Extension of Turán's Theorem to the 2-Stability Number

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Abstract

Given a graph G with n vertices and stability number $\alpha(G)$, Turán's Theorem gives a lower bound on the number of edges in G. Furthermore, Turán has proved that the lower bound is only attained if G is the union of $\alpha(G)$ disjoint balanced cliques.

We prove a similar result for the 2-stability number $\alpha_2(G)$ of G, which is defined as the largest number of vertices in a 2-colorable subgraph of G. Given a graph G with n vertices and 2-stability number $\alpha_2(G)$, we give a lower bound on the number of edges in G and characterize the graphs for which this bound is attained. These graphs are the union of isolated vertices and disjoint balanced cliques. We then derive lower bounds on the 2-stability number, and finally discuss the extension of Turán's Theorem to the q-stability number, for $q > 2$.

Keywords: Turán's Theorem; stability number; lower bounds.

1 Introduction

In a graph $G = (V, E)$, a set S of vertices is *stable* if no two vertices in S are linked by an edge. The maximum number of vertices in a stable set of G is denoted $\alpha(G)$ and is called the *stability number* of G. A *q-coloring* of G is a partition of its vertex set into q stable sets. G is said to be *q-colorable* if it admits a q-coloring. The largest number of vertices in a q-colorable subgraph of G is called the *q-stability number* of G and is denoted $\alpha_q(G)$. Hence, $\alpha_1(G) = \alpha(G)$, and $\alpha_q(G) = |V|$ if and only if G is q-colorable. The union of q disjoint stable sets $S_1, \ldots, S_q$ in G is called a *q-stable* set, and is denoted $(S_1, \ldots, S_q)$. The *complement* of a graph G, which is denoted $\overline{G}$, has the same vertex set as G, while two vertices are linked in $\overline{G}$ if and only if they are not linked in G. A *clique* is a set of pairwise adjacent vertices. Hence, a set of vertices is a clique in G if and only if it is stable

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in $\overline{G}$. A clique with k vertices will be called a *k-clique*. The largest number of vertices in a clique of G is called the *clique number* of G. The *order* of a clique or of a q-stable set is its number of vertices. For undefined terms, we follow the terminology of Berge [2].

Let $G_{n,k}$ be the graph made of k disjoint cliques of order $\lceil \frac{n}{k} \rceil$ or $\lfloor \frac{n}{k} \rfloor$. The cliques in $G_{n,k}$ are said to be *balanced*, in that sense that their order differs by at most one unit. Notice that if $n < k$ then $G_{n,k}$ is made of n isolated vertices. Turán's Theorem [8] states that a graph G with n vertices and stability number $\alpha(G) = k$ has at least as many edges as $G_{n,k}$. Moreover, Turán proved that this lower bound on the number of edges in G is only attained if G is isomorphic to $G_{n,k}$. In the complement $\overline{G}$ of G, Turán's Theorem corresponds to giving an upper bound on the number of edges (i.e. 2-cliques) in a graph with given clique number. An extension of this result has been proposed by Roman [6] who has determined an upper bound on the number of q-cliques ($q \geq 2$) in a graph with given clique number. A different extension is studied in this paper. We compute a lower bound on the number of edges in a graph with given q-stability number. This extremal graph problem can also be formulated in the complementary graph as follows: determine the maximum number of edges in a graph with n vertices that does not contain q cliques of order $r_1, \ldots, r_q$ such that $\sum_{i=1}^{q} r_i = k+1$. The special case where $r_1 = \ldots = r_q$ is discussed in [7], and similar extremal graph problems are studied in [3].

When $q = 1$, the above problem reduces to Turán's Theorem. We solve the case $q = 2$, and prove that the lower bound on the number of edges in G is attained only if G is the union of isolated vertices and disjoint balanced cliques. The case $q > 2$ leads to some open questions.

The paper is organized as follows. In the next section, we first rewrite the proof of Turán's Theorem using our terminology, and then prove the extended result. We give in Section 3 several lower bounds for the 2-stability number of a graph with given number of edges and vertices. Section 4 will be devoted to a discussion of our extension of Turán's Theorem to graphs with given q-stability number, with $q > 2$.

2 Turán's Theorem and an extension

Turán's Theorem has been rediscovered many times and some beautiful proofs of this result are presented in [1]. The proof given below is written in such a way that it should help understanding how it can be extended to deal with graphs with given 2-stability number. Given a subset W of vertices in a graph G, we denote G_W the subgraph induced by W in G. Given two disjoint subsets A and B of vertices in G, $\omega(A, B)$ denotes the set of edges linking a vertex in A to a vertex in B, and $m(A, B) = |\omega(A, B)|$, while $m(G)$ denotes the number of edges in G. The neighborhood of a vertex x in G is denoted $N_G(x)$, and its degree $\deg_G(x) = |N_G(x)|$.

Turán's Theorem *Let G be a graph with n vertices and stability number $\alpha(G) = k$. Then*

(a) $m(G) \geq m(G_{n,k})$, *and*
(b) *if* $m(G) = m(G_{n,k})$, *then G is isomorphic to $G_{n,k}$.*

Proof.
The proof is by induction on n. For $n = 1$, G is an isolated vertex and the result clearly holds since $\alpha(G) = 1$ and $G_{1,1} = G$. Assume the result holds for $n - 1$ vertices and consider a graph $G = (V, E)$ with $|V| = n$ vertices and $\alpha(G) = k$.

Proof of (a)
Let S be a stable set of order k in G, and let $R = V \backslash S$. If $\alpha(G_R) < k$, then either $n - k < k$, which means that $m(G_R) \geq 0 = m(G_{n-k,k})$, or else edges can be removed from G_R in order to obtain a graph with stability number k. We therefore know, by induction, that $m(G_R) \geq m(G_{n-k,k})$. Moreover, each vertex in R has at least one neighbor in S, else S is not maximal. Hence, $m(G) = m(G_R) + m(R, S) \geq m(G_{n-k,k}) + (n - k) = m(G_{n,k})$.

Proof of (b)
Assume that $m(G) = m(G_{n,k})$. The above inequalities imply that $m(G_R) = m(G_{n-k,k})$ and $m(R, S) = (n - k)$, which means that G_R is the union of disjoint balanced cliques, and each vertex in R has exactly one neighbor in S. Moreover, the neighborhood of a vertex in S is a clique, else there exist two non-adjacent neighbors y and z of a vertex x in S and $S \cup \{y, z\} \backslash \{x\}$ is a stable set of order $k + 1$ in G. We now prove that the neighborhood of each vertex in R is a clique. If this is not the case, then R contains a vertex x which has two non-adjacent neighbors y and z. Since G_R is the union of disjoint cliques, y and z cannot be both in R. So assume, without loss of generality, that y is in S and z in R, and consider the neighbor z' of z in S. Replacing y by x in S yields a stable set of order k containing two vertices, z' and x, which are neighbors of y, a contradiction.

Up to this point, we have proved that if $m(G) = m(G_{n,k})$, then G is the union of k disjoint cliques. We now show that these cliques are balanced. Assume that G contains two cliques of order a and b, with $a \geq b + 2$. Then, the total number of edges in these two cliques is equal to $\frac{a(a-1)}{2} + \frac{b(b-1)}{2} = \frac{(a-1)(a-2)}{2} + \frac{b(b+1)}{2} + (a - b - 1) > \frac{(a-1)(a-2)}{2} + \frac{b(b+1)}{2}$. Hence, the graph obtained by replacing these two cliques by cliques of order $(a - 1)$ and $(b + 1)$ has n vertices and stability number k, while it has a smaller number of edges than G, a contradiction. ∎

We now prove a similar result for graphs G with given 2-stability number. We divide the proof into two parts. We first consider connected graphs and give a lower bound on their number of edges. We then extend this result to non-connected graphs, and characterize the graphs for which the bound is attained.

Theorem 1 *Let G be a connected graph with $n \geq 2$ vertices and 2-stability number $\alpha_2(G) = k$. Define $n' = n - (k \bmod 2)$. Then*

(a) $m(G) \geq m(G_{n',\lfloor \frac{k}{2} \rfloor})$, *and*

(b) *if* $m(G) = m(G_{n',\lfloor \frac{k}{2} \rfloor})$, *then G is a clique.*

Proof.

Proof of (a)

Let (S_1, S_2) be a 2-stable set in G with $|S_1| + |S_2| = k$ and $|S_1| \leq |S_2|$, and let $R = V \backslash (S_1 \cup S_2)$. Since the union of S_2 and a stable set in R is a 2-stable set in G, it follows that $\alpha(G_R) \leq \lfloor \frac{k}{2} \rfloor$. Moreover, if $\alpha(G_R) < \lfloor \frac{k}{2} \rfloor$, then either $n - k < \lfloor \frac{k}{2} \rfloor$, which means that $m(G_R) \geq 0 = m(G_{n-k,\lfloor \frac{k}{2} \rfloor})$, or else edges can be removed from G_R in order to obtain a graph with stability number $\lfloor \frac{k}{2} \rfloor$. Hence, we know by Turán's Theorem that $m(G_R) \geq m(G_{n-k,\lfloor \frac{k}{2} \rfloor})$. Moreover, each vertex in R has at least one neighbor in S_1 and one in S_2, else (S_1, S_2) is not maximal, which would mean that $\alpha_2(G) > k$.

Consider the partition of S_2 into two parts P and Q such that a vertex of S_2 is in P if and only if it is adjacent to at least one vertex in S_1. Since G is connected, each vertex in Q is adjacent to at least one vertex in R. Let R_Q be the subset of vertices in R which are adjacent to at least one vertex in Q. Each vertex x in R_Q is also adjacent to at least one vertex in P, else $\big(S_1 \cup (N(x) \cap Q), (S_2 \cup \{x\}) \backslash (N(x) \cap Q)\big)$ is a 2-stable set of order $k+1$ in G. Now, we can conclude part (a) as follows:

$$
\begin{aligned}
m(G) &= m(G_R) + m(S_2, S_1) + m(S_1, R) + m(S_2, R) \\
&= m(G_R) + m(S_2, S_1) + m(S_1, R) + m(S_2, R_Q) + m(S_2, R \backslash R_Q) \\
&\geq m(G_{n-k,\lfloor \frac{k}{2} \rfloor}) + |P| + |R| + (|Q| + |R_Q|) + (|R| - |R_Q|) \\
&= m(G_{n-k,\lfloor \frac{k}{2} \rfloor}) + 2|R| + |S_2| \\
&\geq m(G_{n-k,\lfloor \frac{k}{2} \rfloor}) + 2(n-k) + \lfloor \frac{k}{2} \rfloor = m(G_{n-k+2\lfloor \frac{k}{2} \rfloor,\lfloor \frac{k}{2} \rfloor}) = m(G_{n',\lfloor \frac{k}{2} \rfloor}).
\end{aligned}
$$

Proof of (b)

Assume that $m(G) = m(G_{n',\lfloor \frac{k}{2} \rfloor})$. The above inequalities imply that $m(G_R) = m(G_{n-k,\lfloor \frac{k}{2} \rfloor})$, $m(S_1, R) = n - k$, $m(S_2, S_1) + m(S_2, R) = (n-k) + \lfloor \frac{k}{2} \rfloor$, and $|S_2| = \lfloor \frac{k}{2} \rfloor$. Hence, G_R is the union of disjoint balanced cliques. Moreover, since $|S_1| \leq |S_2|$, we know that k is even and $|S_1| = |S_2| = \frac{k}{2}$. This means that each maximum 2-stable set (of order k) in G is made of two disjoint stable sets of order $\frac{k}{2}$. It follows that each vertex x in S_1 has a neighbor in S_2, else $(S_1 \backslash \{x\}, S_2 \cup \{x\})$ is a 2-stable set in G which contradicts the above property. The equality $m(S_2, S_1) + m(S_2, R) = (n-k) + \lfloor \frac{k}{2} \rfloor$ now implies that $m(S_2, S_1) = \frac{k}{2}$, and $m(S_2, R) = n - k$. Hence, we know that each vertex in R has exactly one neighbor in S_1 and one in S_2, each vertex in S_1 has exactly one neighbor in S_2, and each vertex in S_2 has exactly one neighbor in S_1. The set $\omega(S_1, S_2)$ is therefore a matching, and this is true for any 2-stable set (S_1, S_2) of order k in G.

We now prove that the neighborhood of each vertex x in S_1 induces a clique in G. Let y and z be two non-adjacent vertices in $N(x)$. Since $\omega(S_1, S_2)$ is a matching, at least one of y and z is in R, say y. If both y and z are in R, then $(S_1 \cup \{y,z\} \backslash \{x\}, S_2)$ is a 2-stable set of order $k+1$ in G, a contradiction. If z belongs to S_2, then $(S_1 \cup \{y\} \backslash \{x\}, S_2)$ is a 2-stable set of order k in G while $\omega(S_1 \cup \{y\} \backslash \{x\}, S_2)$ is not a matching (z has no neighbor in $S_1 \cup \{y\} \backslash \{x\}$), a contradiction. Since S_1 and S_2 play a symmetric role, we know that the neighborhood of each vertex x in S_2 also induces a clique in G.

We finally prove that the neighborhood of each vertex in R induces a clique in G. If this is not the case, then R contains a vertex x which has two non-adjacent neighbors y and z. Since G_R is the union of disjoint cliques, y and z cannot be both in R. So assume, without loss of generality, that y is in S_1 and z in $R \cup S_2$. Let z' denote the neighbor of z in S_1. Replacing y by x in S_1 yields a 2-stable set of order k containing two vertices in S_1, z' and x, which are neighbors of z, a contradiction. In conclusion, we have proved that G is the union of disjoint cliques, and since G is connected, it is a clique. ■

Notice that Theorem 1 does not necessarily hold for non-connected graphs. Indeed, it is easy to construct non-connected graphs G with n vertices, 2-stability number $\alpha_2(G) = k$, and such that $m(G) < m(G_{n',\lfloor \frac{k}{2} \rfloor})$. For example, if G is the union of a clique of order three and two isolated vertices, then G has 5 vertices and $\alpha_2(G) = 4$, while $3 = m(G) < m(G_{5,2}) = 4$.
A graph $G = (V, E)$ is called *α_q-minimal* if all graphs with $|V|$ vertices and q-stability number $\alpha_q(G)$ have at least $|E|$ edges. The following result is a corollary of the above theorem.

Corollary 1 *An α_2-minimal graph is the union of disjoint cliques.*

Proof.
Let G be an α_2-minimal graph with n vertices and $\alpha_2(G) = k$. Assume first that G is connected. If k is even, then set $G' = G_{n',\lfloor \frac{k}{2} \rfloor} = G_{n,\frac{k}{2}}$, else set G' equal to the graph obtained by adding an isolated vertex to $G_{n',\lfloor \frac{k}{2} \rfloor}$. Since G' has n vertices and $\alpha_2(G') = k$, we have $m(G) \leq m(G')$. But $m(G') = m(G_{n',\lfloor \frac{k}{2} \rfloor})$, which means that the lower bound in Theorem 1 can be attained. We therefore conclude from Theorem 1 that G is a clique. Now, if G is not connected, then its number of edges is the total number of edges in its connected components. Hence, G is the union of disjoint cliques. ■

It follows from the above corollary that if G is an α_2-minimal graph with an odd 2-stability number, then G contains at least one isolated vertex. In order to completely describe all α_2-minimal graphs, it is therefore sufficient to consider graphs with an even 2-stability number.
Define $\underline{G}_{n,k}$ as the graph obtained from $G_{n,k}$ by removing the edges in the maximal cliques of order 2. Hence, $\underline{G}_{n,k} = G_{n,k}$ if $k \leq \frac{n}{3}$, and $\underline{G}_{n,k}$ has n isolated vertices if $k \geq \frac{n}{2}$.

Notice that $\underline{G}_{n,k}$ is the union of isolated vertices and disjoint balanced cliques.

Theorem 2 *Let G be an α_2-minimal graph with n vertices and an even 2-stability number $\alpha_2(G) = k$. Then G is either isomorphic to $\underline{G}_{n,\frac{k}{2}}$, or it can be obtained from $\underline{G}_{n,\frac{k}{2}}$ by replacing an even number p of maximal 3-cliques, by $\frac{p}{2}$ disjoint 4-cliques and p isolated vertices. (Notice that the latter case is only possible if $k+2 \le n \le 2k-2$.)*

Proof.

It follows from the above corollary that G is the union of disjoint cliques. Notice that none of these cliques can be of order 2. We first prove that all cliques of order > 2 are balanced. Assume that G contains two cliques A and B of order a and b, respectively, with $a \ge b+2 \ge 5$. The total number of edges in A and B is equal to $\frac{a(a-1)}{2} + \frac{b(b-1)}{2} = \frac{(a-1)(a-2)}{2} + \frac{b(b+1)}{2} + (a-b-1)$. Hence, the graph obtained by replacing A and B by two disjoint cliques of order $(a-1)$ and $(b+1)$ has a smaller number of edges than G, but the same 2-stability number, a contradiction. Hence, G is the union of x isolated vertices and $\frac{k-x}{2}$ disjoint balanced cliques of order at least 3.

- If $x = 0$, then G is isomorphic to $G_{n,\frac{k}{2}}$, and since $\frac{k}{2} \le \frac{n}{3}$ we have $\underline{G}_{n,\frac{k}{2}} = G_{n,\frac{k}{2}}$.
- Otherwise, G contains at least two isolated vertices. In such a case, all cliques in G have at most four vertices. Indeed, assume that G contains a clique A with $a > 4$ vertices. This clique has $\frac{a(a-1)}{2} = \frac{(a-1)(a-2)}{2} + (a-1)$ edges, and the graph obtained by replacing A and two isolated vertices by two disjoint cliques of order $(a-1)$ and three has a smaller number of edges than in G, but the same 2-stability number, a contradiction.

It remains to prove that if G is made of $x \ge 2$ isolated vertices, y 4-cliques and z 3-cliques, then G is either isomorphic to $\underline{G}_{n,\frac{k}{2}}$, or it can be obtained from $\underline{G}_{n,\frac{k}{2}}$ by replacing an even number p of maximal 3-cliques, by $\frac{p}{2}$ disjoint 4-cliques and p isolated vertices.

- If $y = 0$, then G is isomorphic to $\underline{G}_{n,\frac{k}{2}}$.
- If $x \ge 2y > 0$, then one can replace the y 4-cliques and $2y$ isolated vertices by $2y$ 3-cliques, without modifying the number of edges and the 2-stability number of G. One gets a graph made of the union of isolated vertices and disjoint 3-cliques, which is isomorphic to $\underline{G}_{n,\frac{k}{2}}$.
- If $2y > x > 0$, then one can replace $\frac{x}{2}$ 4-cliques and the x isolated vertices by x 3-cliques, without modifying the number of edges and the 2-stability number of G. One gets a graph made of balanced cliques of order three and four, which is isomorphic to $G_{n,\frac{k}{2}} = \underline{G}_{n,\frac{k}{2}}$.

■

3 Lower bounds on the 2-stability number

Given two graphs G_1 and G_2 with vertex set V_1 and V_2, respectively, their *sum* $G = G_1 + G_2$ is the graph with vertex set $V = \{(x,y) \mid x \in V_1 \text{ and } y \in V_2\}$ and in which two vertices (x,y) and (z,w) are linked by an edge if either $x = z$ and y is linked to w in G_2, or else $y = w$ and x is linked to z in G_1. It is well-known that $\alpha_q(G) = \alpha(G + K_q)$, where K_q is a q-clique. Hence, lower bounds on the q-stability number of a graph can easily be derived from lower bounds on the stability number. For example, the following lower bound on $\alpha(G)$ can be obtained from Turán's Theorem (see for example [4, 5]):

$$\alpha(G) \geq \frac{2n - \frac{2m}{\lceil \frac{2m}{n} \rceil}}{\lceil \frac{2m}{n} \rceil + 1}.$$

We therefore derive the following lower bound for the q-stability number.

Property 1 *Let G be a graph with n vertices and m edges. Then*

$$\alpha_q(G) \geq \frac{2nq - \frac{2mq + nq(q-1)}{\lceil \frac{2m}{n} \rceil + q - 1}}{\lceil \frac{2m}{n} \rceil + q}.$$

Proof.
It is sufficient to observe that $G + K_q$ has nq vertices and $mq + \frac{nq(q-1)}{2}$ edges. ■

Consider now a graph G with n vertices and 2-stability number $\alpha_2(G) = k$. We have proved in Section 2 that $m(G) \geq m(\underline{G}_{n',\lfloor \frac{k}{2} \rfloor})$ where $n' = n - (k \bmod 2)$. We can therefore derive the following lower bound on the number of edges in G.

Property 2 *Let G be a graph with n vertices and 2-stability number $\alpha_2(G) = k$. Then*

$$m(G) \geq \begin{cases} 3(n-k) & \text{if } n < \lfloor \frac{3k}{2} \rfloor \\ (\overline{p} - 1)(n' - \frac{\overline{p}}{2} \lfloor \frac{k}{2} \rfloor) & \text{if } n \geq \lfloor \frac{3k}{2} \rfloor \end{cases}$$

where $n' = n - (k \bmod 2)$ and $\overline{p} = \left\lceil \frac{n'}{\lfloor \frac{k}{2} \rfloor} \right\rceil$.

Proof.
Let $\underline{p} = \left\lfloor \frac{n'}{\lfloor \frac{k}{2} \rfloor} \right\rfloor$ and $\overline{p} = \left\lceil \frac{n'}{\lfloor \frac{k}{2} \rfloor} \right\rceil$ be the smallest and largest order of a maximal clique in $G_{n',\lfloor \frac{k}{2} \rfloor}$, respectively. Since $\underline{G}_{n',\lfloor \frac{k}{2} \rfloor}$ is obtained from $G_{n',\lfloor \frac{k}{2} \rfloor}$ by removing the edges of all maximal 2-cliques, we divide the proof into two parts, depending on whether $\underline{p} \geq 3$ or not. Notice first that $\underline{p} \geq 3$ if and only if $n' \geq 3\lfloor \frac{k}{2} \rfloor$, that is $n \geq \lfloor \frac{3k}{2} \rfloor$.

(i) If $\underline{p} \geq 3$, then $G_{n',\lfloor \frac{k}{2} \rfloor}$ is isomorphic to $\underline{G}_{n',\lfloor \frac{k}{2} \rfloor}$. Let r be the number of $\overline{p}$-cliques in $G_{n',\lfloor \frac{k}{2} \rfloor}$. We have $n' = \lfloor \frac{k}{2} \rfloor(\overline{p}-1) + r$, which means that

$$\begin{aligned} m(G) \geq m(G_{n',\lfloor \frac{k}{2} \rfloor}) &= r\frac{\overline{p}(\overline{p}-1)}{2} + (\lfloor \frac{k}{2} \rfloor - r)\frac{(\overline{p}-1)(\overline{p}-2)}{2} \\ &= \frac{(\overline{p}-1)}{2}(\lfloor \frac{k}{2} \rfloor(\overline{p}-2) + 2r) \\ &= \frac{(\overline{p}-1)}{2}(2n' - \lfloor \frac{k}{2} \rfloor \overline{p}) = (\overline{p}-1)(n' - \frac{\overline{p}}{2}\lfloor \frac{k}{2} \rfloor). \end{aligned}$$

(ii) If $\underline{p} < 3$, then $\underline{G}_{n',\lfloor \frac{k}{2} \rfloor}$ is the union of $n' - 2\lfloor \frac{k}{2} \rfloor$ cliques of order three, and $6\lfloor \frac{k}{2} \rfloor - 2n'$ isolated vertices. Hence, $m(G) \geq m(\underline{G}_{n',\lfloor \frac{k}{2} \rfloor}) = 3(n' - 2\lfloor \frac{k}{2} \rfloor) = 3(n-k)$. ■

Property 2 gives a lower bound on the number of edges in a graph with n vertices and 2-stability number $\alpha_2(G) = k$. We now give an upper bound which is valid when $n < \lfloor \frac{3k}{2} \rfloor$.

Property 3 *Let G be a graph with n vertices, 2-stability number $\alpha_2(G) = k$, and such that $n < \lfloor \frac{3k}{2} \rfloor$. Then*

$$m(G) < \frac{7n^2 - 3n}{18}.$$

Proof.
Let (S_1, S_2) be a 2-stable set in G with $|S_1| + |S_2| = k$. Then

$$\begin{aligned} m(G) &\leq \frac{n(n-1)}{2} - \left(\frac{|S_1|(|S_1|-1)}{2} + \frac{|S_2|(|S_2|-1)}{2}\right) \\ &\leq \frac{n(n-1)}{2} - \frac{|S_1|+|S_2|}{2}\left(\frac{|S_1|+|S_2|}{2} - 1\right) = \frac{n(n-1)}{2} - \frac{k}{2}(\frac{k}{2} - 1). \end{aligned}$$

Since $n < \lfloor \frac{3k}{2} \rfloor$, we have $\frac{k}{2} > \frac{n}{3}$, and we get $m(G) < \frac{n(n-1)}{2} - \frac{n}{3}(\frac{n}{3} - 1) = \frac{7n^2-3n}{18}$. ■

By combining Properties 2 and 3, one gets the following theorem which gives new bounds on the 2-stability number of a graph.

Theorem 3 *Let G be a graph with n vertices and m edges.*

(a) If $m \geq \frac{7n^2-3n}{18}$, then $\min\left\{\frac{2(n-1)^2}{2m+n-1}+1, \frac{2n^2}{2m+n}\right\} \leq \alpha_2(G) \leq \lfloor\frac{2n+1}{3}\rfloor$.

(b) Else $\alpha_2(G) \geq \min\left\{\frac{2(n-1)^2}{2m+n-1}+1, \frac{2n^2}{2m+n}, \max\{\lceil\frac{2n+1}{3}\rceil, n-\frac{m}{3}\}\right\}$.

Proof.

(a) If $m \geq \frac{7n^2-3n}{18}$, then we know from Property 3 that $n \geq \lfloor\frac{3\alpha_2(G)}{2}\rfloor$. Hence, $\alpha_2(G) \leq \lfloor\frac{2n+1}{3}\rfloor$, and it follows from Property 2 that $m \geq (\overline{p}-1)(n'-\frac{\overline{p}}{2}\lfloor\frac{k}{2}\rfloor)$, where $k=\alpha_2(G)$, $n'=n-(k \bmod 2)$, and $\overline{p}=\left\lceil\frac{n'}{\lfloor\frac{k}{2}\rfloor}\right\rceil$.

As in the proof of Property 2, define $r=n'-\lfloor\frac{k}{2}\rfloor(\overline{p}-1)$. We get $m \geq \frac{n'-r}{\lfloor\frac{k}{2}\rfloor}(n'-\frac{n'-r+\lfloor\frac{k}{2}\rfloor}{2})$, and r can take any value in $\{1,2,\ldots,\lfloor\frac{k}{2}\rfloor\}$. This function in r reaches its minimal value in $r=\lfloor\frac{k}{2}\rfloor$. Hence, $m \geq \frac{(n'-\lfloor\frac{k}{2}\rfloor)n'}{2\lfloor\frac{k}{2}\rfloor}$, which means that $\lfloor\frac{k}{2}\rfloor \geq \frac{n'^2}{2m+n'}$.

Since $k=2\lfloor\frac{k}{2}\rfloor+(n-n')$, we conclude that $\alpha_2(G)=k \geq \min\left\{\frac{2(n-1)^2}{2m+n-1}+1, \frac{2n^2}{2m+n}\right\}$.

(b) If $m < \frac{7n^2-3n}{18}$, then we do not know whether $n \geq \lfloor\frac{3\alpha_2(G)}{2}\rfloor$ or not. If this is the case, then we get the same lower bound as in (a). So assume that $n < \lfloor\frac{3\alpha_2(G)}{2}\rfloor$. This means that $\alpha_2(G) > \frac{2n}{3}$, hence $\alpha_2(G) \geq \lceil\frac{2n+1}{3}\rceil$. Moreover, it follows from Property 2 that $m \geq 3(n-k)$. We therefore have $\alpha_2(G)=k \geq \max\left\{\lceil\frac{2n+1}{3}\rceil, n-\frac{m}{3}\right\}$.

■

4 Open questions

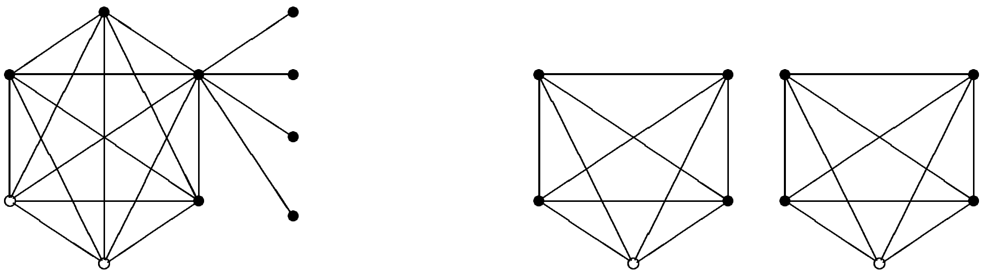
Define $G^q_{n,k}$ as the graph obtained from $G_{n,k}$ by removing all edges in the maximal cliques of order at most q. Hence, $G^1_{n,k}=G_{n,k}$ and $G^2_{n,k}=\underline{G}_{n,k}$. Given any positive integer q, one could think about extending the theorems in Section 2 by proving that all graphs with n vertices and q-stability number $\alpha_q(G)=k$ satisfy $m(G) \geq m(G^q_{n',\lfloor\frac{k}{q}\rfloor})$, where $n'=n-(k \bmod q)$. This property is true for $q \leq 2$, as shown in Section 2, but it does not hold when $q \geq 3$. Indeed, consider the graph which is the union of a clique of order $q+2$ and q isolated vertices. This graph has $2q+2$ vertices, $\frac{(q+2)(q+1)}{2}$ edges, and its q-stability number is equal to $2q$. For comparison, $G^q_{2q+2,2}$ has $q(q+1)$ edges, and this number is strictly larger than $\frac{(q+2)(q+1)}{2}$ when $q \geq 3$.

If we add the constraint that G has to be connected, then it may still happen that $m(G) < m(G^q_{n',\lfloor\frac{k}{q}\rfloor})$ when $q \geq 4$. Indeed, the graph obtained by linking each vertex of a stable set of order q to one vertex of a clique of order $q+2$ (see Figure 1) has $2q+2$

vertices, $\frac{(q+2)(q+1)}{2} + q$ edges, and its q-stability number is equal to $2q$. For comparison, $G^q_{2q+2,2}$ has $q(q+1)$ edges, and this number is strictly larger than $\frac{(q+2)(q+1)}{2} + q$ when $q \geq 4$. For $q = 3$, we have however not been able to prove or disprove this property. We therefore mention our first open question.

Open question

Let G be a connected graph with n vertices and 3-stability number $\alpha_3(G) = k$. Is it true that $m(G) \geq m(G^3_{n',\lfloor \frac{k}{3} \rfloor})$, where $n' = n - (k \bmod 3)$?



This graph is connected and has 19 edges. Graph $G^4_{10,2}$ has 20 edges.

Figure 1: These two graphs have 10 vertices and their 4-stability number is equal to 8. The black vertices correspond to those of a maximum 4-stable set.

Another interesting point would be to characterize all α_q-minimal graphs. We conjecture that each such graph is the union of isolated vertices and disjoint balanced cliques. As observed above, it may happen that a graph which is the union of two disjoint cliques of order $y > q$ has more edges than the graph which is the union of q isolated vertices and a clique of order $2y - q$. It is not difficult to show that this happens if and only if y is an integer strictly smaller than $Q_1 = q + \frac{\sqrt{2q(q-1)}}{2}$. Hence, an α_q-minimal graph cannot contain two disjoint maximal cliques of order y with $q + 1 \leq y < Q_1$.

Similarly, by defining $Q_2 = q + \frac{\sqrt{2q(q-1)+1}-1}{2}$, it is not difficult to show that an α_q-minimal graph cannot contain two disjoint maximal cliques of order y and $y+1$, respectively, with $q + 1 \leq y < Q_2$.

Define $Q = \lfloor Q_1 \rfloor$ and $n' = n - (k \bmod q)$. Notice that Q is either equal to $\lfloor Q_2 \rfloor$ or to $\lfloor Q_2 \rfloor + 1$. If $n' \geq (Q+1)\lfloor \frac{k}{q} \rfloor$, then all cliques in $G^q_{n',\lfloor \frac{k}{q} \rfloor}$ are of order larger than Q, and we therefore conjecture that $G^q_{n',\lfloor \frac{k}{q} \rfloor} = G_{n',\lfloor \frac{k}{q} \rfloor}$ is an α_q-minimal graph. When $n' < (Q+1)\lfloor \frac{k}{q} \rfloor$, we believe that a graph with n vertices and $\alpha_q(G) = k$ is α_q-minimal if it is the union of isolated vertices and disjoint balanced cliques of order $\geq Q$. Our precise conjecture can be stated as follows.

Conjecture

Let G be an α_q-minimal graph with n vertices and q-stability number $\alpha_q(G) = k$. Define $n' = n - (k \bmod q)$, $Q_1 = q + \frac{\sqrt{2q(q-1)}}{2}$, $Q_2 = q + \frac{\sqrt{2q(q-1)+1}-1}{2}$, and $Q = \lfloor Q_1 \rfloor$.

(a) If $\left(Q > \lfloor Q_2 \rfloor \text{ and } n' \geq (Q+1)\lfloor \frac{k}{q} \rfloor - 1\right)$ or $\left(Q = \lfloor Q_2 \rfloor \text{ and } n' \geq (Q+1)\lfloor \frac{k}{q} \rfloor\right)$, then G is the disjoint union of

$$\begin{cases} n' \bmod \lfloor \frac{k}{q} \rfloor & \text{cliques of order } \lceil \frac{n'}{\lfloor \frac{k}{q} \rfloor} \rceil \\ \lfloor \frac{k}{q} \rfloor - (n' \bmod \lfloor \frac{k}{q} \rfloor) & \text{cliques of order } \lfloor \frac{n'}{\lfloor \frac{k}{q} \rfloor} \rfloor \\ k \bmod q & \text{isolated vertices} \end{cases}$$

(b) If $\left(Q > \lfloor Q_2 \rfloor \text{ and } n' < (Q+1)\lfloor \frac{k}{q} \rfloor - 1\right)$ or $\left(Q = \lfloor Q_2 \rfloor \text{ and } n' < (Q+1)\lfloor \frac{k}{q} \rfloor\right)$, then

(b1) If $Q = Q_2 > 1$ and $n = k - 2q + 2Q + 1$, then G is either the disjoint union of

$$\begin{cases} 1 & \text{clique of order } 2Q - q + 1 \\ k - q & \text{isolated vertices} \end{cases}$$

or the disjoint union of

$$\begin{cases} 1 & \text{clique of order } Q \\ 1 & \text{clique of order } Q + 1 \\ k - 2q & \text{isolated vertices} \end{cases}$$

(b2) If $Q = Q_1 > 3$ and $n = k - 2q + 2Q$, then G is either the disjoint union of

$$\begin{cases} 1 & \text{clique of order } 2Q - q \\ k - q & \text{isolated vertices} \end{cases}$$

or the disjoint union of

$$\begin{cases} 2 & \text{cliques of order } Q \\ k - 2q & \text{isolated vertices} \end{cases}$$

(b3) If $Q = Q_1 = 3$ (hence $q = 2$), then there exists an integer x such that $0 \leq x \leq \min\left\{\lfloor \frac{n-k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor - \lfloor \frac{n-k+1}{2} \rfloor\right\}$ and G is the disjoint union of

$$\begin{cases} y = 2x + ((n-k) \bmod 2) & \text{cliques of order } 3 \\ \frac{n-k-y}{2} & \text{cliques of order } 4 \\ 2k - n - y & \text{isolated vertices} \end{cases}$$

(b4) Else define $R = n' - q\lfloor \frac{k}{q} \rfloor$ and $P = Q + 1 - q$. Then G is the disjoint union of

$$\begin{cases} y = (R \bmod P) \bmod \lfloor \frac{R}{P} \rfloor & \text{cliques of order } Q + 1 + \lceil \frac{R \bmod P}{\lfloor \frac{R}{P} \rfloor} \rceil \\ \lfloor \frac{R}{P} \rfloor - y & \text{cliques of order } Q + 1 + \lfloor \frac{R \bmod P}{\lfloor \frac{R}{P} \rfloor} \rfloor \\ k - q\lfloor \frac{R}{P} \rfloor & \text{isolated vertices.} \end{cases}$$

Notice that given any two numbers n and k with $k \leq n$, the above conjecture states that in most cases, there is only one α_q-minimal graph with n vertices and q-stability number $\alpha_q = k$. The exceptions occur

- when $Q = Q_2 > 1$ and $n = k - 2q + 2Q + 1$, in which case there are two α_q-minimal graphs,
- when $Q = Q_1 > 3$ and $n = k - 2q + 2Q$, in which case there are two α_q-minimal graphs,
- when $q = 2$ and $n \leq k + 2\lfloor \frac{k}{2} \rfloor - 8$, in which case there are $1 + \min\left\{\lfloor \frac{n-k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor - \lfloor \frac{n-k+1}{2} \rfloor\right\}$ α_2-minimal graphs.

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