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Stable sets in two subclasses of banner-free graphs

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Abstract

The maximum stable set problem is NP-hard, even when restricted to *banner*-free graphs. In this paper, we use the augmenting graph approach to attack the problem in two subclasses of *banner*-free graphs. We first provide both classes with the complete characterization of minimal augmenting graphs. Based on the obtained characterization, we prove polynomial solvability of the problem in the class of (*banner*, P_8)-free graphs, improving several existing results.

Keywords: Stable set; Augmenting graphs; Polynomial algorithm

1. Introduction

A *stable* set S in a graph G is a set of pairwise non-adjacent vertices. The *stability number* of G, denoted by $\alpha(G)$, is the size of a largest stable set in G. The problem of finding a stable set of maximum cardinality in a graph is referred to as the *maximum stable set problem* (MSP). It is well-known that the MSP is NP-hard in general graphs. Moreover, it remains difficult even under substantial restrictions, for instance, for triangle-free [16] or ($K_{1,4}$, diamond)-free graphs [5]. On the other hand, efficient, i.e., polynomial time, algorithms have been developed for many special classes, such as claw-free [12,17] or (bull, chair)-free graphs [7]. We investigate the gap between "hard" and "simple" cases by studying the problem on graph classes, which have the potential for admitting efficient algorithms. As a result, we conclude that the MSP has

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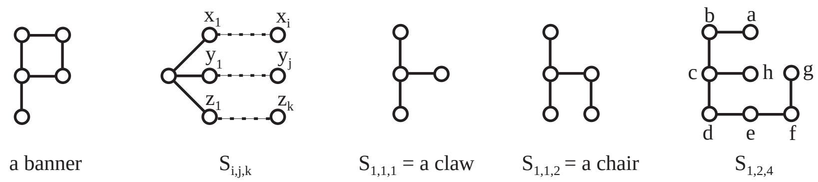

Fig. 1.

a polynomial time solution in $(banner, P_8)$-free graphs, extending several previously studied classes [3,4,10,11,14]. Here a *banner* is the graph with vertices a,b,c,d,e and edges ab,ac,bd,cd and de. As usual, P_k and C_k denote, respectively, a chordless path and a chordless cycle on k vertices. Also, $K_{r,s}$ denotes a complete bipartite graph whose parts have, respectively, r and s vertices. A graph $S_{i,j,k}$ is a tree with exactly three vertices of degree one, being at distance i, j and k from the unique vertex of degree three. Notice that $S_{i,j,0}$ is a path on $i+j+1$ vertices, while $S_{1,1,1}$ is called a *claw* and $S_{1,1,2}$ is called a *chair*. All these graphs are depicted in Fig. 1.

All graphs considered are undirected, without loops and multiple edges. The vertex set and the edge set of a graph G are, respectively, denoted by $V(G)$ and $E(G)$. For a vertex $x \in V(G)$, we denote by $N(x)$ the neighborhood of x, i.e., the set of vertices adjacent to x. For $A \subseteq V(G)$, we denote $G[A]$ the subgraph of G induced by the vertex set A, and $N_A(x) = N(x) \cap A$ the neighborhood of x in $G[A]$. The number of vertices in $N_A(x)$ is called the degree of x in $G[A]$. For two subsets A and B of vertices, we will use the notation $N_A(B) = \bigcup_{b \in B} N_A(b)$. If a graph G contains a graph H as induced subgraph, we simply say that G contains H.

Many classes of graphs, for which polynomial algorithms have been developed to solve the MSP, can be defined by a set $\{H_1, \ldots, H_k\}$ of forbidden induced subgraphs. A graph in such a class is said to be $(H_1, \ldots, H_k)$-free (or simply H_1-free when $k = 1$). Alekseev [1] has proved that if a graph H has a connected component which is not of the form $S_{i,j,k}$, then the stable set problem is NP-hard in the class of H-free graphs. As an immediate consequence, we conclude that the MSP remains NP-hard in *banner*-free graphs (the same conclusion follows from the result of Murphy for graphs with large girth [15]). The class of *banner*-free graphs is of particular interest, since it contains two important subclasses where the problem can be solved efficiently, namely claw-free graphs and P_4-free graphs. In order to make the boundary between NP-hard and polynomially solvable cases more precise, we study the complexity of the problem in $(banner, S_{i,j,k})$-free graphs for increasing values of i,j and k.

There is a trivial algorithm to solve the problem for $S_{i,j,k}$-free graphs when $i+j+k \leqslant 2$, since any graph in this class is simply the union of disjoint cliques.

Up to isomorphism, there are exactly two graphs $S_{i,j,k}$ when $i+j+k=3$, namely the *claw* $S_{1,1,1}$ and the path P_4. Minty [12] and Sbihi [17] have proposed polynomial algorithms for claw-free graphs by applying the *augmenting graph technique*. Corneil et al. [6] have developed a polynomial algorithm for P_4-free graphs (also known as *cographs*) by using the *modular decomposition*.

Table 1

Complexity of the stable set problem in $(banner, S_{i,j,k})$-free graphs

$i+j+k \leqslant 2$	Union of disjoint cliques	Polynomial	
$i+j+k=3$	P_4-free graphs (cographs)	Polynomial	[5]
	claw-free graphs	Polynomial	[12,17]
$i+j+k=4$	$(banner, P_5)$-free graphs	Polynomial	[10]
	chair-free graphs	Polynomial	[2]
$i+j+k=5$	$(banner, P_6)$-free graphs	Polynomial	[3]
	$(banner, S_{1,1,3})$-free graphs	?	
	$(banner, S_{1,2,2})$-free graphs	Polynomial	[8]
$i+j+k=6$	$(banner, P_7)$-free graphs	Polynomial	[3]
	$(banner, S_{1,1,4})$-free graphs	?	
	$(banner, S_{1,2,3})$-free graphs	?	
	$(banner, S_{2,2,2})$-free graphs	Polynomial	[8]
$i+j+k=7$	$(banner, P_8)$-free graphs	Polynomial	Section 4
	$(banner, S_{1,1,5})$-free graphs	?	
	$(banner, S_{1,2,4})$-free graphs	?	
	$(banner, S_{1,3,3})$-free graphs	?	
	$(banner, S_{2,2,3})$-free graphs	?	

The *chair* $S_{1,1,2}$ and the path P_5 are the two possible graphs $S_{i,j,k}$ when $i+j+k=4$. Alekseev [2] proposed a polynomial algorithm for the MSP in *chair*-free graphs also using the augmenting graph technique. The complexity status of the MSP in P_5-free graphs is still unknown. However, the problem becomes polynomial when restricted to $(banner, P_5)$-free graphs [10].

As shown in Table 1, there are, respectively, 3, 4 and 5 different graphs when $i+j+k$ equals, respectively, 5, 6 and 7. Gerber et al. [8] proposed a polynomial algorithm based on graph reductions to solve the MSP in $(banner, S_{2,2,2})$-free graphs. Furthermore, Alekseev et al. [3] recently developed a polynomial algorithm for $(banner, P_7)$-free graphs by characterizing augmenting graphs in this class. In the present paper, we generalize this result to $(banner, P_8)$-free graphs. The complexity status of the MSP in $(banner, S_{i,j,k})$-free graphs is unknown for all values of i, j and k that are not mentioned above.

The augmenting graph technique has proven to be a useful approach to solve the MSP in various classes of graphs [2,3,10,12–14,17]. Below, we describe this approach which we will use in Sections 2 and 3.

An induced bipartite subgraph $H=(W,B,E)$ of G with parts W and B is called *augmenting* for a stable set S in G if $|B|>|W|$, $W \subseteq S$, $B \subseteq V(G)-S$ and $N_S(b) \subseteq W$ for all b in B.

Clearly, if $H=(W,B,E)$ is an augmenting graph for S, then S is not maximum since set $S'=(S-W)\cup B$ is a stable set of size $|S'|>|S|$. Now, assume S is not a maximum stable set, and let S' be a stable set such that $|S'|>|S|$. Then, the subgraph of G induced by set $(S-S')\cup(S'-S)$ is augmenting for S. Hence, we have the following theorem.

Theorem of augmenting graphs. *A stable set S in a graph G is maximum if and only if there are no augmenting graphs for S.*

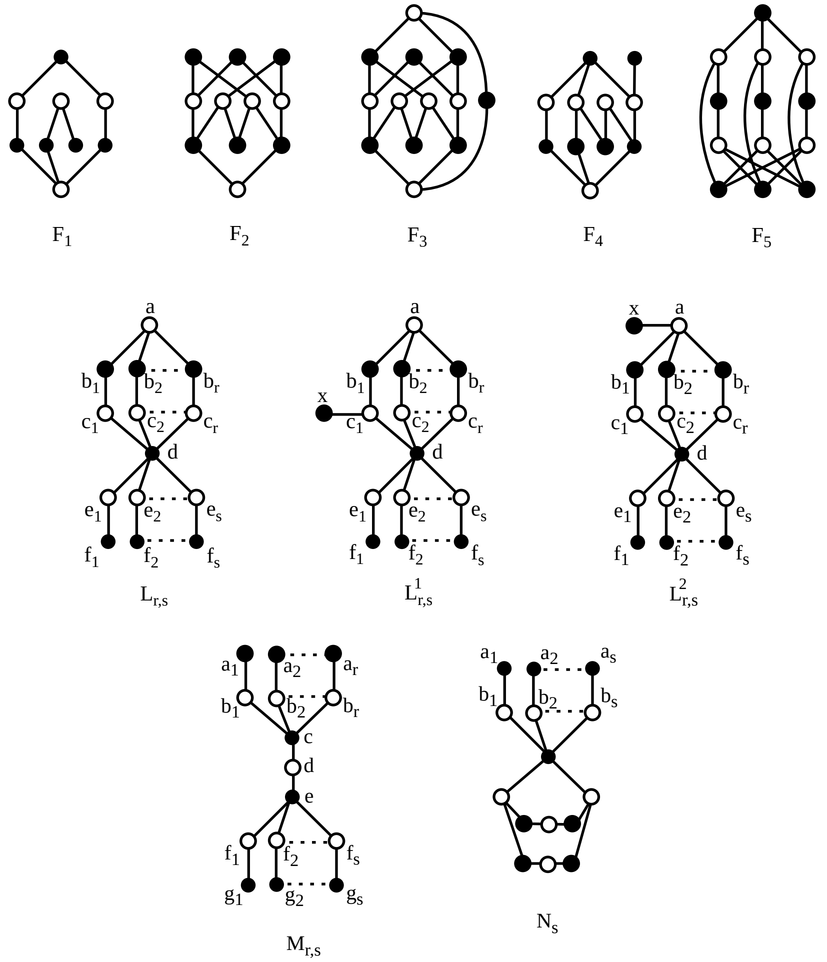

Fig. 2. Minimal augmenting $(C_4, S_{1,2,4})$-free graphs with a vertex of degree $\geqslant 3$.

In the following, we will restrict our attention to minimal (inclusionwise) augmenting graphs. Obviously, any minimal augmenting graph $H = (W, B, E)$ is connected and $|B| = |W| + 1$.

In Sections 2 and 3, we will characterize all minimal augmenting graphs for $(banner, S_{1,2,4})$-free graphs, and $(banner, P_8)$-free graphs. These augmenting graphs are all depicted in Fig. 2, with the exception of complete bipartite graphs and paths of odd length. In Section 4, we will show how to find these augmenting graphs in $(banner, P_8)$-free graphs, which will lead to a polynomial algorithm to solve the MSP in this class of graphs.

Before characterizing all minimal augmenting graphs, we state two helpful lemmas from [3].

Lemma 1. *If $H=(W,B,E)$ is a minimal augmenting graph for a stable set S, then $|A|<|N_B(A)|$ for each subset $A\subseteq W$.*

Lemma 2. *Let H be a connected bipartite banner-free graph. If H contains a C_4, then it is complete bipartite.*

2. Minimal augmenting (*banner*, $S_{1,2,4}$)-free graphs

According to Lemma 2, we know that if a minimal augmenting *banner*-free graph H contains a C_4, then H is complete bipartite. Notice that if a minimal augmenting graph H has no vertex of degree 3 or more, then it is a path with an odd number of vertices. In the following, we characterize minimal augmenting $(C_4,S_{1,2,4})$-free graphs $H=(W,B,E)$ which contain at least one vertex of degree 3 or more. All these graphs are depicted in Fig. 2 along with corresponding notations. We call vertices in W *white* and those in B *black*. In order to characterize minimal augmenting $(C_4,S_{1,2,4})$-free graphs H, we first consider the case where all black vertices are of degree at most 2 in H.

Lemma 3. *Let H be a minimal augmenting $(C_4,S_{1,2,4})$-free graph such that all black vertices are of degree at most 2 in H, and at least one white vertex has degree $\geqslant 3$ in H. Then H is either an $L^2_{2,0}$, an F_1, or an N_0.*

Proof. Let a be a white vertex in H with degree 3 or more. Let $\{b_1,\dots,b_k\}$ $(k\geqslant 3)$ be the neighborhood of a in H. There is at most one vertex of degree 1 among $\{b_1,\dots,b_k\}$, say b_k, else H strictly contains an augmenting P_3. Let c_i denote the second white neighbor of b_i $(1\leqslant i<k)$. In case b_k also has two white neighbors, then its second white neighbor is also denoted by c_k. Clearly, $c_i\neq c_j$ when $i\neq j$, otherwise H contains a C_4. By Lemma 1, each c_j has a second black neighbor, denoted by d_j (it is possible that $d_i=d_j$ for $i\neq j$), other than b_j.

Suppose there is a black vertex d_i that is adjacent to exactly two white vertices c_i and c_j. Then $k=3$, else vertices $c_i,d_i,c_j,b_j,a,b_r,c_r$ and b_s $(r,s\neq i,j)$ induce an $S_{1,2,4}$ in H. Let r be the index in $\{1,2,3\}$ different from i and j. Vertex b_r has a second white neighbor c_r, else H is an augmenting $L^2_{2,0}$. Moreover, vertex d_r has a second white neighbor x, else H is an augmenting F_1. If x belongs to $\{c_i,c_j\}$, then H is an augmenting N_0. Otherwise, vertices x,d_r,c_r,b_r,a,b_i,c_i and b_j induce an $S_{1,2,4}$ in H.

Now suppose that no black vertex d_i is adjacent to a c_j with $i\neq j$. If d_1 and d_2 are both of degree 1, then vertices d_1,c_1,b_1,a,b_2,c_2 and d_2 induce an augmenting P_7, which contradicts the minimality of H. So assume d_1 and/or d_2 has two white neighbors. Say, by symmetry, that d_1 has a second neighbor $e\neq c_1$. Then, vertices e,d_1,c_1,b_1,a,b_2,c_2 and b_3 induce an $S_{1,2,4}$ in H, a contradiction. $\square$

From now on, we will only consider minimal augmenting $(C_4,S_{1,2,4})$-free graphs $H=(W,B,E)$ which contain at least one black vertex b of degree $\geqslant 3$. We denote $x_1,x_2,\dots,x_k$ $(k\geqslant 3)$ the white neighbors of b. From Lemma 1 and Hall's theorem [9], we know that there is a perfect matching, denoted by $M(b)$, in the subgraph of H induced by $V(H)-\{b\}$. We denote $B(b)=\{m(x_1),m(x_2),\dots,m(x_k)\}$ the set of black vertices such that $m(x_i)$ is the vertex matched with x_i in $M(b)$. Notice that no x_i is adjacent to an $m(x_j)$ with $i\neq j$, else H contains a C_4. In what follows, we denote $W(b)$ the set of white vertices which are not neighbors of b but which are adjacent to at least one vertex in $B(b)$.

Lemma 4. *Let $H=(W,B,E)$ be a minimal augmenting $(C_4,S_{1,2,4})$-free graph with a black vertex b of degree $k\geqslant 3$. If $W(b)=\emptyset$, then H is an $M_{k-1,0}$.*

Proof. Since $W(b)=\emptyset$ and H is minimal augmenting, we know that $B=B(b)\cup\{b\}$ and $W=\{x_1,\dots,x_k\}$. Hence, H is an $M_{k-1,0}$. □

Lemma 5. *Let $H=(W,B,E)$ be a minimal augmenting $(C_4,S_{1,2,4})$-free graph with a black vertex b of degree $k\geqslant 4$. If $W(b)\neq\emptyset$, then H is an $L^2_{k,0}$.*

Proof. Let b be a vertex of degree $k\geqslant 4$ in B and assume $W(b)\neq\emptyset$. Consider an arbitrary vertex $z\in W(b)$ and suppose, without loss of generality, that z is adjacent to $m(x_1)$. Let $m(z)$ denote the vertex matched with z in $M(b)$. Since H is C_4-free, vertex $m(z)$ is not adjacent to x_1 and has at least two non-neighbors in $\{x_2,x_3,x_4\}$, say x_2 and x_3.

Vertex z is adjacent to $m(x_2)$, else vertices $m(z),z,m(x_1),x_1,b,x_2,m(x_2)$ and x_3 induce an $S_{1,2,4}$ in H. Furthermore, if z is not adjacent to some vertex $m(x_i)$ in $B(b)$, then vertices $m(x_2),z,m(x_1),x_1,b,x_i,m(x_i)$ and x_j $(j\neq 1,2,i)$ induce an $S_{1,2,4}$ in H. Hence, z and therefore all vertices in $W(b)$ are adjacent to all vertices in $B(b)$. Since H is C_4-free, we now know that $W(b)=\{z\}$ and $m(z)$ has no neighbor in $\{x_1\dots,x_k\}$.

If $m(z)$ has a second white neighbor $z'\neq z$, then vertices $x_1,b,x_2,m(x_2),z,m(z),z'$ and $m(x_3)$ induce an $S_{1,2,4}$ in H, a contradiction. Hence, H is an $L^2_{k,0}$. □

We will now assume that the maximum degree of a black vertex in a minimal augmenting $(C_4,S_{1,2,4})$-free graph is three.

Lemma 6. *Let H be a minimal augmenting $(C_4,S_{1,2,4})$-free graph with no black vertex of degree >3, and at least one black vertex b of degree 3.*

If a vertex in $B(b)$ has two neighbors in $W(b)$, then H is either an F_2 or an F_3.

Proof. Without loss of generality, we may assume that vertex $m(x_1)$ has two white neighbors z,z' in $W(b)$. Let $m(z)$ and $m(z')$ be the vertices matched with z and z' in $M(b)$. Since H is C_4-free, we know that x_1 is neither adjacent to $m(z)$ nor to $m(z')$, and that z is not adjacent to $m(z')$, and z' is not adjacent to $m(z)$. For the same reason, neither $m(z)$ nor $m(z')$ is adjacent to both x_2 and x_3.

Assume that $m(z)$ is neither adjacent to x_2 nor to x_3. Then z is adjacent to $m(x_2)$, else vertices $m(x_2),x_2,b,x_1,m(x_1),z,m(z)$ and x_3 induce an $S_{1,2,4}$ in H. By symmetry, z is also adjacent to $m(x_3)$. Hence, z' is neither adjacent to $m(x_2)$ nor to $m(x_3)$, else H contains a C_4. But now, vertices $x_3,b,x_2,m(x_2),z,m(x_1),z'$ and $m(z)$ induce an $S_{1,2,4}$ in

H, a contradiction. So, we now know that $m(z)$ has exactly one neighbor in $\{x_2,x_3\}$. By symmetry, $m(z')$ also has exactly one neighbor in $\{x_2,x_3\}$.

Assume now that x_2 is adjacent to both $m(z)$ and $m(z')$. Then $m(x_2)$ is neither adjacent to z nor to z', else H contains a C_4. But now, vertices $z', m(x_1), z, m(z), x_2, b, x_3$ and $m(x_2)$ induce an $S_{1,2,4}$ in H, a contradiction. Hence, we may assume that $m(z)$ is adjacent to x_2 (but not to x_3), and that $m(z')$ is adjacent to x_3 (but not to x_2). Hence, z is not adjacent to $m(x_2)$ and z' is not adjacent to $m(x_3)$, else H contains a C_4. Also, z is adjacent to $m(x_3)$, else vertices $m(x_3), x_3, b, x_1, m(x_1), z, m(z)$ and z' induce an $S_{1,2,4}$ in H. By symmetry, z' is adjacent to $m(x_2)$. If none of the vertices $m(z), m(z'), m(x_2)$ and $m(x_3)$ has a third white neighbor, then H is an F_2.

Notice that one can exchange the role of the pair $(m(z), m(z'))$ of vertices with the pair $(m(x_2), m(x_3))$ by replacing the edges $x_2m(x_2)$, $x_3m(x_3)$, $zm(z)$ and $z'm(z')$ in $M(b)$ by $x_2m(z)$, $x_3m(z')$, $zm(x_3)$ and $z'm(x_2)$. Hence, if H is not an F_2, we may assume, by symmetry, that $m(z)$ has a third white neighbor $y \neq z, x_3$. Vertex y is not adjacent to $m(x_i)$ $(i=1,2,3)$, else H contains a C_4. Moreover, y is adjacent to $m(z')$, else vertices $y, m(z), z, m(x_3), x_3, b, x_1$ and $m(z')$ induce an $S_{1,2,4}$ in H. Now, let $m(y)$ be the black vertex matched with y in $M(b)$. Vertex $m(y)$ has no neighbor in $\{x_2,x_3,z,z'\}$, else H contains a C_4. Also, $m(y)$ is adjacent to x_1, else vertices $x_1, m(x_1), z, m(z), y, m(z'), x_3$ and $m(y)$ induce an $S_{1,2,4}$ in H.

If none of the vertices $m(y), m(x_2)$ and $m(x_3)$ has a third white neighbor, then H is an F_3. Otherwise, these three vertices play a symmetric role in F_3 and we can therefore assume that $m(y)$ has a third white neighbor $w \neq y, x_1$. Vertex w is not adjacent to $m(x_1), m(z)$ or $m(z')$ since these three black vertices have already three white neighbors. So, w is adjacent to $m(x_2)$, else vertices $m(x_2), z', m(z'), y, m(y), x_1, b$ and w induce an $S_{1,2,4}$ in H. By symmetry, w is also adjacent to $m(x_3)$. Now, let $m(w)$ be the black vertex matched with w in $M(b)$. Vertex $m(w)$ has no neighbor in $\{x_1,x_2,x_3,z,z',y\}$, else H contains a C_4. Hence, vertices $x_1, b, x_2, m(x_2), w, m(x_3), z$ and $m(w)$ induce an $S_{1,2,4}$ in H, a contradiction. □

From now on, we assume that each vertex in $B(b)$ has at most one neighbor in $W(b)$. Since we also assume that the maximum degree of a black vertex is 3, this means that $W(b)$ contains at most three vertices. The case where $W(b)$ is empty has already been studied in Lemma 4. It remains to consider the cases where $W(b)$ contains one, two and three vertices.

Lemma 7. *Let H be a minimal augmenting $(C_4, S_{1,2,4})$-free graph with no black vertex of degree >3, and at least one black vertex b of degree 3.*

If $|W(b)|=1$, then H is an $L^2_{3,0}$, $L^1_{3,0}$, $L^1_{2,1}$ or an F_4.

Proof. Without loss of generality, assume that the unique vertex z in $W(b)$ is adjacent to $m(x_1)$. Let $m(z)$ be the black vertex matched with z in $M(b)$. Since H is C_4-free, we know that $m(z)$ is not adjacent to x_1.

Suppose $m(z)$ is neither adjacent to x_2 nor to x_3. Then, z is adjacent to $m(x_2)$, else vertices $m(z), z, m(x_1), x_1, b, x_2, m(x_2)$ and x_3 induce an $S_{1,2,4}$ in H. By symmetry, z is also adjacent to $m(x_3)$, and $m(z)$ cannot have a second white neighbor $y \neq z$, else vertices $x_1, b, x_2, m(x_2), z, m(z), y$ and $m(x_3)$ induce an $S_{1,2,4}$ in H. Hence, H is an $L^2_{3,0}$.

Suppose now that $m(z)$ has at least one neighbor among x_2 and x_3, say x_2. Then $m(z)$ is not adjacent to x_3 and z is not adjacent to $m(x_2)$, else H contains a C_4. If $m(z)$ does not have a third white neighbor, then H is either an $L^1_{2,1}$ or an $L^1_{3,0}$ (depending on the existence of the edge between z and $m(x_3)$). Otherwise, let $y \neq z, x_2$ be the third white neighbor of $m(z)$, and let $m(y)$ be the vertex matched with y in $M(b)$. Vertex z is adjacent to $m(x_3)$, else vertices $m(x_3), x_3, b, x_2, m(z), z, m(x_1)$ and y induce an $S_{1,2,4}$ in H. Moreover, vertex $m(y)$ is neither adjacent to x_2 nor to z, else H contains a C_4. Vertex $m(y)$ is adjacent to x_1 or/and x_3, else vertices $m(y), y, m(z), x_2, b, x_1, m(x_1)$ and x_3 induce an $S_{1,2,4}$ in H. By symmetry between x_1 and x_3, we may assume that $m(y)$ is adjacent to x_1. Then, $m(y)$ is not adjacent to x_3, else H contains a C_4. Notice now that $m(y)$ cannot have a third white neighbor $w \neq y, x_1$, else vertices $m(x_3), x_3, b, x_1, m(y), y, m(z)$ and w induce an $S_{1,2,4}$ in H. Hence, H is an F_4. □

Lemma 8. *Let H be a minimal augmenting $(C_4, S_{1,2,4})$-free graph with no black vertex of degree > 3, and at least one black vertex b of degree* 3.

If $|W(b)| = 2$ while no vertex in $B(b)$ has two neighbors in $W(b)$, then H is an F_4 or an N_1.

Proof. Without loss of generality, assume that $W(b) = \{z, z'\}$ such that z is adjacent to x_1 and z' is adjacent to x_2. Let $m(z)$ and $m(z')$ be the vertices matched with z and z' in $M(b)$. Then, $m(z)$ is not adjacent to x_1 and $m(z')$ is not adjacent to x_2, else H contains a C_4.

Suppose that $m(x_3)$ is adjacent to z. Then, $m(x_3)$ is not adjacent to z' since no vertex in $B(b)$ has two neighbors in $W(b)$. Moreover, $m(z)$ is not adjacent to x_3, else H contains a C_4. Now, $m(z)$ is adjacent to x_2, else vertices $m(z), z, m(x_1), x_1, b, x_2, m(x_2)$ and x_3 induce an $S_{1,2,4}$ in H. If $m(z)$ is adjacent to z', then H contains a C_4, else vertices $z', m(x_2), x_2, m(z), z, m(x_1)$, x_1 and $m(x_3)$ induce an $S_{1,2,4}$ in H, a contradiction. Hence, $m(x_3)$ is not adjacent to z. By symmetry, $m(x_3)$ is not adjacent to z'.

Suppose now that z is adjacent to $m(z')$. Then, z' is not adjacent to $m(z)$ and x_1 is not adjacent to $m(z')$, else H contains a C_4. Also, $m(z)$ is adjacent to x_2, else vertices $x_2, m(x_2), z', m(z'), z, m(x_1), x_1$ and $m(z)$ induce an $S_{1,2,4}$ in H. Now, $m(z)$ is not adjacent to x_3, else H contains a C_4, and $m(z')$ is adjacent to x_3, else vertices $m(z'), z, m(x_1), x_1, b, x_2, m(x_2)$ and x_3 induce an $S_{1,2,4}$ in H. Vertex $m(z)$ cannot have a third white neighbor $y \neq z, x_2$, else vertices $y, m(z), z, m(z'), x_3, b, x_1$ and $m(x_3)$ induce an $S_{1,2,4}$ in H. Hence, H is an F_4.

So, assume now that z is not adjacent to $m(z')$. By symmetry, we can also assume that z' is not adjacent to $m(z)$. If x_3 is adjacent to $m(z)$, then x_2 is not adjacent to $m(z)$ (else H contains a C_4), and vertices $z', m(x_2), x_2, b, x_3, m(z), z$ and $m(x_3)$ induce an $S_{1,2,4}$ in H, a contradiction. Hence, x_3 is not adjacent to $m(z)$. By symmetry, x_3 is not adjacent to $m(z')$. Moreover, x_2 is adjacent to $m(z)$, else vertices $m(z), z, m(x_1), x_1, b, x_3, m(x_3)$ and x_2 induce an $S_{1,2,4}$ in H. By symmetry, x_1 is adjacent to $m(z')$. Now, if $m(z)$ has a third white neighbor $y \neq z, x_2$, then vertices $m(x_3), x_3, b, x_2, m(z), z, m(x_1)$ and y

induce an $S_{1,2,4}$ in H, a contradiction. Hence, $m(z)$ has only two white neighbors. By symmetry, $m(z')$ does not have a third white neighbor, and H is an N_1. □

Lemma 9. *Let H be a minimal augmenting $(C_4, S_{1,2,4})$-free graph with no black vertex of degree > 3, and at least one black vertex b of degree 3.*

If $|W(b)| = 3$ while no vertex in $B(b)$ has two neighbors in $W(b)$, then H is an F_5.

Proof. Assume that $W(b) = \{z_1, z_2, z_3\}$ such that z_i is adjacent to $m(x_i)$ $(i = 1, 2, 3)$. No vertex $m(x_i)$ is adjacent to a z_j with $i \neq j$ since no vertex in $B(b)$ has two neighbors in $W(b)$. Let $m(z_1), m(z_2)$ and $m(z_3)$ be the vertices matched with z_1, z_2 and z_3 in $M(b)$. No vertex x_i is adjacent to $m(z_i)$, else H contains a C_4.

Vertex $m(z_1)$ cannot be adjacent to both x_2 and x_3, else H contains a C_4. Hence, assume without loss of generality that $m(z_1)$ is not adjacent to x_2. Now, $m(z_1)$ is adjacent to x_3, else vertices $m(z_1), z_1, m(x_1), x_1, b, x_2, m(x_2)$ and x_3 induce an $S_{1,2,4}$ in H. Hence, $m(z_1)$ is not adjacent to z_3, else H contains a C_4. Moreover, $m(z_1)$ is adjacent to z_2, else vertices $z_2, m(x_2), x_2, b, x_3, m(z_1), z_1$ and $m(x_3)$ induce an $S_{1,2,4}$ in H. Hence, $m(z_2)$ is not adjacent to x_3, else H contains a C_4. Now, by symmetry, we know that $m(z_2)$ is adjacent to x_1 and z_3, but not to z_1, while $m(z_3)$ is adjacent to x_2 and z_1 but not to x_1 and z_2. Since no vertex in $B(b)$ can have a third white neighbor while vertices $b, m(z_1), m(z_2)$ and $m(z_3)$ have three white neighbors, we can conclude that H is an F_5. □

As a consequence of the above lemmas, and by observing that P_1 and P_3 are complete bipartite while $P_5 = M_{1,0}$, we obtain the following characterization of all minimal augmenting graphs in the class of $(banner, S_{1,2,4})$-free graphs.

Theorem 1. *A minimal augmenting $(banner, S_{1,2,4})$-free graph is one of the following graphs*:

- *a complete bipartite graph $K_{r,r+1}$ with $r \geqslant 0$,*
- *a path P_k with k odd $\geqslant 7$,*
- *an $L^2_{r,0}$ with $r \geqslant 2$,*
- *an $M_{r,0}$ with $r \geqslant 1$,*
- *one of the graphs $F_1, \ldots, F_5, L^1_{3,0}, L^1_{2,1}, N_0, N_1$.*

It is proved in [3] that the problem of finding a complete bipartite augmenting graph in a *banner*-free graph is polynomially solvable. Also, the same authors have designed a polynomial algorithm for finding an $L^2_{k,0}$ with $k \geqslant 2$ (these graphs being called *plants*) or an $M_{k,0}$ with $k \geqslant 1$ (these graphs being called *simple augmenting trees*) in a $(banner, S_{1,2,4})$-free graph. Since graphs $F_1, \ldots, F_5, L^1_{3,0}, L^1_{2,1}, N_0, N_1$ have a finite number of vertices, they can be detected in polynomial time. Hence, in order to obtain a polynomial algorithm to solve the MSP in $(banner, S_{1,2,4})$-free graphs, it is enough to design a polynomial algorithm to find an augmenting path P_k for any given stable set in these graphs. Such an algorithm is not yet available. We show in the next section that the above description of all minimal augmenting graphs in $(banner, S_{1,2,4})$-free graphs does lead to a polynomial algorithm for the MSP in $(banner, P_8)$-free graphs.

3. Minimal augmenting (*banner*, P_8)-free graphs

With the exception of graph F_1 and paths with at least nine vertices, all graphs mentioned in Theorem 1 are also P_8-free. Hence, a minimal augmenting $(banner, S_{1,2,4}, P_8)$-free graph $H=(B,W,E)$ is either a complete bipartite graph, a P_7, an $M_{k,0}$ $(k \geqslant 1)$, an $L^2_{k,0}$ $(k \geqslant 2)$, or one of the graphs $F_2,\ldots,F_5, L^1_{3,0}, L^1_{2,1}, N_0, N_1$.

Remember that a minimal augmenting *banner*-free graph is C_4-free, unless it is a complete bipartite graph. We now characterize all minimal augmenting (C_4,P_8)-free graphs which contain an $S_{1,2,4}$. We first consider the case where a minimal augmenting (C_4,P_8)-free graph contains an $L_{3,1}$. As usual, the white vertices belong to the stable set and are replaced by the black vertices to increase the size of the stable set.

Lemma 10. *Let H be a minimal augmenting (C_4,P_8)-free graph. If H contains an $L_{3,1}$, then, H is an $L^1_{r,s}$ or an $L^2_{r,s}$ with $r \geqslant 3$ and $s \geqslant 1$, or an $L^1_{r,0}$ with $r \geqslant 4$.*

Proof. Let $r \geqslant 3$ be the largest integer such that H contains an $L_{r,1}$, and let $s \geqslant 1$ be the largest integer such that H contains an $L_{r,s}$. Let a, b_i, c_i, d, e_j, f_j $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$ be the vertices of such an $L_{r,s}$, being labeled as in Fig. 2.

If f_1 has a second neighbor $x \neq e_1$, then x is not adjacent to d and cannot be adjacent to more than one vertex among b_1, b_2 and b_3, else H contains a C_4. Assume without loss of generality that x is neither adjacent to b_1 nor to b_2. Then vertices $x, f_1, e_1, d, c_1, b_1, a$ and b_2 induce a P_8 in H, a contradiction. Hence, no f_j $(1 \leqslant j \leqslant s)$ has a second neighbor. By Lemma 1, vertices b_i, d and f_j are black, while vertices a, c_i and e_j are white $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$.

If b_1 has a third white neighbor $x \neq a, c_1$, then x is not adjacent to d or to a b_i $(2 \leqslant i \leqslant r)$, else H contains a C_4. Then vertices $x, b_1, a, b_2, c_2, d, e_1$ and f_1 induce a P_8 in H, a contradiction. Hence, no b_i $(1 \leqslant i \leqslant r)$ has a third white neighbor.

If d has an additional white neighbor $x \neq c_i, e_j$ $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$, then we know from Lemma 1 that x has a second black neighbor $x' \neq d$. As observed above, x' is not a black vertex of the $L_{r,s}$ under consideration. Vertex x' is not adjacent to a c_i or an e_j $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$, else H contains a C_4. Now H contains an $L_{r+1,s}$ (if x' is adjacent to a) or an $L_{r,s+1}$ (if x' is not adjacent to a), which contradicts the choice of r and s. Hence, d has exactly $r+s$ white neighbors.

Since an $L_{r,s}$ has as many black vertices as white ones, it is not an augmenting graph. Hence, H must contain an additional black vertex x which is adjacent to at least one white vertex in $L_{r,s}$. Notice that x cannot have more than two white neighbors in $L_{r,s}$, else H contains a C_4.

Assume first that x has two white neighbors in $L_{r,s}$. Without loss of generality, we may suppose that x is adjacent to a and e_1. The graph obtained by removing f_1 from $L_{r,s}$ and adding x is an $L_{r+1,s-1}$. Hence, by maximality of r, we know that $s=1$. If x has a third white neighbor $x' \neq a, e_1$, then we have observed above that x' has no black neighbor in $L_{r,s}$. We know from Lemma 1 that x' has a second black neighbor $x'' \neq x$. Since x'' cannot be adjacent to more than one vertex among c_1, c_2 and c_3 (else H contains a C_4), we may assume that x'' is neither adjacent to c_1 nor to c_2. Then vertices $x'', x', x, a, b_1, c_1, d$ and c_2 induce a P_8 in H, a contradiction. Hence, x has only two white neighbors and H is an $L^1_{r+1,0}$.

Assume now that x has only one white neighbor in $L_{r,s}$. As observed above, if x has a second white neighbor x' outside $L_{r,s}$, then x' has no black neighbor in $L_{r,s}$. If x is adjacent to an e_j ($1 \leqslant j \leqslant s$), then vertices $x', x, e_j, d, c_1, b_1, a$ and b_2 induce a P_8 in H, a contradiction. If x is adjacent to a c_i, say c_1, then vertices $x', x, c_1, d, c_2, b_2, a$ and b_3 induce a P_8 in H, a contradiction. If x is adjacent to a, then vertices $x', x, a, b_1, c_1, d, e_1$ and f_1 induce a P_8 in H, a contradiction. Hence, we now know that x cannot have a white neighbor outside $L_{r,s}$. Now, x is not adjacent to an e_j else vertices e_j, f_j and x induce an augmenting P_3, which contradicts the minimality of H. Therefore, H is either an $L^1_{r,s}$ (if x is adjacent to a c_j) or an $L^2_{r,s}$ (if x is adjacent to a). $\square$

Lemma 11. *Let H be a minimal augmenting $(C_4, P_8, L_{3,1})$-free graph. If H contains an $L_{2,2}$, then H is an $L^1_{2,s}$, an $L^2_{2,s}$ or an N_s with $s \geqslant 2$.*

Proof. Let $s \geqslant 2$ be the largest integer such that H contains an $L_{2,s}$, and let a, b_i, c_i, d, e_j and f_j ($1 \leqslant i \leqslant 2, 1 \leqslant j \leqslant s$) be the vertices of such an $L_{2,s}$, labeled as in Fig. 2.

If f_1 has a second neighbor $x \neq e_1$, then x is not adjacent to d and is adjacent to at most one vertex among b_1 and b_2, else H contains a C_4. Now, one of the vertex sets $\{a, b_1, b_2, c_1, d, e_1, f_1, x\}$ or $\{a, b_1, b_2, d, e_1, e_2, f_1, x\}$ induces a P_8 in H, a contradiction. Hence, no f_j ($1 \leqslant j \leqslant s$) has a second neighbor. According to Lemma 1, vertices b_1, b_2, d and f_j are black while vertices a, c_1, c_2 and e_j are white ($1 \leqslant j \leqslant s$).

If d has an additional white neighbor x outside $L_{2,s}$, then x has no other black neighbor in $L_{2,s}$, else H contains a C_4. By Lemma 1, x has a second black neighbor $x' \neq d$. Then x' is not adjacent to c_1, c_2 or e_j ($1 \leqslant j \leqslant s$), else H contains a C_4. Vertex x' cannot be adjacent to a, else H contains an $L_{3,1}$. Now H contains an $L_{2,s+1}$ which contradicts the maximality of s. Hence, d has exactly $s+2$ white neighbors.

If b_1 has a third white neighbor $x \neq a, c_1$, then x is not adjacent to b_2, else H contains a C_4. Hence, vertices $x, b_1, a, b_2, c_2, d, e_1$ and f_1 induce a P_8 in H, a contradiction. So, both b_1 and b_2 have exactly two white neighbors.

Notice that $L_{2,s}$ contains as many black vertices as white ones. By Lemma 1, there exists an additional black vertex y which is adjacent to at least one vertex in $L_{2,s}$. Notice that y cannot be adjacent to more than two white vertices in $L_{2,s}$ else H contains a C_4. If y is adjacent to exactly two white vertices in $L_{2,s}$, then we may assume that a and e_1 are the two neighbors of y. The graph obtained by removing f_1 from $L_{2,s}$ and adding y is $L_{3,s-1}$ which contains an $L_{3,1}$, a contradiction. Hence, y has only one white neighbor in $L_{2,s}$.

If y is adjacent to an e_j ($1 \leqslant j \leqslant s$), then y and f_j play a symmetric role and we therefore know that y does not have a second white neighbor. Hence, vertices f_j, e_j and y induce an augmenting P_3, which contradicts the minimality of H.

If y is adjacent to a, then y does not have a second white neighbor $y' \neq a$, else vertices $y', y, a, b_1, c_1, d, e_1$ and f_1 induce a P_8 in H. Hence, H is an $L^2_{2,s}$.

If y is adjacent to c_1 (or symmetrically to c_2), then either H is an $L^1_{2,s}$, or y has a second white neighbor $y' \neq c_1$. In the latter case, we know from Lemma 1 that y' has a second black neighbor $y'' \neq y'$. Vertex y'' is not adjacent to c_1 else H contains a C_4. Hence, y'' is adjacent to c_2, else vertices $y'', y', y, c_1, b_1, a, b_2$ and c_2 induce a P_8 in H. Vertex y cannot have a third white neighbor $w \neq y', c_1$, else w is not adjacent to y'' (to avoid a C_4 in H) and vertices $w, y, c_1, b_1, a, b_2, c_2$ and y'' then induce a P_8 in H. By symmetry between y and y'', we also know that y'' has exactly two white neighbors. Hence, H is an N_s. □

Lemma 12. *Let H be a minimal augmenting $(C_4, P_8, L_{3,1}, L_{2,2})$-free graph.*
If H contains an $S_{1,2,4}$, then H is either an $L^2_{2,1}$ or an $M_{r,s}$ with $r \geqslant 2$ and $r \geqslant s \geqslant 1$.

Proof. Assume that vertices a, b, c, d, e, f, g and h induce an $S_{1,2,4}$ in H, the vertices being labeled as in Fig. 1.

If a has a second neighbor $x \neq b$, then x is not adjacent to c, and is adjacent to at most one vertex among e and g, else H contains a C_4. Now, one of the vertex sets $\{a, b, c, d, e, f, g, x\}$ or $\{a, b, c, e, f, g, h, x\}$ induces a P_8 in H, a contradiction. So vertex a has a unique neighbor and we know from Lemma 1 that vertices a, c, e and g are black while vertices b, d, f and h are white. The same lemma tells us that h has second black neighbor $i \neq c$.

If g has a second white neighbor $x \neq f$, then x is not adjacent to e, else H contains a C_4. Hence, x is adjacent to c, else vertices a, b, c, d, e, f, g and x induce a P_8 in H. Now H contains an $L_{3,1}$ (if i is adjacent to f) or an $L_{2,2,}$ (if i is not adjacent to f), a contradiction. We therefore know that g has exactly one white neighbor.

Suppose that i is adjacent to f. Then e does not have a third white neighbor $x \neq d, f$, else x is not adjacent to c or i (to avoid a C_4 in H) and vertices a, b, d, h, i, f, e and x induce a P_8 in H. By symmetry between e and i, we know that i does not have a third white neighbor. Moreover, if c has a fourth white neighbor $x \neq d, f, h$, then we know from Lemma 1 that x has a second black neighbor $x' \neq c$. Since x' is not adjacent to b, d or h (to avoid a C_4 is H), graph H contains an $L_{3,1}$ (if x' is adjacent to f) or an $L_{2,2}$ (if x' is not adjacent to f), a contradiction. Hence, c has exactly three white neighbors, and H is an $L^2_{2,1}$.

Suppose now that i is not adjacent to f. Vertices a, b, c, d, e, f, g, h and i induce an $M_{2,1}$ in H. So let $r \geqslant 2$ be the largest integer such that H contains an $M_{r,1}$, and let $s \geqslant 1$ be the largest integer such that H contains an $M_{r,s}$. Notice that $r \geqslant s$ since $M_{r,s}$ is isomorphic to $M_{s,r}$. Let a_i, b_i, c, d, e, f_j and g_j $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$ be the vertices of such an $M_{r,s}$, the vertices being labeled as in Fig. 2. Since vertices $a_i, b_i, c, d, e, f_j, g_j$ and f_k $(j \neq k)$ induce an $S_{1,2,4}$ in H, we know from above that vertices a_i, c, e and g_j are black while b_i, d and f_j are white. Also, no a_i $(1 \leqslant i \leqslant r)$ and no g_j $(1 \leqslant j \leqslant s)$ has a second white neighbor.

If e has an additional white neighbor $x \neq d, f_j$, $(1 \leqslant j \leqslant s)$, then x is not adjacent to c, else H contains a C_4. By Lemma 1, there exists a black vertex $x' \neq e$ adjacent to x. Vertex x' cannot be adjacent to d or an f_j $(1 \leqslant j \leqslant s)$ else H contains a C_4. If x' is adjacent to b_1, then x' cannot be adjacent to a b_i $(2 \leqslant i \leqslant r)$ else H contains a C_4. Then vertices $f_1, e, x, x', b_1, c, b_2$ and a_2 induce a P_8 in H, a contradiction. Hence, x' is not adjacent to a b_i ($1 \leqslant i \leqslant r$) and H therefore contains an $M_{r,s+1}$, which contradicts the maximality of s. Hence, e has exactly $s+1$ white neighbors.

If c has exactly $r+1$ white neighbors, then H is an $M_{r,s}$. Otherwise, c has an additional white neighbor $x \neq d, b_i$ ($1 \leqslant i \leqslant r$), and we have observed above that x has no other black neighbor in $M_{r,s}$. By Lemma 1, vertex x has a second black neighbor x'. Vertex x' is not adjacent to d or a b_i ($1 \leqslant i \leqslant r$), else H contains a C_4. Moreover, x' is adjacent to an f_j ($1 \leqslant j \leqslant s$), say f_1 else H contains an $M_{r+1,s}$ which contradicts the maximality of r. Now vertices f_1, e, d, c, x, x', b_i and a_i ($1 \leqslant i \leqslant 2$) induce an $L_{2,2}$ in H, a contradiction. □

As a consequence of Theorem 1 and Lemmas 10–12, and observing that $P_7 = M_{1,1}$, we obtain the following characterization of all minimal augmenting (*banner*, P_8)-free graphs.

Theorem 2. *A minimal augmenting* (*banner*, P_8)*-free graph is one of the following graphs*:

- *a complete bipartite graph* $K_{r,r+1}$ *with* $r \geqslant 0$,
- *an* $L^1_{r,s}$ *or an* $L^2_{r,s}$ *with* $r \geqslant 2$ *and* $s \geqslant 0$,
- *an* $M_{r,s}$ *with* $r \geqslant 1$ *and* $r \geqslant s \geqslant 0$,
- *an* N_s *with* $s \geqslant 0$,
- *one of the graphs* $F_2, \ldots, F_5$.

4. Augmentation in the class of (*banner*, P_8)-free graphs

An augmenting F_i ($2 \leqslant i \leqslant 5$) can be found in polynomial time since these graphs have a number of vertices which does not depend on the size of G. Moreover, it is proved in [3] that complete bipartite augmenting graphs can be detected in polynomial time in *banner*-free graphs. In the present section, we study the problem for the remaining graphs listed in Theorem 2. As usual, given a stable set S in G, we call vertices in S white and those in $V-S$ black. We will denote B^i the set of black vertices having exactly i white neighbors. Given a black vertex b, we denote $W(b) = N(b) \cap S$ the set of white neighbors of b. We first show how to find an augmenting $M_{r,s}$ with $r \geqslant 1$ and $r \geqslant s \geqslant 0$.

Lemma 13. *If* G *contains no augmenting* P_3, *then an augmenting* $M_{r,s}$ *with* $r \geqslant 1$ *and* $r \geqslant s \geqslant 0$ *can be found in polynomial time.*

Proof. Consider three black mutually non-adjacent vertices a_1, c, e such that $a_1 \in B^1$, $|W(c)| \geqslant |W(e)|$, $|W(a_1) \cap W(c)| = 1$, $|W(c) \cap W(e)| = 1$ and $|W(a_1) \cap W(e)| = 0$. Let b_1 be the unique vertex in $W(a_1) \cap W(c)$ and let d be the unique vertex in $W(c) \cap N(e)$. Notice that we have chosen, on purpose, the same labeling as in Fig. 2. We now show how to determine in polynomial time whether this initial structure can be extended to an augmenting $M_{r,s}$ in G (with $r = |W(c)| - 1$ and $s = |W(e)| - 1$).

Let $A=(W(c)\cup W(e))-\{b_1,d\}$. For a vertex $w\in A$, we denote $N_1(w)$ the set of black neighbors of w which are in B^1, and which are not adjacent to a_1,c or e. Notice that the desired $M_{r,s}$ exists only if $N_1(w)\neq\emptyset$ for all w in A. Finally, let $V'=\bigcup_{w\in A} N_1(w)$.

Consider any vertex w in A. If $N_1(w)$ contains two non-adjacent vertices y and y', then y,w and y' induce an augmenting P_3 in G, a contradiction. Hence, each $N_1(w)$ induces a clique in G. It follows that the desired augmenting $M_{r,s}$ exists if and only if $\alpha(G[V'])=|A|$.

Consider any induced $P_4=(p_1,p_2,p_3,p_4)$ in $G[V']$ (if any), and let w be the vertex in A such that $p_1\in N_1(w)$. Notice that neither p_3 nor p_4 is adjacent to w, since $N_1(w)$ is a clique. Now, $p_2\in N_1(w)$, else the vertex set $\{c,d,e,w,p_1,p_2,p_3,p_4\}$ induces a P_8 in G. Hence, if $G[V']$ contains a $P_4=(p_1,p_2,p_3,p_4)$, then p_1 and p_2 have the same white neighbor, while p_2 and p_3 have a different white neighbor. This implies that $G[V']$ is $(banner,P_5,C_5,chair)$-free and several polynomial algorithms are available for the computation of $\alpha(G[V'])$ (e.g. [2,10]). □

Finding an augmenting N_s with $s\geqslant 0$ is even simpler as shown in the following lemma.

Lemma 14. *If G contains no augmenting P_3, then an augmenting N_s with $s\geqslant 0$ can be found in polynomial time.*

Proof. Consider five black non-adjacent vertices $x_1,\dots,x_5$ such that $x_i\in B^2$ $(i=1,\dots,4)$, $\bigcup_{i=1}^4(\{x_i\}\cup W(x_i))$ induces a $C_8=(x_1,w_1,x_2,w_2,x_3,w_3,x_4,w_4)$ in G, and $W(x_5)\cap\{w_1,\dots,w_4\}=\{w_2,w_4\}$.

Let $A=W(x_5)-\{w_2,w_4\}$. For a vertex $w\in A$, we denote $N_1(w)$ the set of black neighbors of w which are in B^1, and which are not adjacent to $x_1,\dots,x_4$. Notice that the desired N_s exists only if $N_1(w)\neq\emptyset$ for all w in A. Finally, let $V'=\bigcup_{w\in A} N_1(w)$.

Consider any vertex w in A. If $N_1(w)$ contains two non-adjacent vertices y and y', then y,w and y' induce an augmenting P_3 in G, a contradiction. Hence, each $N_1(w)$ induces a clique in G, and the desired augmenting N_s exists if and only if $\alpha(G[V'])=|A|$.

Let w and w' be any two vertices in A. No vertex $y\in N_1(w)$ is adjacent to a vertex $y'\in N_1(w')$, else vertices x_1,w_1,x_2,w_2,x_5,w,y and y' induce a P_8 in G. Hence, $G[V']$ is the union of $|A|$ disjoint cliques and $\alpha(G[V'])=|A|$. □

We finally show how augmenting $L_{r,s}$ with $r\geqslant 2$ and $s\geqslant 0$ can be found in polynomial time.

Lemma 15. *If G contains no augmenting $P_3,P_5=M_{1,0}$ or $P_7=M_{1,1}$, then an augmenting $L^1_{r,s}$ or $L^2_{r,s}$ with $r\geqslant 2$ and $s\geqslant 0$ can be found in polynomial time.*

Proof. Consider four black non-adjacent vertices b_1,b_2,d and x such that x belongs to B^1, b_1 and b_2 belong to B^2, $\{b_1,b_2\}\cup W(b_1)\cup W(b_2)$ induces a $P_5=(c_1,b_1,a,b_2,c_2)$ in G, d is adjacent to c_1 and c_2 but not to a, and x is adjacent to a or (exclusive) c_1. Notice that we have chosen, on purpose, the same labeling as in Fig. 2. We now show

how to determine in polynomial time whether this initial structure can be extended to an augmenting $L^1_{r,s}$ or $L^2_{r,s}$ in G (with $r+s=|W(d)|$).

Let $A=W(d)-\{c_1,c_2\}$ and let $\bar{B}$ be the set of black vertices which are not adjacent to x, b_1, b_2 or d. For a vertex $w\in A$, we denote $N_1(w)$ the set of black neighbors of w which are in $B^1\cap\bar{B}$, and we denote $N_2(w)$ the set of black vertices in $B^2\cap\bar{B}$ which are adjacent to both a and w. Notice that the desired $L^1_{r,s}$ or $L^2_{r,s}$ exists only if $N_1(w)\cup N_2(w)\neq\emptyset$ for all w in A. Finally, let $V'=\bigcup_{w\in A}N_1(w)$.

Consider any vertex w in A. If $N_1(w)$ contains two non-adjacent vertices y and y', then y, w and y' induce an augmenting P_3 (i.e., a complete bipartite augmenting graph) in G, a contradiction. If $N_2(w)$ contains two non-adjacent vertices y and y', then vertices b_1, a, y, y' and w induce a banner in G, a contradiction. Each vertex y in $N_1(w)$ is adjacent to all vertices y' in $N_2(w)$, else G contains an augmenting $P_5=(x,a,y',w,y)$ (if x is adjacent to a) or an augmenting $P_7=(x,c_1,b_1,a,y',w,y)$ (if x is adjacent to c_1). Hence, each $N_1(w)\cup N_2(w)$ induces a clique in G. It follows that the desired augmenting $L^1_{r,s}$ or $L^2_{r,s}$ exists if and only if $\alpha(G[V'])=|A|$.

Consider any induced $P_4=(p_1,p_2,p_3,p_4)$ in $G[V']$ (if any), and let w be the vertex in $W(d)$ such that $p_1\in N_1(w)\cup N_2(w)$. Notice that neither p_3 nor p_4 is adjacent to w, since $N_1(w)\cup N_2(w)$ is a clique. Now, p_2 is adjacent to w, else vertices $b_1, c_1, d, w, p_1, p_2, p_3$ and p_4 induce a P_8 in G. In summary, if $G[V']$ contains a $P_4=(p_1,p_2,p_3,p_4)$, then p_1 and p_2 have the same white neighbor, while p_2 and p_3 have a different white neighbor. This implies that $G[V']$ is $(banner, P_5, C_5, chair)$-free and several polynomial algorithms are available for the computation of $\alpha(G[V'])$ (e.g. [2,10]). □

As a consequence of the above lemmas, we state the following theorem.

Theorem 3. *The MSP can be solved in polynomial time in the class of* $(banner, P_8)$*-free graphs.*

5. Conclusion

In this paper, we first characterized all minimal augmenting graphs in the class of $(banner, S_{1,2,4})$-free graphs. In order to solve the MSP in polynomial time in this class of graphs, we observed that a polynomial algorithm is needed for finding augmenting paths in $(banner, S_{1,2,4})$-free graphs.

We then observed that all minimal augmenting $(banner, S_{1,2,4})$-free graphs are also P_8-free, with the exception of the odd paths with at least nine vertices and graph F_1 in Fig. 2. This led us to a characterization of all minimal augmenting $(banner, P_8)$-free graphs. Moreover, we have shown that all these augmenting graphs can be found in polynomial time. As a result, we have concluded that a polynomial time algorithm can be developed to solve the MSP in the class of $(banner, P_8)$-free graphs. It should be noticed that we have not analyzed, on purpose, the complexity of the algorithm, since better algorithms can probably be designed. We are mainly interested in the elaboration of the border between NP-hard and polynomially solvable cases. From this point of view, the obtained result is of interest not only because it improves several previously studied cases. In addition, it gives some ideas for future research. First, we conjecture that for arbitrary $k > 0$, the class of $(banner, P_k)$-free graphs contains finitely many minimal augmenting graphs with vertex degree at most 3. When looking at minimal augmenting graphs with some vertices of degree larger than 3, we observe that a P_7 can be viewed as a "pattern" for augmenting graphs of the form $L^2_{r,s}$ or $M_{r,s}$, since any such graph can be obtained from a P_7 by "parallelizing" some of its parts. In a similar way, $L^1_{2,1}$ and N_1 can be viewed as patterns for augmenting graphs $L^1_{r,s}$ $(r > 2, s > 1)$ and N_s $(s > 1)$. We believe that there is a finite number of such patterns which generate all minimal augmenting graphs in $(banner, P_k)$-free graphs when $k \geqslant 9$.

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