is a *posiform*. To a posiform f we associate a weighted *conflict graph* $G = (V, E)$ defined as follows:

$$V = \{1, \ldots, p\} \text{ and each vertex } i \text{ has a weight } w_i,$$
$$E = \{[i, j] \mid \exists\, k \in ((A_i \cap B_j) \cup (A_j \cap B_i))\}.$$

In other words, two vertices i and j of G are linked by an edge if x_k appears in T_i (or T_j) while $\bar{x}_k$ appears in T_j (or T_i). It is clear from the definition of G that the maximum value of f is equal to the maximum weight $\alpha_w(G)$ of a stable set in G.

Conversely, for each graph $G = (V, E)$ with positive weights w_u associated with each vertex $u \in V$, there exist posiforms f such that G is the conflict graph of f [7]. Indeed, consider an arbitrary covering of the edge set E by complete bipartite partial subgraphs $G_i((V_{i_1}, V_{i_2}), E_i)$ of G, $i = 1, \ldots, q$. Then set

$$f = \sum_{u \in V} w_u T_u,$$

where $T_u = \prod_{j \in A_u} x_j \prod_{k \in B_u} \bar{x}_k$ with $A_u = \{i \mid u \in V_{i_1}\}$, $B_u = \{i \mid u \in V_{i_2}\}$.

Let T_u and T_v be two terms of the posiform f such that x_i appears in T_u and $\bar{x}_i$ appears in T_v. Then $u \in V_{i_1}$ and $v \in V_{i_2}$. Hence, the edge $[u, v]$ belongs to $E_i \subseteq E$, showing that G is the conflict graph associated with f.

Notice that given a graph $G = (V, E)$, there might exist different coverings of E by complete bipartite partial subgraphs. Each covering corresponds to a posiform f such that G is the conflict graph of f. Consider, for example, the two different coverings of the graph G represented in Fig. 1. These coverings induce two posiforms f and f' having the same maximum value which is equal to 5. The maximum value of f is obtained by setting $x_1 = x_2 = 0$, which means that $T_a = T_c = x_1 = 0$, $T_b = \bar{x}_1 \bar{x}_2 = 1$, $T_d = x_2 \bar{x}_1 = 0$ and $T_e = \bar{x}_2 = 1$. For f', the maximum value is reached for $x_1 = 0$ and

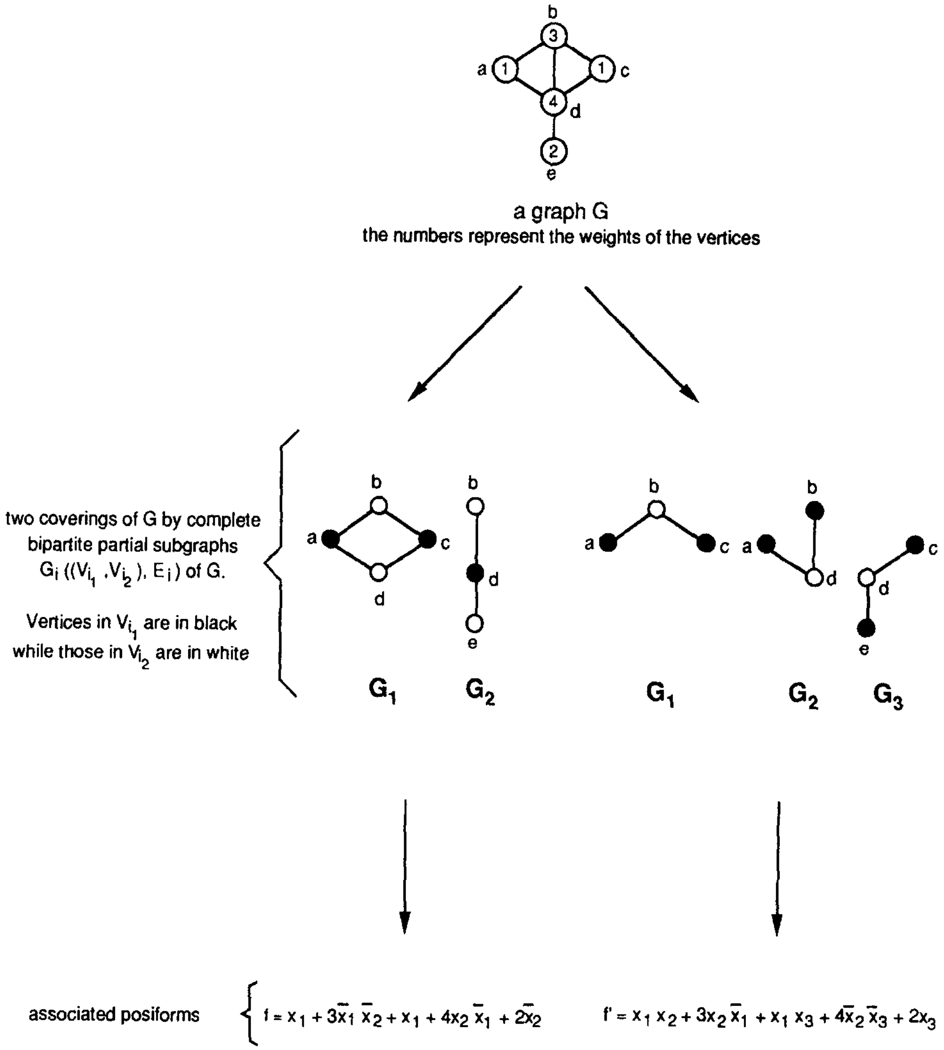


Fig. 1.

$x_2 = x_3 = 1$, hence, $T_a = x_1x_2 = 0$, $T_b = x_2\bar{x}_1 = 1$, $T_c = x_1x_3 = 0$, $T_d = \bar{x}_2x_3 = 0$ and $T_e = x_3 = 1$. Both settings of the variables of f and f' correspond to the stable set $S = \{b, e\}$ of weight 5 in G.

3. Known graph transformations based on Boolean methods

We describe, in this section, two known graph transformations which are based on Boolean methods, and which have been exploited for the computation of the stability number of graphs. The first one can be applied to any graph $G = (V, E)$ and amounts to building a new graph G' with $\alpha(G') = \alpha(G) - 1$. The second transformation, when applicable, amounts to constructing from a graph $G = (V, E)$ another graph $G' = (V', E')$ with $|V'| = |V| - 1$ and $\alpha(G') = \alpha(G)$.

3.1. The Struction

The Struction (for STability number RedUCTION) [7] is a procedure which, given a graph $G = (V, E)$, constructs a new graph $G' = (V', E')$ with $\alpha(G') = \alpha(G) - 1$. Let a_0 be any vertex in V and let $N_G(a_0) = \{a_1, \dots, a_p\}$ and $W = V \setminus N_G[a_0] = \{a_{p+1}, \dots, a_{|V|-1}\}$. The Struction is based on the following covering of E with $|V| - 1$ stars:

– For each vertex $a_i \in N_G(a_0)$ $(1 \leqslant i \leqslant p)$ consider the star $G_i = ((V_{i_1}, V_{i_2}), E_i)$ with $V_{i_1} = \{a_i\}$ and $V_{i2} = \{a_0\} \cup \{a_j \in N_G(a_i) \cap N_G(a_0) | j > i\} \cup \{a_j \in N_G(a_i) \cap W\}$.
– For each vertex $a_i \in W$ $(p + 1 \leqslant i \leqslant |V| - 1)$ consider the star $G_i = ((V_{i_1}, V_{i_2}), E_i)$ with $V_{i_1} = \{a_i\}$ and $V_{i_2} = \{a_j \in N_G(a_i) \cap W \mid j > i\}$.

This covering defines the following terms of the associated posiform $f = \sum_{i=0}^{|V|-1} w_{a_i} T_{a_i}$:

$$T_{a_0} = \prod_{i=1}^{p} \bar{x}_i,$$

$$T_{a_i} = \begin{cases} x_i \prod\limits_{\substack{a_j \in N_G(a_i) \\ 1 \leqslant j < i \leqslant p}} \bar{x}_j & (1 \leqslant i \leqslant p), \\ x_i \prod\limits_{a_j \in N_G(a_i) \cap N_G(a_0)} \bar{x}_j \prod\limits_{\substack{a_j \in N_G(a_i) \cap W \\ p < j < i < |V|}} \bar{x}_j & (p < i < |V|). \end{cases}$$

It is proved in [7] that

$$\sum_{i=0}^{p} T_{a_i} = 1 + \sum_{\substack{a_q \notin N_G(a_r) \\ 1 \leqslant q < r \leqslant p}} x_q x_r \prod_{1 \leqslant s < q} \bar{x}_s \prod_{\substack{a_t \notin N_G(a_r) \\ q < t < r}} \bar{x}_t.$$

Hence, in the case where all weights are equal to 1 (unweighted case), f can be rewritten as $1 + g$ where g is also a posiform. The conflict graph G' associated with g satisfies $\alpha(G') = \alpha(G) - 1$ and it is shown in [7] that it can be obtained directly from G by the following transformation:

(a) For each q $(1 \leqslant q < p)$ define the *layer* $L_q = \{(q, r) | q < r \leqslant p \text{ and } a_q \notin N_G(a_r)\}$.
(b) The vertex set V' of G' is equal to $W \cup (\bigcup_{1 \leqslant q < p} L_q)$.
(c) The edge set of G' consists of
– all edges of $G[W]$;
– all edges linking vertices in different layers;
– edges linking two vertices (q, r) and (q, s) in the same layer L_q if $a_r \notin N_G(a_s)$;
– edges linking a vertex (q, r) in L_q $(1 \leqslant q < p)$ with a vertex $a_i \in W$ if $a_i \in N_G(a_q) \cup N_G(a_r)$.

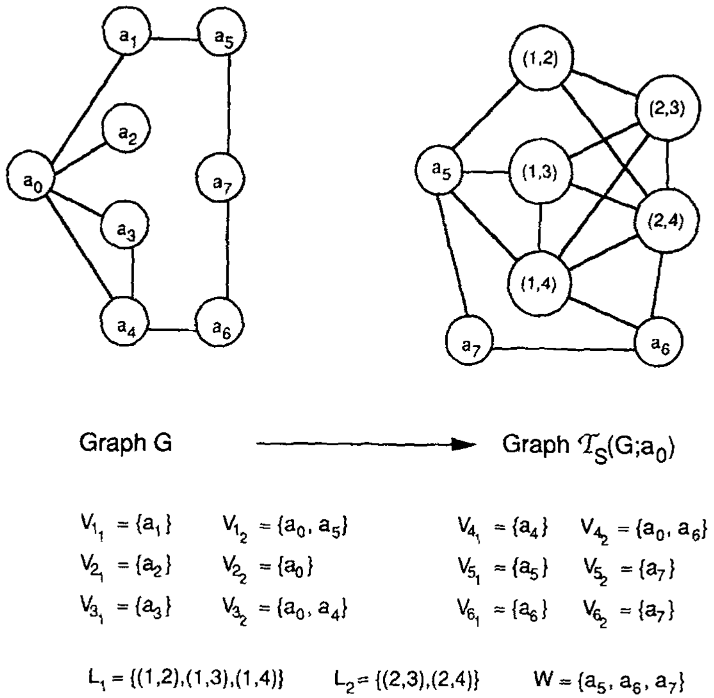


Fig. 2.

We shall denote $G' = \mathcal{T}_S(G; a_0)$. An example of this transformation is represented in Fig. 2. Notice that the number of vertices of G' is in $O(|V|^2)$. For certain classes of graphs, one can avoid this potentially exponential growth of the number of vertices, thus giving a polynomial time algorithm for the computation of the stability number of graphs in those classes [11, 14, 15, 18]. The Struction can be extended to the weighted case as well. For more details, the reader is referred to [7].

3.2. *Magnets*

Hammer and Hertz [13] have studied a transformation which, when applicable, amounts to building from a graph $G = (V, E)$ a new graph $G' = (V', E')$ with $|V'| = |V| - 1$ and $\alpha(G') = \alpha(G)$.

A *magnet* in a graph $G = (V, E)$ is defined as a pair (a, b) of adjacent vertices with the same weight and such that each vertex in $N_G(a)\backslash N_G(b)$ is adjacent to each vertex in $N_G(b)\backslash N_G(a)$. Let G be a graph containing a magnet (a, b). The edges incident to a or b can be covered by the two following complete bipartite partial subgraphs G_1 and G_2 of G:

$G_1 = ((V_{1_1}, V_{1_2}), E_1)$ with $V_{1_1} = N_G(b)\backslash N_G(a)$ and $V_{1_2} = N_G(a)\backslash N_G(b)$,

$G_2 = ((V_{2_1}, V_{2_2}), E_2)$ with $V_{2_1} = \{a, b\}$ and $V_{2_2} = N_G(a)\cap N_G(b)$.

Let us now consider any covering of the edges in $E\backslash(E_1 \cup E_2)$ by complete bipartite partial subgraphs $G_3, \ldots, G_q$. The graphs $G_1, \ldots, G_q$ cover all the edges of E and the

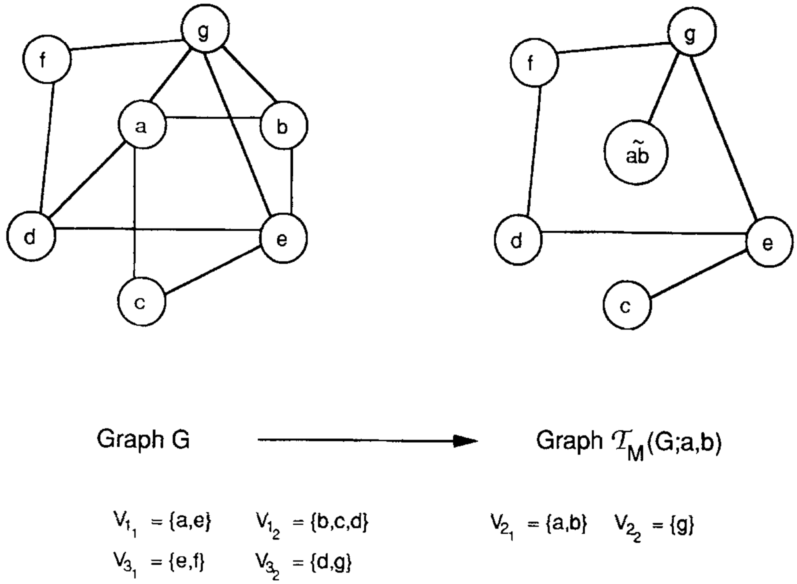


Fig. 3.

associated posiform $f = \sum_{v\in V} w_v T_v$ satisfies $T_a = x_1 x_2$ and $T_b = \bar{x}_1 x_2$. Hence, $T_a + T_b = (x_1 + \bar{x}_1)x_2 = x_2$. It follows that f has the same maximum value as the posiform $g = \sum_{v\in V\setminus\{a,b\}} w_v T_v + w_a x_2$. This means that the conflict graph $G' = (V', E')$ associated with g satisfies $\alpha_w(G') = \alpha_w(G)$ and $|V'| = |V| - 1$.

The graph G' can be obtained directly from G by replacing the vertices a and b by a new vertex $\widetilde{ab}$ having the same weight as a and b, and linked to every common neighbor of a and b in G (hence, $N_{G'}(\widetilde{ab}) = N_G(a) \cap N_G(b)$). We shall denote $G' = \mathcal{T}_M(G;\, a, b)$. An example of this transformation is represented in Fig. 3. More details on magnets are given in [13].

4. A new graph transformation

We describe in this section a new graph transformation which is based on the following Boolean equality:

$$\bar{x}\bar{y} + x + y = 1 + xy.$$

Let us consider a posiform $f = \sum_{i=1}^{p} w_i T_i$ with $T_i = \prod_{j\in A_i} x_j \prod_{k\in B_i} \bar{x}_k$ and let us assume that there exist three terms T_a, T_b, T_c with equal weight $w_a = w_b = w_c$ and two indices q and r such that the two following conditions are satisfied:

(i) $A_b = A_a \cup \{q\}$ and $A_c = A_a \cup \{r\}$;
(ii) $r \notin B_b$, $q \notin B_c$ and $B_a = B_b \cup \{q, r\} = B_c \cup \{q, r\}$.

The sum of the three terms T_a, T_b and T_c of the above posiform f can be reduced to the sum of two terms $\tilde{T}_a$ and $\tilde{T}_{bc}$ as follows:

$$T_a + T_b + T_c = \bar{x}_q\bar{x}_r\left(\prod_{j\in A_a} x_j \prod_{k\in B_a\setminus\{q,r\}} \bar{x}_k\right) + x_q\left(\prod_{j\in A_b\setminus\{q\}} x_j \prod_{k\in B_b} \bar{x}_k\right)$$

$$+ x_r\left(\prod_{j\in A_c\setminus\{r\}} x_j \prod_{k\in B_c} \bar{x}_k\right)$$

$$= (\bar{x}_q\bar{x}_r + x_q + x_r)\left(\prod_{j\in A_a} x_j \prod_{k\in B_b} \bar{x}_k\right)$$

$$= (1 + x_q x_r)\left(\prod_{j\in A_a} x_j \prod_{k\in B_b} \bar{x}_k\right)$$

$$= \left(\prod_{j\in A_a} x_j \prod_{k\in B_b} \bar{x}_k\right) + \left(\prod_{j\in A_a\cup\{q,r\}} x_j \prod_{k\in B_b} \bar{x}_k\right)$$

$$= \tilde{T}_a + \tilde{T}_{bc}.$$

Now, let us consider the posiform $g = \sum_{\substack{1\leqslant i\leqslant p\\ i\neq a,b,c}} w_i T_i + w_a(\tilde{T}_a + \tilde{T}_{bc})$.

The conflict graph $G' = (V', E')$ associated with g has one vertex less than the conflict graph $G = (V, E)$ associated with f and $\alpha_w(G) = \alpha_w(G')$, since f and g have the same maximum value. The vertex set V' is obtained from V by replacing vertices a, b and c by two new vertices $\tilde{a}$ and $\widetilde{bc}$ associated with the terms $\tilde{T}_a$ and $\tilde{T}_{bc}$ of g.

In the graph G', the vertex $\tilde{a}$ is adjacent to each vertex i ($1 \leqslant i \leqslant p$, $i \neq a, b, c$) such that $A_a \cap B_i \neq \emptyset$ or $B_b \cap A_i \neq \emptyset$, and the vertex $\widetilde{bc}$ is adjacent to each vertex i ($1 \leqslant i \leqslant p$, $i \neq a, b, c$) such that $(A_a \cup \{q, r\}) \cap B_i \neq \emptyset$ or $B_b \cap A_i \neq \emptyset$.

This means that the edge set E' of G' is obtained from G by removing all edges incident to a, b or c and by linking $\tilde{a}$ to each vertex in $N_G(a) \cap N_G(b) \cap N_G(c) = \mathscr{C}_G(a, b, c)$ and $\widetilde{bc}$ to each vertex in $(N_G(b) \cup N_G(c)) \setminus \{a, b, c\}$.

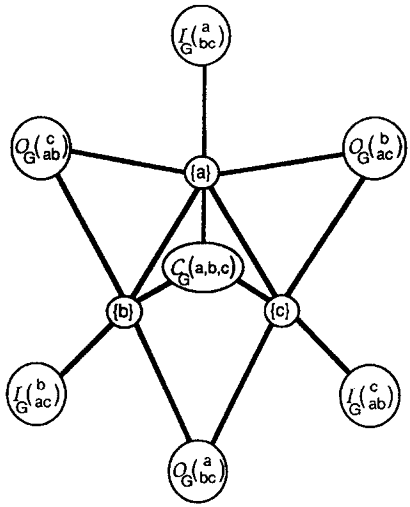
Relations (i) and (ii) satisfied by the terms T_a, T_b and T_c of f induce relations between a, b and c in the conflict graph $G = (V, E)$ associated with f. First notice that a is adjacent to b and c since $q \in A_b \cap B_a$ and $r \in A_c \cap B_a$. Moreover, b is not adjacent to c, since $B_b = B_c$. Hence, G contains an induced $P_3(b, a, c)$.

If v is a vertex in $\mathscr{I}_G(\substack{a\\bc})$ then q or (not exclusive) r belongs to A_v. So, let us partition the set $\mathscr{I}_G(\substack{a\\bc})$ into the three following subsets:

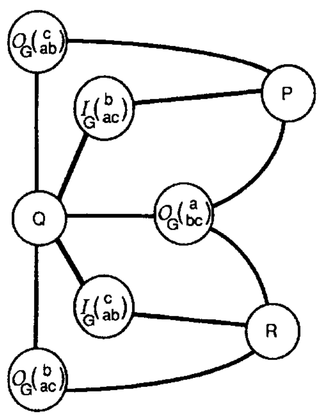
$$P = \{v \in \mathscr{I}_G(\substack{a\\bc}) \mid A_v \cap \{q, r\} = \{q\}\},$$

$$Q = \{v \in \mathscr{I}_G(\substack{a\\bc}) \mid A_v \cap \{q, r\} = \{q, r\}\},$$

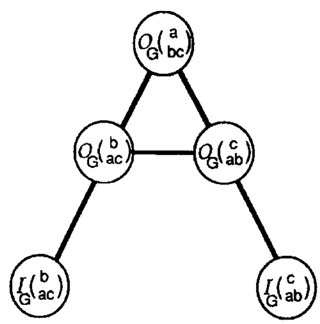
$$R = \{v \in \mathscr{I}_G(\substack{a\\bc}) \mid A_v \cap \{q, r\} = \{r\}\}.$$



4.1 considered partition of the edges of G incident to a, b or c



4.2 constraints on the set P∪Q∪R= $\mathscr{I}_G\binom{a}{bc}$



4.3 constraints on the sets $\mathscr{O}_G\binom{b}{ac}$ and $\mathscr{O}_G\binom{c}{ab}$

(A)——(B) means that each vertex in A is adjacent to each vertex in B

a BAT $\binom{a}{bc}$

Fig. 4.

According to conditions (i) and (ii), we have $q \in B_v$ for each $v \in \mathscr{I}_G\binom{b}{ac} \cup \mathscr{O}_G\binom{a}{bc} \cup \mathscr{O}_G\binom{c}{ab}$ and $r \in B_v$ for each $v \in \mathscr{I}_G\binom{c}{ab} \cup \mathscr{O}_G\binom{a}{bc} \cup \mathscr{O}_G\binom{b}{ac}$. Hence, we have the following configuration (see Fig. 4.2):

– each vertex in P is adjacent to each vertex in $\mathscr{I}_G\binom{b}{ac} \cup \mathscr{O}_G\binom{a}{bc} \cup \mathscr{O}_G\binom{c}{ab}$;

– each vertex in Q is adjacent to each vertex in $\mathscr{I}_G(^{b}_{ac}) \cup \mathscr{I}_G(^{c}_{ab}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{b}_{ac}) \cup \mathscr{O}_G(^{c}_{ab})$;
– each vertex in R is adjacent to each vertex in
$\mathscr{I}_G(^{c}_{ab}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{b}_{ac})$.

Let us now consider any vertex v in $\mathscr{O}_G(^{b}_{ac})$. Since, v is adjacent to a and c, but not to b we know that $q \in A_v$. It follows that each vertex in $\mathscr{O}_G(^{b}_{ac})$ is adjacent to each vertex in $\mathscr{I}_G(^{b}_{ac}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{c}_{ab})$. By symmetry between b and c, each vertex in $\mathscr{O}_G(^{c}_{ab})$ is adjacent to each vertex in $\mathscr{I}_G(^{c}_{ab}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{b}_{ac})$ (see Fig. 4.3).

It can be observed in Fig. 4 that the vertices in $V \setminus \{a, b, c\}$ which are adjacent to a, b or c induce two configurations in G which look like the letters B and A. We shall say that a, b and c induce a BAT$(^{a}_{bc})$ in G (for BA-triplet).

Definition. A BAT$(^{a}_{bc})$ in a graph G is a set of three vertices a, b, c with equal weight and satisfying the following properties:
– G contains an induced $P_3(b, a, c)$;
– each vertex in $\mathscr{I}_G(^{a}_{bc})$ is adjacent to each vertex in $\mathscr{I}_G(^{b}_{ac}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{c}_{ab})$ or (not exclusive) to each vertex in $\mathscr{I}_G(^{c}_{ab}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{b}_{ac})$;
– each vertex in $\mathscr{O}_G(^{b}_{ac})$ is adjacent to each vertex in $\mathscr{I}_G(^{b}_{ac}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{c}_{ab})$;
– each vertex in $\mathscr{O}_G(^{c}_{ab})$ is adjacent to each vertex in $\mathscr{I}_G(^{c}_{ab}) \cup \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{O}_G(^{b}_{ac})$.

Notice that there are graphs which contain a BAT but no magnet. This is, for example, the case for any chordless cycle with at least five vertices. Conversely, the graph $G = (V, E)$ with $V = \{a, b, c, d, e\}$ and $E = \{[a, b], [a, c], [a, d], [a, e], [b, c], [d, e]\}$ has no BAT while each pair of adjacent vertices is a magnet.

Let us now consider the transformation which has been applied to the conflict graph G of f for getting the conflict graph G' of g (see Fig. 5 for an example of such a transformation).

Given a graph $G = (V, E)$ and a BAT$(^{a}_{bc})$ in G, we define $G' = (V', E') = \mathscr{T}_B(G; ^{a}_{bc})$ as follows:
– the vertex set V' of G' is obtained from V by removing the vertices a, b, c and adding two new vertices $\tilde{a}$ and $\widetilde{bc}$ with weight $w_a (= w_b = w_c)$;
– the edge set E' of G' is obtained from E by removing all edges incident to a, b or c and by linking $\tilde{a}$ to each vertex in $C_G(a, b, c)$ and $\widetilde{bc}$ to each vertex in $(N_G(b) \cup N_G(c)) \setminus \{a\}$.

Up to this point, we have observed that if f is a posiform containing three terms T_a, T_b, T_c satisfying conditions (i) and (ii) then the associated conflict graph G contains a BAT$(^{a}_{bc})$. Moreover, we have noticed that $\mathscr{T}_B(G; ^{a}_{bc})$ is the conflict graph of a posiform g having the same maximum value as f; hence, $\alpha_w(G) = \alpha_w(\mathscr{T}_B(G; ^{a}_{bc}))$ in that case. We prove now the following stronger theorem.

Theorem 1. Let $G = (V, E)$ be a graph containing a BAT$(^{a}_{bc})$. Then $\alpha_w(G) = \alpha_w(\mathscr{T}_B(G; ^{a}_{bc}))$.

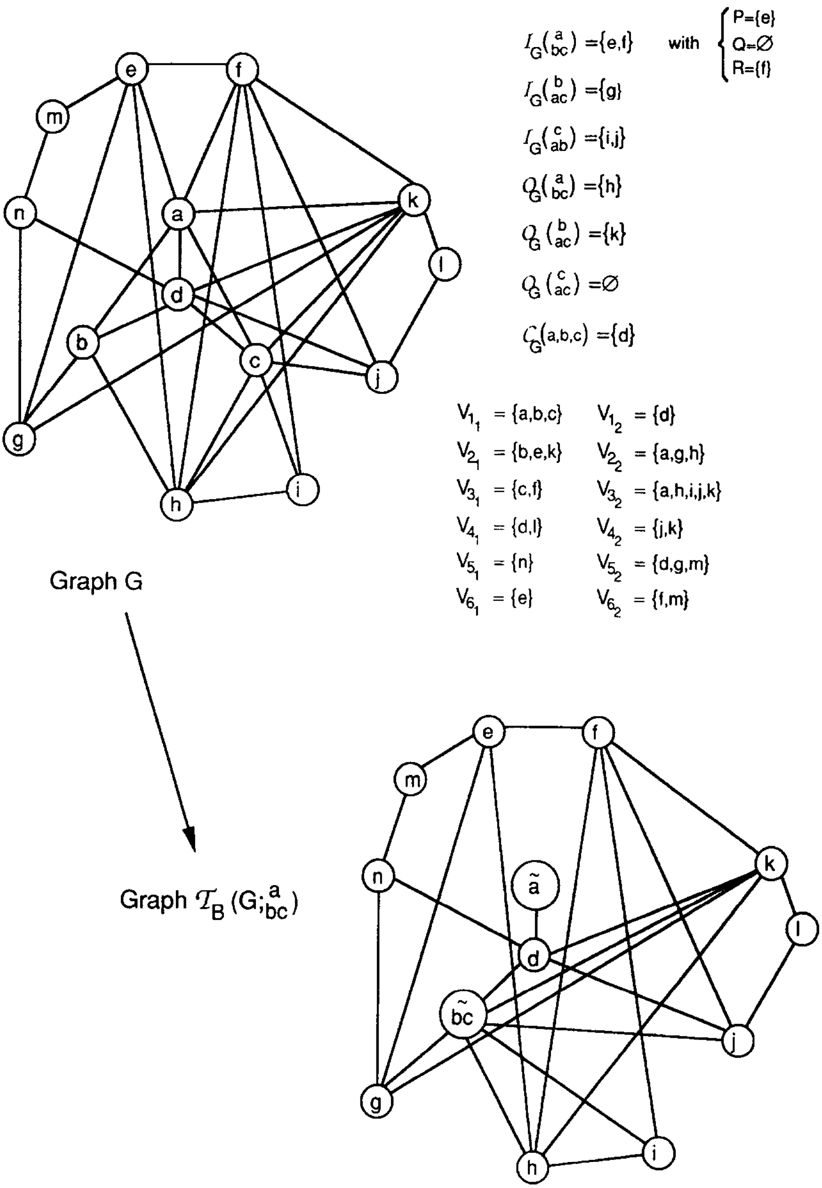


Fig. 5.

Proof. We shall give two proofs of this theorem, one having a Boolean and one having a graph theoretical flavour.

Boolean proof: Let us consider the two following subsets of V:

– $K = \{v \in N_G(a) \backslash \{b, c\} \mid v$ is adjacent to all vertices in $\mathscr{I}_G\binom{b}{ac} \cup \mathscr{O}_G\binom{a}{bc} \cup \mathscr{O}_G\binom{c}{ab}\}$;

– $L = \{v \in N_G(a) \backslash \{b, c\} \mid v$ is adjacent to all vertices in $\mathscr{I}_G\binom{c}{ab} \cup \mathscr{O}_G\binom{a}{bc} \cup \mathscr{O}_G\binom{b}{ac}\}$.

Since the vertices a, b and c induce a BAT$(\substack{a\\bc})$ in G, it follows that $\mathscr{I}_G(\substack{a\\bc}) \subseteq K \cup L$, $\mathscr{O}_G(\substack{b\\ac}) \subseteq K$ and $\mathscr{O}_G(\substack{c\\ab}) \subseteq L$. So, let us consider the following covering of the edges incident to a, b or c by three complete bipartite partial subgraphs $G_i = ((V_{i1}, V_{i2}), E_i)$ $(1 \leqslant i \leqslant 3)$ of G:

$$V_{1_1} = \{a, b, c\}, \qquad V_{1_2} = C_G(a, b, c),$$

$$V_{2_1} = \{b\} \cup K, \qquad V_{2_2} = \{a\} \cup \mathscr{I}_G(\substack{b\\ac}) \cup \mathscr{O}_G(\substack{a\\bc}) \cup \mathscr{O}_G(\substack{c\\ab}),$$

$$V_{3_1} = \{c\} \cup L, \qquad V_{3_2} = \{a\} \cup \mathscr{I}_G(\substack{c\\ab}) \cup \mathscr{O}_G(\substack{a\\bc}) \cup \mathscr{O}_G(\substack{b\\ac}).$$

Let us now consider any covering of the edges in $E \backslash (E_1 \cup E_2 \cup E_3)$ by complete bipartite partial subgraphs $G_i((V_{i_1}, V_{i_2}), E_i)$, $(4 \leqslant i \leqslant q)$ of G. For each $v \in V$ we define

$$A_v = \{i \mid 1 \leqslant i \leqslant q \text{ and } v \in V_{i_1}\} \quad \text{and} \quad B_v = \{i \mid 1 \leqslant i \leqslant q \text{ and } v \in V_{i_2}\}$$

$$T_v = \textstyle\prod_{j \in A_v} x_j \prod_{k \in B_v} \bar{x}_k.$$

As observed in Section 2, G is the conflict graph associated with the posiform $f = \sum_{v \in V} w_v T_v$. By construction, we have $A_a = \{1\}$, $B_a = \{2, 3\}$, $A_b = \{1, 2\}$, $A_c = \{1, 3\}$ and $B_b = B_c = \emptyset$. It follows that $T_a = x_1 \bar{x}_2 \bar{x}_3$, $T_b = x_1 x_2$, $T_c = x_1 x_3$. Hence, $T_a + T_b + T_c = x_1(\bar{x}_2 \bar{x}_3 + x_2 + x_3) = x_1(1 + x_2 x_3)$ and the posiform f has the same maximum value as the posiform $g = \sum_{v \in V\, v \neq a,b,c} w_v T_v + w_a x_1 + w_a x_1 x_2 x_3$.

The conflict graph G' associated with g is the graph obtained from $G[V \backslash \{a, b, c\}]$ by adding two vertices $\tilde{a}$ and $\widetilde{bc}$ of weight w_a $(= w_b = w_c)$ and such that

– $\tilde{a}$ is adjacent to every vertex v $(v \neq a, b, c)$ such that $\bar{x}_1$ appears in T_v. So, by construction, $\tilde{a}$ is adjacent to v if and only if $v \in V_{1_2} = \mathscr{C}_G(a, b, c)$.
– $\widetilde{bc}$ is adjacent to every vertex v $(v \neq a, b, c)$ such that $\bar{x}_1$, $\bar{x}_2$ or $\bar{x}_3$ appears in T_v. Hence, $\widetilde{bc}$ is adjacent to v if and only if $v \in (V_{1_2} \cup V_{2_2} \cup V_{3_2}) \backslash \{a, b, c\} = (N_G(b) \cup N_G(c)) \backslash \{a\}$.

This shows that $G' = \mathscr{T}_B(G; \substack{a\\bc})$. Hence G and G' are conflict graphs associated with two posiforms having the same maximum value.

Graph-theoretical proof: Let us denote $G' = \mathscr{T}_B(G; \substack{a\\bc})$ and let us consider any stable set S of G. If $S \cap \{a, b, c\} = \emptyset$, then S is stable in G' and if $|S \cap \{a, b, c\}| = 1$, then $(S \backslash \{a, b, c\}) \cup \{\tilde{a}\}$ is stable in G'; otherwise $S \cap \{a, b, c\} = \{b, c\}$ and $(S \backslash \{b, c\}) \cup \{\tilde{a}, \widetilde{bc}\}$ is stable in G'. In each case, we have found a stable set S' of G' having the same weight as S. This shows that $\alpha_w(G) \leqslant \alpha_w(G')$.

Let us prove now that $\alpha_w(G) \geqslant \alpha_w(G')$. For this purpose, consider any stable set S' having a maximum weight in G'. Notice that $S' \cap \{\tilde{a}, \widetilde{bc}\} \neq \{\widetilde{bc}\}$ else $S' \cup \{\tilde{a}\}$ would be a stable set of G' having a weight larger than S'.

– If $S' \cap \{\tilde{a}, \widetilde{bc}\} = \emptyset$ then S' is stable in G.
– If $S' \cap \{\tilde{a}, \widetilde{bc}\} = \{\tilde{a}, \widetilde{bc}\}$ then $S' \backslash \{\tilde{a}, \widetilde{bc}\} \cup \{b, c\}$ is stable in G.

– If $S' \cap \{\tilde{a}, \widetilde{bc}\} = \{\tilde{a}\}$ and $S' \cap (\mathcal{O}_G(^{b}_{ac}) \cup \mathcal{O}_G(^{a}_{bc}) \cup \mathcal{O}_G(^{c}_{ab})) \neq \emptyset$, then there is a vertex v in $S' \cap (V' \setminus \{\tilde{a}, \widetilde{bc}\})$ such that $|N_G(v) \cap \{a, b, c\}| = 2$ and $(S' \setminus \{\tilde{a}\}) \cup (\{a, b, c\} - N_G(v))$ is stable in G.
– If $S' \cap \{\tilde{a}, \widetilde{bc}\} = \{\tilde{a}\}$ and $S' \cap (\mathcal{O}_G(^{a}_{bc}) \cup \mathcal{O}_G(^{b}_{ac}) \cup \mathcal{O}_G(^{c}_{ab})) = \emptyset$, then
  . if $S' \cap \mathcal{I}_G(^{b}_{ac}) = \emptyset$ then $S' \setminus \{\tilde{a}\} \cup \{b\}$ is stable in G;
  . if $S' \cap \mathcal{I}_G(^{c}_{ab}) = \emptyset$ then $S' \setminus \{\tilde{a}\} \cup \{c\}$ is stable in G;
  . if $S' \cap (\mathcal{I}_G(^{b}_{ac}) \cup \mathcal{I}_G(^{c}_{ab})) \neq \emptyset$ then $S' \setminus \{\tilde{a}\} \cup \{a\}$ is stable in G.

In each case, we have found a stable set S of G having the same weight as S'. □

5. Some applications

Let G and H be two graphs. We shall say that the graph G is H-free if it does not contain an induced subgraph isomorphic to H. We denote $2K_2(a, c; b, d)$ the complement of the chordless cycle $C_4(a, b, c, d)$ on four vertices and *claw*$(a; b, c, d)$ the star $((V_1, V_2), E)$ with $V_1 = \{a\}$ and $V_2 = \{b, c, d\}$ (see Fig. 6).

In this section, we first prove that each connected C_4-free and $2K_2$-free graph G which is not a clique contains a BAT$(^{a}_{bc})$ such that $\mathcal{T}_B(G; ^{a}_{bc})$ is also C_4-free and $2K_2$-free. As a corollary, the stability and clique numbers of any C_4-free and $2K_2$-free graph can be computed by repeatedly applying transformation $\mathcal{T}_B$.

We shall then prove that if a claw-free graph G contains a BAT$(^{a}_{bc})$ but no pair of adjacent vertices u, v with $N_G(u) \subseteq N_G(v)$, then $\mathcal{T}_B(G; ^{a}_{bc})$ is also claw-free.

5.1. C_4-free and $2K_2$-free graphs

The structure of C_4-free and $2K_2$-free graphs has been characterized by Blázsik, et al. [3]. This class of graphs contains all threshold [10, 4] and split graphs [8, 17]. A linear-time recognition algorithm has been proposed by Maffray and Preissmann [23]: this algorithm provides a maximum stable set with no extra work. We propose in this section a different approach for the computation of the stability number of C_4-free and $2K_2$-free graphs. Instead of developing specific tools which take into account the structure of these graphs, we prove that their stability number can be obtained by repeatedly applying transformation $\mathcal{T}_B$ until each connected component of the transformed graph is a clique: the number of connected components in the final graph is then equal to the stability number of the original one. The proposed algorithm is more complex than the one described in [23]. It provides, however, an example of the various contexts in which transformation $\mathcal{T}_B$ can be applied. Another application of this general tool will be given in the next section.

Notice that the complement $\bar{G}$ of a C_4-free and $2K_2$-free graph is also C_4-free and $2K_2$-free. Hence, every algorithm which computes the stability number of a C_4-free and $2K_2$-free graph G can also be applied to its complement $\bar{G}$ for the computation of the clique number $\omega(G) = \alpha(\bar{G})$.

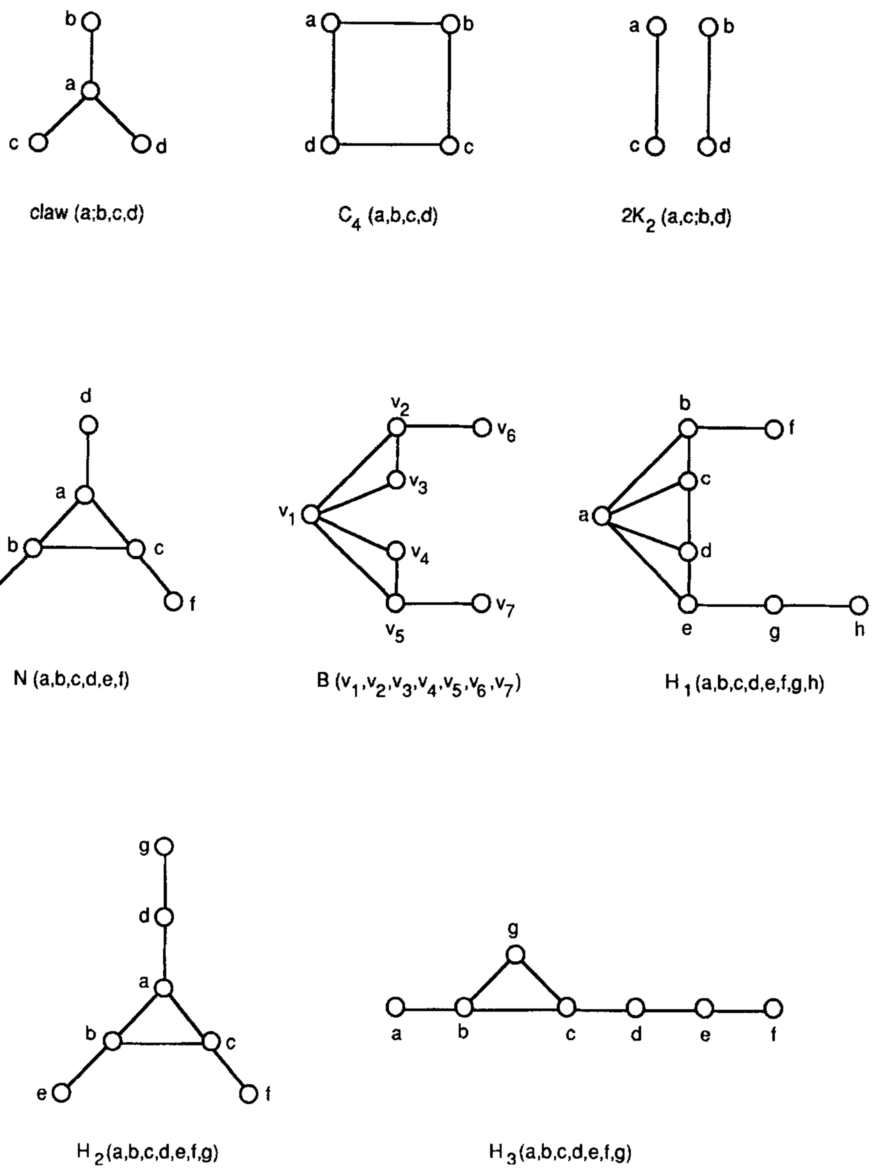


Fig. 6.

In order to prove that the stability number of C_4-free and $2K_2$-free graph can be obtained by repeatedly applying transformation $\mathscr{T}_B$, we have to show that every connected C_4-free and $2K_2$-free graph which is not a clique contains a BAT. We prove the following stronger lemma.

Lemma 1. *Every connected C_4-free and $2K_2$-free graph G which is not a clique contains a $BAT\binom{a}{bc}$ such that $\mathcal{O}_G\binom{b}{ac} = \emptyset$ or $\mathcal{O}_G\binom{c}{ab} = \emptyset$.*

Proof. Let G be a connected C_4-free and $2K_2$-free graph which is not a clique and let us consider any subset $W = \{a, b, c\}$ of vertices in G such that W induces a $P_3(b, a, c)$ in G. We first notice that $\mathcal{O}_G\binom{a}{bc} = \emptyset$ else G contains an induced $C_4(b, a, c, v)$ for any v in $\mathcal{O}_G\binom{a}{bc}$. Moreover, if $\mathcal{O}_G\binom{b}{ac}$ is not empty, then we have

– $\mathcal{I}_G\binom{b}{ac} = \emptyset$ else G contains an induced $2K_2(c, u;\ b, v)$ or $C_4(b, a, u, v)$ for any u in $\mathcal{O}_G\binom{b}{ac}$ and any v in $\mathcal{I}_G\binom{b}{ac}$;
– each vertex in $\mathcal{O}_G\binom{b}{ac}$ is adjacent to each vertex in $\mathcal{O}_G\binom{c}{ab}$ else G contains an induced $2K_2(c, u;\ b, v)$ for any u in $\mathcal{O}_G\binom{b}{ac}$ and any v in $\mathcal{O}_G\binom{c}{ab}$ such that $[u, v] \notin E$.

By symmetry between b and c, we also have $\mathcal{I}_G\binom{c}{ab} = \emptyset$ if $\mathcal{O}_G\binom{c}{ab} \neq \emptyset$. Hence, each vertex in $\mathcal{O}_G\binom{b}{ac}$ is adjacent to each vertex in $\mathcal{I}_G\binom{b}{ac} \cup \mathcal{O}_G\binom{a}{bc} \cup \mathcal{O}_G\binom{c}{ab}$ and each vertex in $\mathcal{O}_G\binom{c}{ab}$ is adjacent to each vertex in $\mathcal{I}_G\binom{c}{ab} \cup \mathcal{O}_G\binom{a}{bc} \cup \mathcal{O}_G\binom{b}{ac}$. This means that the vertices a, b and c induce a $\mathrm{BAT}\binom{a}{bc}$ in G if and only if each vertex in $\mathcal{I}_G\binom{a}{bc}$ is either adjacent to each vertex in $\mathcal{I}_G\binom{b}{ac} \cup \mathcal{O}_G\binom{c}{ab}$ or (not exclusively) to each vertex in $\mathcal{I}_G\binom{c}{ab} \cup \mathcal{O}_G\binom{b}{ac}$.

Let us assume that $\mathcal{O}_G\binom{b}{ac} = \emptyset$. If $\mathcal{I}_G\binom{c}{ab} = \emptyset$ then we are done, since $\mathcal{I}_G\binom{c}{ab} \cup \mathcal{O}_G\binom{b}{ac} = \emptyset$. Otherwise, we have observed that $\mathcal{O}_G\binom{c}{ab} = \emptyset$, and we have $\mathcal{I}_G\binom{a}{bc} = \emptyset$ or $\mathcal{I}_G\binom{b}{ac} = \emptyset$ else G contains an induced $2K_2(b, v;\ c, t)$, $C_4(a, b, v, u)$, $C_4(a, c, t, u)$ or $2K_2(a, u;\ v, t)$ for any $u \in \mathcal{I}_G\binom{a}{bc}$, $v \in \mathcal{I}_G\binom{b}{ac}$ and $t \in \mathcal{I}_G\binom{c}{ab}$; so, either $\mathcal{I}_G\binom{a}{bc} = \emptyset$ or $\mathcal{I}_G\binom{b}{ac} \cup \mathcal{O}_G\binom{c}{ab} = \emptyset$ and G contains a $\mathrm{BAT}\binom{a}{bc}$.

Up to this point, we have proved that if G contains an induced $P_3(b, a, c)$ with $\mathcal{O}_G\binom{b}{ac} = \emptyset$ then it contains a $\mathrm{BAT}\binom{a}{bc}$. So let us consider any induced $P_3(b, a, c)$ in G. If $\mathcal{O}_G\binom{b}{ac} = \emptyset$ or $\mathcal{O}_G\binom{c}{ab} = \emptyset$ then the lemma is proved. Otherwise, let us consider any vertices $u \in \mathcal{O}_G\binom{b}{ac}$ and $v \in \mathcal{O}_G\binom{c}{ab}$. We have already shown that $[u, v] \in E$. Hence, G contains an induced $P_3(u, v, b)$ and $\mathcal{O}_G\binom{u}{vb} = \emptyset$ else G contains an induced $2K_2(b, t;\ c, u)$ or $C_4(c, u, v, t)$ for any $t \in \mathcal{O}_G\binom{u}{vb}$. □

It can be observed in Fig. 7 that if G is a C_4-free and $2K_2$-free graph which contains a $\mathrm{BAT}\binom{a}{bc}$, then the transformed graph $\mathcal{T}_B(G; \binom{a}{bc})$ is not necessarily $2K_2$-free. However, we prove now that $\mathcal{T}_B(G; \binom{a}{bc})$ is $2K_2$-free if the $\mathrm{BAT}\binom{a}{bc}$ in G satisfies $\mathcal{O}_G\binom{b}{ac} = \emptyset$ or $\mathcal{O}_G\binom{c}{ab} = \emptyset$. We next show that $\mathcal{T}_B(G; \binom{a}{bc})$ is C_4-free for any $\mathrm{BAT}\binom{a}{bc}$ in G.

Lemma 2. *Let G be a C_4-free and $2K_2$-free graph and let a, b and c be three vertices which induce a $\mathrm{BAT}\binom{a}{bc}$ in G such that $\mathcal{O}_G\binom{b}{ac} = \emptyset$ or $\mathcal{O}_G\binom{c}{ab} = \emptyset$. Then $\mathcal{T}_B(G; \binom{a}{bc})$ is $2K_2$-free.*

Proof. Let us assume without loss of generality that $\mathcal{O}_G\binom{b}{ac} = \emptyset$ and suppose that $\mathcal{T}_B(G; \binom{a}{bc}) = (V', E')$ contains an induced $2K_2(p, q;\ r, s)$.

– If $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \emptyset$ then G contains an induced $2K_2(p, q;\ r, s)$, a contradiction.
– If $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \{\tilde{a}\}$ then we may assume that $p = \tilde{a}$. The vertex q is adjacent to a, b and c in G, since $[\tilde{a}, q] \in E'$. Moreover, $[c, r] \in E$ or $[c, s] \in E$ else G contains an

this C_4-free and $2K_2$-free graph contains a BAT$(^{a}_{bc})$ $\mathcal{T}_B(G;^{a}_{bc})$ contains a $2K_2(t,\tilde{a};u,v)$

Fig. 7.

induced $2K_2(c, q; r, s)$. So, let us assume without loss of generality that $[c, r] \in E$. Now, $[a, r] \notin E$ else $[\tilde{a}, r] \in E'$ or $\mathcal{O}_G(^{b}_{ac}) \neq \emptyset$. Hence, $[b, r] \notin E$ else G contains an induced $C_4(b, a, c, r)$. We now have $[b, s] \in E$ and $[a, s] \in E$ else G contains an induced $2K_2(b, q; r, s)$ or $2K_2(a, q; r, s)$. Since $[\tilde{a}, s] \notin E'$, we have $[c, s] \notin E$. Hence, G contains an induced $C_4(a, c, r, s)$, a contradiction.

– If $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \{\widetilde{bc}\}$ then we may assume that $p = \widetilde{bc}$. Since $[\widetilde{bc}, q] \in E'$, $[\widetilde{bc}, r] \notin E'$ and $[\widetilde{bc}, s] \notin E'$, we know that r and s are adjacent neither to b nor to c in G while $[b, q] \in E$ or $[c, q] \in E$. Hence, G contains an induced $2K_2(b, q; r, s)$ or $2K_2(c, q; r, s)$, a contradiction.

– The case $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc},\} = \{\tilde{a}, \widetilde{bc}\}$ is not possible since $[\tilde{a}, \widetilde{bc}] \notin E'$ and each vertex adjacent to $\tilde{a}$ is also adjacent to $\widetilde{bc}$. □

Lemma 3. *Let G be a C_4-free and $2K_2$-free graph and let a, b and c be three vertices which induce a* BAT$(^{a}_{bc})$ *in G. Then $\mathcal{T}_B(G; ^{a}_{bc})$ is C_4-free.*

Proof. Let us assume that $\mathcal{T}_B(G; ^{a}_{bc}) = (V', E')$ contains an induced $C_4(p, q, r, s)$.

– If $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \emptyset$ then G contains an induced $C_4(a, b, c, d)$, a contradiction.

– If $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \{\tilde{a}\}$ then we may assume that $p = \tilde{a}$. Since $[\tilde{a}, q] \in E'$ and $[\tilde{a}, s] \in E'$, we have $[u, q] \in E$ and $[u, s] \in E$ for all u in $\{a, b, c\}$. Moreover, there exists a vertex u in $\{a, b, c\}$ such that $[u, r] \notin E$ else $[\tilde{a}, r] \in E'$. Hence G contains an induced $C_4(u, q, r, s)$, a contradiction.

– If $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \{\widetilde{bc}\}$ then we may assume that $p = \widetilde{bc}$. Now $[b, r] \notin E$ and $[c, r] \notin E$, since $[\widetilde{bc}, r] \notin E'$ and we know that q and s belong to $(N_G(b) \cup N_G(c)) \setminus \{a\}$, since $[\widetilde{bc}, q] \in E'$ and $[\widetilde{bc}, s] \in E'$. We may suppose that $[b, q] \in E$, the case $[c, q] \in E$ being symmetrical. Now $[b, s] \notin E$ else G contains an induced $C_4(b, q, r, s)$. Hence, $[c, s] \in E$ and G contains an induced $C_4(c, q, r, s)$ or $2K_2(b, q; c, s)$, a contradiction.

– If $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}] = \{\tilde{a}, \widetilde{bc}\}$ then we may assume that $p = \widetilde{bc}$ and $r = \tilde{a}$. Since $[\tilde{a}, q] \in E'$ and $[\tilde{a}, s] \in E'$, we know that each vertex in $\{a, b, c\}$ is adjacent to q and s. Hence, G contains an induced $C_4(b, q, c, s)$, a contradiction. □

Theorem 2. *The stability and clique numbers of C_4-free and $2K_2$-free graphs can be computed in polynomial time by repeatedly applying transformation $\mathscr{T}_B$.*

Proof. Let G be a C_4-free and $2K_2$-free graph. We know by Lemma 1 that if G contains a connected component which is not a clique then G contains a BAT$\binom{a}{bc}$ such that $\mathscr{C}_G\binom{b}{ac} = \emptyset$ or $\mathscr{C}_G\binom{c}{ab} = \emptyset$. Moreover, for such a BAT$\binom{a}{bc}$, we know by Lemmas 2 and 3 that $\mathscr{T}_B(G; \binom{a}{bc})$ is also C_4-free and $2K_2$-free. By repeatedly applying transformation $\mathscr{T}_B$ with this kind of BAT it is, therefore, possible to transform G into a graph G' in which each connected component is a clique. Since transformation $\mathscr{T}_B$ preserves the stability number, it follows that $\alpha(G) = \alpha(G')$. Hence, the stability number of G is equal to the number of connected components in G'.

Since $\omega(G) = \alpha(\bar{G})$ and $\bar{G}$ is also C_4-free and $2K_2$-free, the clique number of G can be determined by computing the stability number of $\bar{G}$ with the above technique.

Given a C_4-free and $2K_2$-free graph in which at least one connected component is not a clique, it is easy to determine a BAT$\binom{a}{bc}$ such that $\mathscr{C}_G\binom{b}{ac} = \emptyset$ or $\mathscr{C}_G\binom{c}{ab} = \emptyset$. Indeed, following the proof of Lemma 1, it is sufficient to find an induced $P_3(b, a, c)$ in G with $\mathscr{C}_G\binom{b}{ac} = \emptyset$. We first detect any induced $P_3(b, a, c)$ in G. If $\mathscr{C}_G\binom{b}{ac} = \emptyset$ or $\mathscr{C}_G\binom{c}{ab} = \emptyset$ then a, b and c induce the desired BAT in G; otherwise, there are vertices $u \in \mathscr{C}_G\binom{b}{ac}$ and $v \in \mathscr{C}_G\binom{c}{ab}$. It has been shown in lemma 1 that G contains an induced $P_3(u, v, b)$ with $\mathscr{C}_G\binom{u}{vb} = \emptyset$ which means that G contains a BAT$\binom{v}{ub}$ with $\mathscr{C}_G\binom{u}{vb} = \emptyset$. This search can thus easily be performed in time complexity $O(|V|^2)$. $\square$

5.2. Claw-free graphs

It has been proved by Minty [24] and Sbihi [25] that the stability number of claw-free graphs can be computed in polynomial time. Both authors have described an algorithm based on augmenting chains techniques. Boolean methods have also been applied to the computation of the stability number of claw-free graphs. They turn to be useful for reducing the size of the considered graph or for finding the stability number in subclasses of claw-free graphs.

As described in Section 3, the Struction is a procedure which, given a graph $G = (V, E)$, constructs a new graph G' such that $\alpha(G') = \alpha(G) - 1$. Given an arbitrary vertex a, the vertex set of the transformed graph $G' = (V', E') = \mathscr{T}_S(G; a)$ consists of all vertices in $V \setminus N_G[a]$ as well as "new" vertices contained in layers L_q $(1 \leqslant q < |N_G(a)|)$. We have noticed that $|V'| \in \mathscr{O}(|V|^2)$. When G is a claw-free graph, it is easy to see that the subgraph induced by the vertices in the layers is a clique. Hence, at most one new vertex may belong to a maximum stable set in G'. In order to avoid the potentially exponential growth of the number of vertices, specialized versions of the Struction have been proposed which ensure that the new graph does not contain more vertices than the original one. It has been proved that by choosing either (i) one vertex per layer [14, 15], or (ii) all the vertices of one layer [86], or (iii) all the vertices of two layers [86], then the stability number of (i) claw-free and N-free, (ii) claw-free, H_1-free and H_2-free and (iii) claw-free, B-free and H_3-free graphs can be computed in polynomial time (see Fig. 6).

Notice that if G is a claw-free graph, then the transformed graph $\mathscr{T}_S(G; a)$ is not necessarily claw-free. Indeed, for the graph $B(v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ (see Fig. 6), the transformed graph $\mathscr{T}_S(G; v_1)$ contains an induced claw((3, 4);(2, 5), v_6, v_7). It can, however, be proved easily that if a graph $G = (V, E)$ is claw-free and B-free then $\mathscr{T}_S(G; a)$ is claw-free for all a in V.

It is a simple exercise to prove that if G is a claw-free graph and (a, b) a magnet in G (if any), then $\mathscr{T}_M(G; a, b)$ is also claw-free. Hence, when applicable, the transformation $\mathscr{T}_M$ can be used for reducing the size of a claw-free graph while preserving its stability number. However, there are claw-free graphs, such as chordless cycles with more than four vertices, which do not contain any magnet.

If a graph G contains two adjacent vertices a and b such that $N_G(a) \subseteq N_G(b)$, we shall say that b *dominates* a. In that case, we have $\alpha(G) = \alpha(G[V\backslash\{b\}])$, since for each stable set S of G containing b, there is a stable set $(S\backslash\{b\})\cup\{a\}$ in $G[V\backslash\{b\}]$. In fact, the pair (a, b) is a magnet in G and $G[V\backslash\{b\}] = \mathscr{T}_M(G; a, b)$. Let $\mathscr{F}$ be a class of graphs characterized by a set of forbidden induced subgraphs. Then, by removing a dominating vertex in a graph $G \in \mathscr{F}$, one gets an induced subgraph of G which also belongs to $\mathscr{F}$. Hence, given any claw-free graph G, one can first remove each dominating vertex of G before applying any procedure for the computation of the stability number of G. We prove in this section that if a claw-free graph contains a BAT$(^{a}_{bc})$ but no dominating vertex, then the transformed graph $\mathscr{T}_B(G; ^{a}_{bc})$ is also claw-free.

Lemma 4. *Let G be a claw-free graph which contains a* BAT$(^{a}_{bc})$ *but no dominating vertex. Then $\mathscr{T}_B(G; ^{a}_{bc})$ is claw-free.*

Proof. Let us assume that $\mathscr{T}_B(G; ^{a}_{bc})$ contains an induced claw($p; q, r, s$).

Case 1: $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \emptyset$. G contains an induced claw($p; q, r, s$), a contradiction.

Case 2: $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \{\tilde{a}\}$. We may assume that either $p = \tilde{a}$ or $q = \tilde{a}$.

Case 2.1: $p = \tilde{a}$. Since $[\tilde{a}, q] \in E'$, $[\tilde{a}, r] \in E'$ and $[\tilde{a}, s] \in E'$, it follows that q, r and s are adjacent to a, b and c. Hence, G contains an induced claw($u; q, r, s$) for all u in $\{a, b, c\}$, a contradiction.

Case 2.2: $q = \tilde{a}$. Since $[\tilde{a}, p] \in E'$, we have $[u, p] \in E$ for all u in $\{a, b, c\}$. At least one vertex u in $\{a, b, c\}$ is not adjacent to r else $[\tilde{a}, r] \in E'$. For this vertex u, we have $[u, s] \in E$ else G contains an induced claw($p; u, r, s$).

Case 2.2.1: $u = b$.

Case 2.2.1.1: $[a, s] \notin E$. We know that $[a, r] \in E$ else G contains an induced claw($p; a, r, s$). Hence, $[c, r] \notin E$ else $r \in \mathscr{O}_G(^{b}_{ac})$ and $s \in \mathscr{I}_G(^{b}_{ac}) \cup \mathscr{O}_G(^{a}_{bc})$ while $[r, s] \notin E$, contradicting the fact that a, b and c induce a $\mathrm{BAT}(^{a}_{bc})$ in G. Now $[c, s] \in E$ else G contains an induced claw(p; c, r, s). So, $r \in \mathscr{I}_G(^{a}_{bc})$ and $s \in \mathscr{O}_G(^{a}_{bc})$ while $[r, s] \notin E$, contradicting the fact that a, b and c induce a $\mathrm{BAT}(^{a}_{bc})$ in G.

Case 2.2.1.2: $[a, s] \in E$. We have $[c, s] \notin E$ else $[\tilde{a}, s] \in E'$. Hence, $[c, r] \in E$ else G contains an induced claw(p; c, r, s). So $r \in \mathscr{I}_G(^{c}_{ab}) \cup \mathscr{O}_G(^{b}_{ac})$ and $s \in \mathscr{O}_G(^{c}_{ab})$ while $[r, s] \notin E$, contradicting the fact that a, b and c induce a $\mathrm{BAT}(^{a}_{bc})$ in G.

Case 2.2.2: $u = c$. This case is symmetrical to case 2.2.1.

Case 2.2.3: $u = a$. It follows from 2.2.1 and 2.2.2 that $[b, r] \in E$ and $[c, r] \in E$. By symmetry between r and s, we have $[b, s] \in E$ and $[c, s] \in E$. It follows that $[\tilde{a}, s] \in E'$, a contradiction.

Case 3: $\{p, q, r, s\} \cap \{\tilde{a}, \widetilde{bc}\} = \{\widetilde{bc}\}$. We may assume that either $p = \widetilde{bc}$ or $q = \widetilde{bc}$.

Case 3.1: $p = \widetilde{bc}$. At least one vertex among q, r and s is not adjacent to a else G contains an induced claw(a; q, r, s). We may assume that $[a, q] \notin E$ (the other cases being symmetrical). Since $[\widetilde{bc}, q] \in E'$, we have $[b, q] \in E$ or $[c, q] \in E$. By symmetry between b and c, we may suppose that $[b, q] \in E$. Now either $[b, r] \notin E$ or $[b, s] \notin E$ else G contains an induced claw(b; q, r, s). By symmetry between r and s, we may assume that $[b, r] \notin E$. We have $[c, r] \in E$ else $[\widetilde{bc}, r] \notin E'$. Moreover, $[a, r] \notin E$ else $q \in \mathscr{O}_G(^{a}_{bc}) \cup \mathscr{I}_G(^{b}_{ac})$ and $r \in \mathscr{O}_G(^{b}_{ac})$ while $[q, r] \notin E$, contradicting the fact that a, b and c induce a $\mathrm{BAT}(^{a}_{bc})$ in G. It follows that $[c, q] \notin E$ else G contains an induced claw(c; a, q, r). Since $[\widetilde{bc}, s] \notin E'$, we have $[b, s] \in E$ or $[c, s] \in E$. So, $[a, s] \in E$ else G contains an induced claw(b; a, q, s) or claw(c; a, r, s). We may assume that $[b, s] \in E$ (the case $[c, s] \in E$ being symmetrical, permuting the role of b and q with c and r). Now $[c, s] \in E$, since each vertex in $\mathscr{O}_G(^{c}_{ab})$ is adjacent to $r \in \mathscr{I}_G(^{c}_{ab})$. There exists a vertex u such that $[a, u] \in E$ and $[s, u] \notin E$ else s dominates a. We have $[b, u] \in E$ or $[c, u] \in E$ else G contains an induced claw(a; b, c, u). We may assume that $[b, u] \in E$ (the case $[c, u] \in E$ being symmetrical, permuting the role of b and q with c and r). Hence, $[q, u] \in E$ else G contains an induced claw(b; q, s, u). So, $[r, u] \notin E$ and $[c, u] \notin E$ else G contains an induced claw(u; a, q, r) or claw(c; u, r, s). Finally, $u \in \mathscr{O}_G(^{c}_{ab})$ and $r \in \mathscr{I}_G(^{c}_{ab})$ while $[r, u] \notin E$, contradicting the fact that a, b and c induce a $\mathrm{BAT}(^{a}_{bc})$ in G.

Case 3.2: $q = \widetilde{bc}$. Since $[\widetilde{bc}, r] \notin E'$, $[\widetilde{bc}, s] \notin E'$ and $[\widetilde{bc}, p] \in E'$, we know that r and s are neither adjacent to b nor to c, while p is adjacent to b or c. Hence, G contains an induced claw(p; b, r, s) or claw(p; c, r, s), a contradiction.

Case 4: $\{p, q, r, s\} \cap \{\widetilde{bc}, \tilde{a}\} = \{\widetilde{bc}, a\}$. Since $[\widetilde{bc}, \tilde{a}] \notin E'$, we may assume that $q = \tilde{a}$ and $r = \widetilde{bc}$. Now p is adjacent to b and c since $[\tilde{a}, p] \in E'$ and s is neither adjacent to b nor to c since $[\widetilde{bc}, s] \notin E'$. Hence, G contains an induced claw(p; b, c, s), a contradiction. □

Notice that transformations $\mathscr{T}_M$, $\mathscr{T}_B$ and the specialized version of $\mathscr{T}_S$ mentioned above are not sufficient for designing a polynomial time algorithm for the

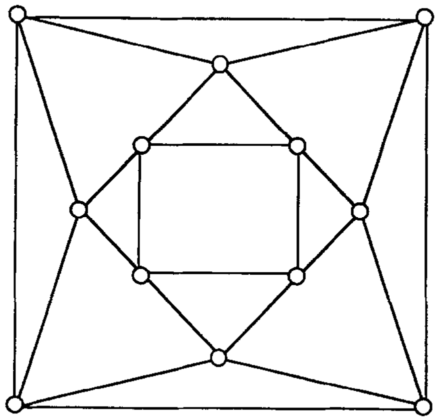


Fig. 8.

computation of the stability number of claw-free graphs. As an example, the graph represented in Fig. 8 has no magnet and no BAT while it contains induced subgraphs isomorphic to a N, B, H_1, H_2 and H_3. Transformations $\mathcal{T}_S$, $\mathcal{T}_M$ and $\mathcal{T}_B$ can, however, be used as preprocessing procedures before applying Sbihi's or Minty's algorithm.

6. Conclusion

We have studied a simplification of posiforms which, when applicable, amounts to reducing the size of the corresponding conflict graph while preserving its stability number. This graph transformation can be used in various contexts. We have proved that it provides a polynomial time algorithm for the computation of the stability and clique numbers of C_4-free and $2K_2$-free graphs. Moreover, we have observed that if the transformation is applied to a claw-free graph G which contains a BAT but no dominating vertex, then the transformed graph is also claw-free.

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