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# Finding augmenting chains in extensions of claw-free graphs

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**Abstract**

Finding augmenting chains is in the heart of the maximum matching problem, which is equivalent to the maximum stable set problem in the class of line graphs. Due to the celebrated result of Edmonds, augmenting chains can be found in line graphs in polynomial time. Minty and Sbihi generalized this result to claw-free graphs. In this paper we extend it to larger classes. As a particular consequence, a new polynomially solvable case for the maximum stable set problem has been detected.


*Keywords:* Stable sets; Augmenting chains; Polynomial algorithms; Combinatorial problems; Graph algorithms

## 1. Introduction

We consider simple undirected graphs without loops and multiple edges. As usual, $P_n$ is the chordless chain (path) on $n$ vertices. By $S_{i,j,k}$ we denote a tree with exactly three vertices of degree one being at distance $i, j, k$ from the only vertex of degree three. In particular, $S_{1,1,1}$ is a *claw*, and $S_{1,1,2}$ is a *fork*. A *banner* is the graph with vertices $a, b, c, d, e$ and edges $(a,b)$, $(b,c)$, $(c,d)$, $(d,e)$ and $(e,b)$. By $N(v)$ we denote the neighborhood of a vertex $v$, i.e., the subset of vertices adjacent to $v$.

A *matching* in a graph is a subset of edges no two of which have a vertex in common, and a *stable set* is a subset of pairwise non-adjacent vertices. The problem of finding a matching of maximum cardinality is a special case of the maximum stable set problem, when restricted to the class of line graphs. In general, the maximum stable set problem is NP-hard, while the maximum matching problem is polynomially solvable. The first polynomial time algorithm to find a maximum matching has been proposed by Edmonds [5]. The algorithm exploits the idea of Berge that a matching $M$ in a graph is maximum if and only if there are no augmenting (alternating) chains for $M$ [3]. Later, this idea has been used independently by Minty [10] and Sbihi [16] in order to extend Edmonds' result to a polynomial

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time algorithm for the maximum stable set problem in the class of claw-free graphs, an extension of the line graphs. In his paper, Minty also proposed a generalization of this result to the weighted case, but more recently Nakamura and Tamura [15] showed that his proof fails some special cases and gave modifications to overcome it.

Finding augmenting chains is a special case of a general approach to solve the stable set problem, known as the augmenting graph technique. In the next section we roughly describe the idea of this approach and mention several particular classes of graphs where it provides a polynomial time solution. For some classes, finding augmenting chains is the only open question to fix a polynomial algorithm for the stable set problem. In the present paper we concentrate on this question and answer it positively for certain extensions of claw-free graphs. As a result, we reveal a new polynomially solvable case for the stable set problem, generalizing the result of Minty and Sbihi as well as some other particular cases. Notice that since our result is only applied to the unweighted case, it does not generalize the one of Nakamura and Tamura.

## 2. Preliminaries

Let $G$ be a graph and $S$ a stable set in $G$. We call the vertices of $S$ *white* and the remaining vertices of the graph *black*. A bipartite graph $H = (W, B, E)$ with parts $W$ and $B$ is called augmenting for $S$ if $|B| > |W|$, $W \subseteq S$, $B \subseteq V(G) - S$, and $N(b) \cap S \subseteq W$ for each vertex $b \in B$. Clearly, if $H$ is augmenting for $S$, then $S$ is not of maximum cardinality, since $S' = (S - W) \cup B$ is a larger stable set. The converse is also true: if $S$ is not a maximum stable set, and $S'$ is a stable set with $|S'| > |S|$, then the subgraph of $G$ induced by the set $(S - S') \cup (S' - S)$ is augmenting for $S$. Thus, the problem of finding a stable set of maximum cardinality is polynomially equivalent to detecting augmenting graphs. In general, this is an NP-hard problem. However, if for a certain class of graphs, we have

(a) a complete list of augmenting graphs,
(b) a polynomial time algorithm for detecting each augmenting graph in the list,

then the maximum stable set problem can be solved efficiently with this approach.

For instance, for the class of claw-free graphs, question (a) has a simple answer. Indeed, by definition, augmenting graphs are bipartite, and each vertex in a claw-free bipartite graph clearly has degree at most two. Hence, every connected claw-free bipartite graph is either an even cycle or a chain. Cycles of even length and chains of odd length cannot be augmenting, since they have equal number of vertices in both parts. Thus, every connected claw-free augmenting graph is a chain of even length. However, finding augmenting chains is not a trivial task. In 1980, Minty proposed a way to determine whether a claw-free graph contains an augmenting chain by reducing the problem to the class of line graphs, i.e., to the maximum matching problem.

In 1999, Alekseev [1] extended the result of Minty to the class of fork-free graphs. He has shown that every connected fork-free augmenting graph is either a chain or an almost complete bipartite graph (i.e., a graph in which every vertex has at most one non-neighbor in the opposite part), and has proven that both types of augmenting graphs can be found in polynomial time in fork-free graphs.

Many other classes have been recently studied for possible application of the augmenting graph technique (see, e.g., [2,4,6–9,11–13]). For many of them, polynomial algorithms have been designed. For certain classes, only partial information has been obtained. For instance, for the class of ($S_{1,2,4}$, *banner*)-free graphs (an extension of claw-free graphs) question (a) has been solved completely [6], while for (b), only partial solution is available: the only open problem is how to find augmenting chains in polynomial time in that class of graphs. In the next section we settle this problem even for more general graphs by reducing it to the class of claw-free graphs. To this end, we can make the following assumptions that are helpful for finding augmenting chains in arbitrary graphs.

In order to determine whether $S$ admits an augmenting chain, we consider two black non-adjacent vertices, denoted $x_0$ and $x_k$, each of which has exactly one white neighbor. If $G$ contains no such vertices, then obviously there is no augmenting chain for $S$. Having found such a pair of vertices, we determine whether there exists an augmenting chain with $x_0$ and

$x_k$ being the endpoints. Without loss of generality we assume that

(1) each white vertex has at least two black neighbors,
(2) each black vertex different from $x_0$ and $x_k$ has exactly two white neighbors,
(3) no black vertex is adjacent to $x_0$ or $x_k$.

The vertices not satisfying these assumptions can be simply removed from the graph, since they cannot occur in any augmenting chain connecting $x_0$ to $x_k$.

## 3. Augmenting chains in $(S_{1,2,i}, \mathit{banner})$-free graphs

Let $G = (V, E)$ be a $(S_{1,2,i}, \mathit{banner})$-free graph, and $S$ a maximal stable set in $G$. We look for an augmenting chain of the form $P = (x_0, x_1, x_2, \ldots, x_{k-1}, x_k)$ ($k$ is even) where the even-indexed vertices of $P$ are black, and the odd-indexed vertices are white. To simplify the proof we start with a preprocessing consisting in detecting augmenting chains with at most $i + 3$ vertices. In order to determine whether $S$ admits an augmenting chain with at least $i + 4$ vertices (i.e., $k \geqslant i + 3$), we first find two black non-adjacent vertices $x_0$ to $x_k$, as suggested above, and then two disjoint chordless alternating chains $L = (x_0, x_1, x_2)$ and $R = (x_{k-m}, x_{k-m+1}, \ldots, x_{k-1}, x_k)$ such that no vertex of $L$ is adjacent to any vertex of $R$, and where $m = 2\lfloor i/2 \rfloor$ and each vertex $x_j$ is black if and only if $j$ is even. Such a pair $(L, R)$ of alternating chains is said *candidate*. Our purpose is to find an augmenting chain containing $L$ and $R$ as subchains. Evidently, if there are no such chains, then there is no augmenting chain with at least $i + 4$ vertices between $x_0$ and $x_k$. Having found a candidate pair $(L, R)$ of alternating chains, we may furthermore assume that

(4) no black vertex outside $L$ and $R$ has a neighbor in $L$ or $R$.

Again, the vertices not satisfying the assumption can be removed from the graph, as the desired chain cannot contain them.

**Lemma 1.** *Let $G = (V, E)$ be a $(S_{1,2,i}, \mathit{banner})$-free graph, and $S$ a maximal stable set in $G$. Let $(L, R)$ be a candidate pair of alternating chains with $L = (x_0, x_1, x_2)$ and $R = (x_{k-m}, x_{k-m+1}, \ldots, x_{k-1}, x_k)$, and assume that the vertices of $G$ satisfy* (1)–(4). *If $S$ admits an augmenting chain $P = (x_0, \ldots, x_k)$, then no vertex of $P$ is the center of an induced claw.*

**Proof.** By contradiction, assume that $G$ contains a claw $C(a; b, c, d)$ whose center $a$ is a vertex $x_j$ on $P$. Notice that since each black vertex of $P$ has all its white neighbors defined, each vertex of $C \setminus P$ is black. We shall use the following convention: for a black vertex $v \in \{b, c, d\}$, if only one of the two white neighbors of $v$ is defined explicitly, then the other is denoted $\bar{v}$. Also, for a vertex $v$ belonging to $C \setminus P$, we denote by $r(v)$ the largest index in $\{3, 4, \ldots, k - m - 1\}$ such that $v$ is adjacent to $x_{r(v)}$. We now analyze three cases: exactly one (C1), two (C2) or three (C3) vertices in $\{b, c, d\}$ do not belong to $P$.

*Case* (C1). We may assume $b = x_{j-1}$ and $c = x_{j+1}$. Then $d$ is adjacent neither to $x_{j-2}$ nor to $x_{j+2}$, else there is a $\mathit{Banner}(c, a, b, x_{j-2}, d)$ or a $\mathit{Banner}(b, a, c, x_{j+2}, d)$, respectively. But then we have either an $S_{1,2,i}(x_{j+i}, \ldots, x_{j-2}, d)$ if $r(d) = j$, or an $S_{1,2,i}(x_{r(d)+i-2}, \ldots, x_{r(d)}, d, a, b, x_{j-2}, c)$ if $r(d) > j$, a contradiction.

*Case* (C2). Assume that $b$ belongs to $P$ while $c$ and $d$ are outside $P$. Then vertex $b$ is either equal to $x_{j-1}$ or to $x_{j+1}$. If $b = x_{j-1}$, then $x_{j+1}$ is adjacent both to $c$ and $d$ to avoid (C1), and $x_{j-2}$ is adjacent neither to $c$ nor to $d$, else there is a $\mathit{Banner}(b, x_{j-2}, c, x_{j+1}, d)$, a $\mathit{Banner}(c, a, b, x_{j-2}, d)$ or a $\mathit{Banner}(d, a, b, x_{j-2}, c)$. By symmetry, if $b = x_{j+1}$, then $x_{j-1}$ is adjacent both to $c$ and $d$, while $x_{j+2}$ is adjacent neither to $c$ nor to $d$. In both cases we have $\bar{c} \neq \bar{d}$, else there is a $\mathit{Banner}(b, a, c, \bar{c}, d)$. Moreover, $r(c) \neq r(d)$, since otherwise there is either a $\mathit{Banner}(b, a, c, x_{r(c)}, d)$ (if $r(c) > j + 1$) or an $S_{1,2,i}(x_{j+i+1}, \ldots, x_{j+1}, c, \bar{c}, d)$. But we may then assume $r(c) > r(d)$, and we therefore have an $S_{1,2,i}(x_{r(c)+i-2}, \ldots, x_{r(c)}, c, a, d, \bar{d}, b)$ (if $\bar{d} \neq x_{r(c)-1}$) or an $S_{1,2,i}(x_{r(c)+i}, \ldots, x_{r(c)}, \bar{d}, d, c)$, a contradiction.

*Case* (C3). Without loss of generality suppose that the claw $C(a; b, c, d)$ with center $a = x_j$ minimizes $j$. Notice first that $r(b)$, $r(c)$ and $r(d)$ are three different integers, else we may assume $r(b) = r(c)$ and the claw $C(x_{r(c)}; x_{r(c)+1}, b, c)$ contradicts (C2). Moreover, by minimality of $j$ and by (C2), we know that $x_{j-1}$ has exactly two neighbors in $\{b, c, d\}$, say

$b$ and $c$. To avoid (C1) and (C2) we conclude that $x_{j+1}$ is adjacent to $d$ and has at least one neighbor in $\{b, c\}$, say $c$. Then $x_{j+1}$ is not adjacent to $b$ to avoid a $Banner(d, x_{j+1}, b, x_{j-1}, c)$. We prove now that each white vertex $w \notin \{x_{j-1}, x_j, x_{j+1}\}$ is adjacent to at most one vertex in $\{b, c, d\}$. If this is not the case, then such a white vertex $w$ is adjacent to $b$, $c$ and $d$, otherwise a banner appears. Consequently, $a$ is a white vertex, since otherwise $c$ would have three white neighbors $x_{j-1}, x_{j+1}$ and $w$; but we know by minimality of $j$ that $w \neq x_{j-2}$, and therefore there exists a claw $C(x_{j-1}; x_{j-2}, b, c)$, which contradicts (C2). Now let $v_1$ and $v_2$ be the vertices in $\{b, d\}$ renamed in such a way that $r(v_1) > r(v_2)$, and let $\overline{v_2}$ denote the white neighbor of $v_2$ which is not in $\{x_{j-1}, x_j, x_{j+1}\}$. If $r(c) > r(v_1)$, then there is an $S_{1,2,i}(x_{r(c)+i-2}, \ldots, x_{r(c)}, c, a, v_2, \overline{v_2}, v_1)$. Otherwise, there is a $S_{1,2,i}(x_{r(v_1)+i-2}, \ldots, x_{r(v_1)}, v_1, a, v_2, \overline{v_2}, c)$ (if $\overline{v_2} \neq x_{r(v_1)-1}$) or an $S_{1,2,i}(x_{r(v_1)+i}, \ldots, x_{r(v_1)}, \overline{v_2}, v_2, v_1)$, a contradiction. □

**Theorem 2.** *Given an $(S_{1,2,i}, banner)$-free graph $G$, and a stable set $S$ in $G$, one can determine whether $S$ admits an augmenting chain in time* $O(n^{(i+14)/2})$.

**Proof.** Augmenting chains of small length $k < i + 3$ can be found in a trivial way in time $O(n^{(i+4)/2})$ by inspecting all subsets of black vertices of cardinality at most $(i + 4)/2$. To detect a larger augmenting chain, we first find a pair of black vertices $x_0$ and $x_k$ as described above, and then remove from $G$ all the black vertices not satisfying (2) and (3). For the given pair $x_0$ and $x_k$, we do the following:

Find all candidate pairs $(L, R)$ of alternating chains, and for each such pair, do steps (a) through (d):

(a) remove all black vertices that have a neighbor in $L$ or in $R$,
(b) remove the vertices of $L$ and $R$ except for $x_2$ and $x_{k-m}$,
(c) remove all the vertices that are the center of a claw in the remaining graph,
(d) in the resulting claw-free graph, determine whether there exists an augmenting chain between $x_2$ and $x_{k-m}$.

With an exhaustive search all candidate pairs $(L, R)$ of alternating chains can be found in time $O(n^{(i+2)/2})$. For each such pair, steps (a) through (d) can be implemented in time $O(n^4)$. So, the total time for finding an augmenting chain between $x_0$ and $x_k$ is $O(n^{(i+10)/2})$. In the worst case, we have to check $O(n^2)$ pairs of black vertices for being the endpoints of augmenting chain. Hence the conclusion. □

## 4. Application to $(S_{1,2,4}, banner)$-free graphs

The complete description of minimal (inclusion-wise) augmenting graphs in the class of $(S_{1,2,4}, banner)$-free graphs has been found in [6].

**Theorem 3.** *A minimal augmenting $(S_{1,2,4}, banner)$-free graph is one of the following graphs (see Fig.* 1)

- *a complete bipartite graph,*
- *a chain,*
- *a simple augmenting tree,*
- *a plant,*
- *one of the graphs $F_1, \ldots, F_9$.*

Every graph in the list $F_1, \ldots, F_9$ has at most 7 black vertices. So, these graphs can be detected in time $O(n^7)$. An $O(n^6)$ algorithm is described in [2] for finding simple augmenting trees and plants in $(S_{1,2,4}, banner)$-free graphs. By Theorem 2, an augmenting chain in that class can be determined in time $O(n^9)$. Thus, in at most $O(n^{10})$ steps we can find a stable set in an $(S_{1,2,4}, banner)$-free graph that admits no augmenting graph except possibly complete bipartite graphs. Let $C$ be a subclass of *banner*-free graphs. It is proved in [2] that if for every graph in $C$ one can determine in time $O(n^k)$ a stable set that admits no augmenting graph except possibly complete bipartite graphs, then one can solve the maximum stable set problem in $C$ in time $O(n^{\max\{4,k+1\}})$. Summarizing the above arguments, we conclude that

**Theorem 4.** *Given an $(S_{1,2,4}, banner)$-free graph $G$ with $n$ vertices, one can find a stable set of maximum cardinality in time* $O(n^{11})$.

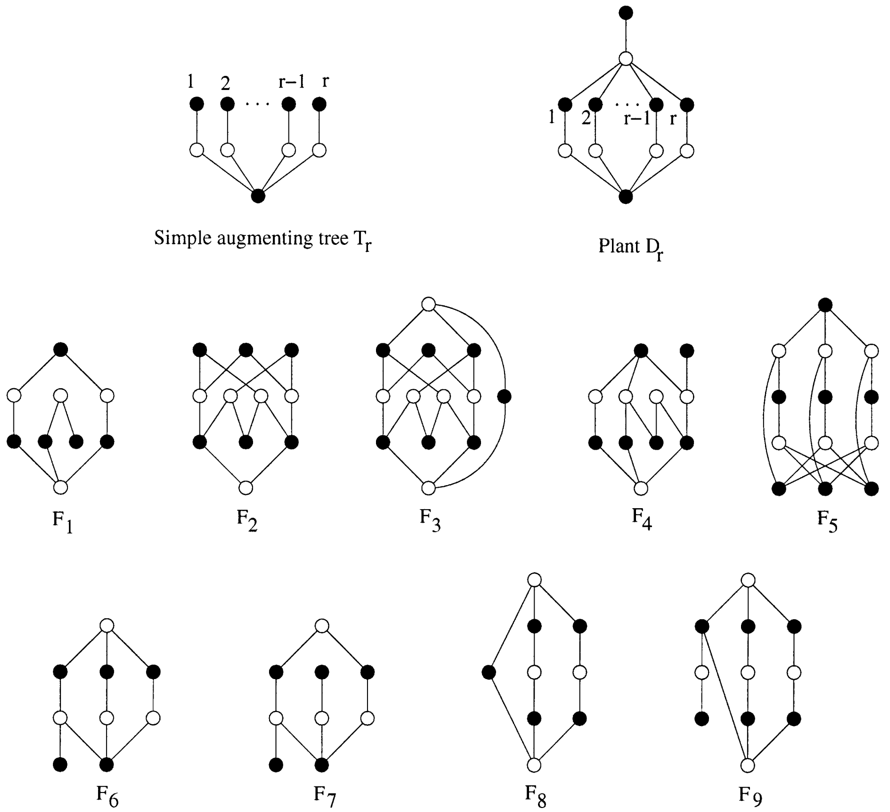


Fig. 1. Augmenting tree $T_r$, plant $D_r$ and graphs $F_1, \ldots, F_9$.

## 5. Conclusion

The paper proves that augmenting chains can be found in polynomial time in the class of $(S_{1,2,i}$, *banner*)-free graphs, for any particular value of $i$. Together with the results in [2] and [6] this leads to the conclusion that the maximum stable set problem is polynomially solvable in the class of $(S_{1,2,4}$, *banner*)-free graphs. Our result generalizes polynomial time algorithms for claw-free graphs [10,16], $(P_6, C_4)$-free graphs [12], and $(P_7$, *banner*)-free graphs [2]. Notice that the maximum stable set problem is NP-hard for $C_4$-free (and hence for banner-free) graphs [14].

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