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Note

On perfect switching classes

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Abstract

We study a graph transformation (defined by Seidel) called switching which, given a graph $G=(V,E)$ and a subset $W \subseteq V$ of its vertices, builds a new graph by exchanging the cocycle linking W to $V \backslash W$ with its complement. Switching is an equivalence relation and the associated equivalence classes are called switching classes. A switching class is perfect if it contains only perfect graphs. We show that a switching class is perfect if and only if some graph in the class is P_4-free, and that whether a graph belongs to such a class can be determined in polynomial time. We also show that a graph belongs to a perfect switching class if and only if it contains no C_5, bull, gem or anti-gem as an induced subgraph.

Keywords: Switching; Perfect graphs

1. Introduction

A graph is perfect if for every induced subgraph G, the chromatic number $\chi(G)$ of G equals the clique number $\omega(G)$ of G. This definition was introduced by Claude Berge [1], who conjectured that a graph is perfect if and only if it contains no induced subgraph isomorphic to a chordless odd cycle with at least five verticces, or the complement of such a cycle. Graphs of such form are called *Berge* while the conjecture is known as the *Strong Perfect Graph Conjecture* and remains open.

Given a graph $G=(V,E)$ and a subset $W \subseteq V$ of its vertices, the set of edges in G linking a vertex of W to a vertex outside W is called a *cocycle*. The purpose of this paper is to study a graph transformation called *switching* that exchanges the edges of a cocyle in G for its complement. More precisely, given any subset $W \subseteq V$ of vertices in $G=(V,E)$, all edges having exactly one endpoint in W are removed

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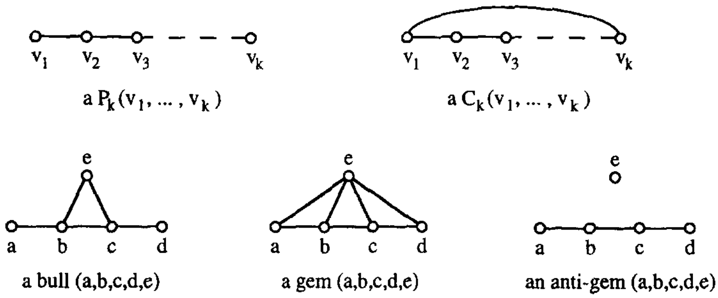

Fig. 1.

from G while an edge is added between a vertex $x \in W$ and a vertex $y \notin W$ if x is not adjacent to y in G. The new graph obtained through this transformation will be denoted $\sigma(G,W)$. Switching has been defined by Seidel [7] and is also referred to as *Seidel switching*.

Switching is an equivalence relation and the associated equivalence classes are called *switching classes*. Two graphs are called *switching equivalent* if they belong to the same switching class. Colbourn and Corneil [3] have shown that determining whether two graphs are switching equivalent is polynomial time equivalent to deciding graph isomorphism. Various studies on switching can be found in [5–7].

We call a switching class *perfect* if it contains only perfect graphs and a graph *switching-perfect* if it belongs to a perfect switching class. The aim of this paper is to characterise perfect switching classes.

A chordless cycle, respectively chain, on k vertices is denoted by C_k, respectively P_k. A *bull* is the (self complementary) graph with five vertices a,b,c,d,e and five edges ab, bc, cd, be, and ce. A *gem* is the graph obtained from a P_4 by adding a vertex adjacent to all four vertices of the P_4. The complement of a gem is called *anti-gem*. All these particular graphs are represented in Fig. 1. If a graph G does not contain another graph H as an induced subgraph, we say that G is H-free.

2. The main result

Complementarity is preserved under switching. Indeed, it directly follows from the definitions that given a graph $G=(V,E)$ and a subset W of V, the complement $\overline{\sigma(G,W)}$ of $\sigma(G,W)$ is equal to $\sigma(\bar{G},W)$ (where $\bar{G}$ denotes the complement of G).

It is known that perfect graphs are Berge; in particular, perfect graphs are C_5-free. Thus, switching-perfect graphs contain no C_5, nor any graph switching equivalent to C_5, as an induced subgraph.

Lemma. C_5, bull, gem and anti-gem are all switching equivalent, and no other graph is switching equivalent to any of these.

Proof. Since switching is an equivalence relation, it suffices to verify that C_5 is switching equivalent to C_5, bull, gem and anti-gem, and no other graph. Let $G=(V,E)$ be isomorphic to C_5, and let W be any subset of V. Since $\sigma(G,W)=\sigma(G,V\backslash W)$, we may assume that $|W|$ is smaller than $|V\backslash W|$. Now, there are up to isomorphism only four cases to consider, namely W is empty, a single vertex, two adjacent vertices, or two non-adjacent vertices. In these cases $\sigma(G,W)$ is respectively C_5, bull, gem and anti-gem. □

Let $\mathbb{C}$ be the class of graphs which are C_5-free, bull-free, gem-free and anti-gem-free. It follows from the lemma that a graph belongs to $\mathbb{C}$ if and only if it is switching equivalent to a graph in $\mathbb{C}$.

Theorem 1. *For a switching class S, the following statements are equivalent:*
(1) *S is perfect.*
(2) *Every graph in S is C_5-free, bull-free, gem-free and anti-gem-free.*
(3) *Some graph in S is P_4-free.*

Proof. (1)⇒(2), as argued at the beginning of the section. To show (2)⇒(3), let x be any vertex in a graph G in $\mathbb{C}$, and let W be the union of x with all vertices adjacent to x. It is sufficient to prove that $H=\sigma(G,W)$ is P_4-free.

To see this, argue by contradiction: suppose H contains $P_4(a,b,c,d)$ as an induced subgraph. Since x is universal in H, x does not belong to $\{a,b,c,d\}$. Let $U=W\cap\{a,b,c,d\}$. By reversing the labelling of P_4 if necessary, there are only ten cases to consider, namely $U=\emptyset$, $\{a\},\{b\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{a,b,c\},\{a,b,d\},\{a,b,c,d\}$, in which case $\{a,b,c,d,x\}$ induces, respectively, an anti-gem, a bull, an anti-gem, a gem, an anti-gem, a C_5, a bull, a gem, a bull and a gem in G.

To show (3)⇒(1), let G be any P_4-free graph, let W be any vertex subset of G, and let $H=\sigma(G,W)$. G is in $\mathbb{C}$ since C_5, the bull, the gem, and the anti-gem each contain P_4 as an induced subgraph. Thus H is in $\mathbb{C}$, by the Lemma. Also, H is Berge, since every graph in $\mathbb{C}$ is C_5-free, C_{2k+1}-free ($k\geqslant 3$) (such graphs contain anti-gems), and $\overline{C_{2k+1}}$-free ($k\geqslant 3$) (such graphs contain gems). Chvátal and Sbihi [2] proved that Berge bull-free graphs are perfect, so H is perfect. □

Corollary. *A graph is switching-perfect if and only if it is C_5-free, bull-free, gem-free, and anti-gem-free.*

The proof of Theorem 1 yields an algorithm for determining whether a graph $G=(V,E)$ is switching-perfect.

RECOGNITION ALGORITHM	
Input.	A graph G=(V,E).
Output	Either a P_4-free graph which is switching equivalent to G or the message 'non switching-perfect'.
1.	Let x be any vertex in V. Consider the subset W of V containing x as well as all vertices adjacent to x in G. Determine the graph H=σ(G,W).
2.	Determine whether H is P_4-free. If H is P_4-free then return H. Else return 'non switching-perfect'.

Constructing H at Step 1 takes $O(|V|^2)$ time. Let E' be the edge set of H. Determining whether H is P_4-free takes $O(|E'|)$ time using the algorithm of Corneil et al. [4]. Since $|E'| \leqslant |V|^2$, the recognition algorithm runs in $O(|V|^2)$ time.

3. An open problem

The corollary characterises switching-perfect graphs. Different results can be obtained if restricted versions of switching are considered, for example, if the size of the switching set W must be one.

Theorem 2. *Let x be any vertex in a Berge bull-free graph G. Then $\sigma(G,\{x\})$ is Berge.*

Proof. Assume $H=\sigma(G,\{x\})$ is not Berge; we wish to show that G is not Berge or not bull-free. If H contains $C_{2k+1}(v_1,v_2,\ldots,v_{2k+1})$ $(k \geqslant 2)$ as an induced subgraph, then x must be a vertex on this cycle, else G is not Berge. Assume without loss of generality that x is equal to v_1. Then, either $\{v_1,v_2,v_3,v_4,v_5\}$ (if $k=2$) or $\{v_1,v_2,v_3,v_4,v_6\}$ (if $k>2$) induces a bull in G.

Since the complement $\bar{H}$ of H is equal to $\sigma(\bar{G},\{x\})$, and since $\bar{G}$ is also Berge and bull-free, it follows that $\bar{H}$ does not contain any C_{2k+1} $(k \geqslant 2)$ as an induced subgraph. □

However, if x is a vertex in a Berge bull-free graph G, then $\sigma(G,\{x\})$ is not necessarily bull-free. Indeed, if G is a gem (a,b,c,d,e), then $\sigma(G,\{b\})$ is a bull. Notice also that if G is a bull (a,b,c,d,e), then $\sigma(G,\{e\})$ is C_5. This means that if x is a vertex in a Berge graph G, then $\sigma(G,\{x\})$ is not necessarily Berge. Thus, the Strong Perfect Graph Conjecture yields the following question.

Open question. Let G be any perfect graph. Is it true that $\sigma(G,\{x\})$ is perfect for every vertex x in G if and only if G is Berge and bull-free?

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