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A Polynomial Time Reduction from Linear to Cyclic

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Abstract

A k-edge-coloring of a graph $G = (V, E)$ is a function c that assigns an integer $c(e)$ (called color) in $\{0, 1, \cdots, k-1\}$ to every edge $e \in E$ so that adjacent edges get different colors. A k-edge-coloring is linear compact if the colors incident to every vertex are consecutive. The problem $k - LCCP$ is to determine whether a given graph admits a linear compact k-edge coloring. A k-edge-coloring is cyclic compact if there are two positive integers a_v, b_v in $\{0, 1, \cdots, k-1\}$ for every vertex v such that the colors incident to v are exactly $\{a_v, (a_v + 1) \bmod k, \cdots, b_v\}$. The problem $k - CCCP$ is to determine whether a given graph admits a cyclic compact k-edge coloring. We show that the $k - LCCP$ with possibly imposed or forbidden colors on some edges is polynomially reducible to the $k - CCCP$ when $k \geq 12$, and to the $12 - CCCP$ when $k < 12$.

Key Words: Consecutive (or interval) edge-colorings; compactness requirements; cyclic production scheduling.

Résumé

Une k-coloration des arêtes d'un graphe $G = (V, E)$ est une fonction c qui affecte un entier $c(e) \in \{0, 1, \cdots, k-1\}$ (appelé couleur) à chaque arête $e \in E$ de telle sorte que les arêtes adjacentes aient des couleurs différentes. Une k-coloration des arêtes est linéaire compacte si les couleurs incidentes à chaque sommets sont consécutives. Le problème $k - LCCP$ consiste à déterminer si un graphe donnné admet une k-coloration linéaire compacte de ses arêtes. Une k-coloration des arêtes d'un graphe G est cyclique compacte si pour chaque sommet v dans G il existe deux entiers a_v, b_v dans $\{0, 1, \cdots, k-1\}$ tel que les couleurs incidentes à v constituent l'ensemble $\{a_v, (a_v + 1) \bmod k, \cdots, b_v\}$ au complet. Le problème $k - LCCP$ consiste à déterminer si un graphe donné admet une k-coloration cyclique compacte de ses arêtes. Nous démontrons que le $k - LCCP$, avec possiblement des couleurs imposées ou interdites sur certaines arêtes, est polynomialement réductible au $k - CCCP$ lorsque $k \geq 12$, et au $12 - CCCP$ lorsque $k < 12$.

1 Introduction

All graphs considered in this paper have no loop but may contain parallel edges. A *k-edge-coloring* of a graph $G = (V, E)$ is a function $c : E \rightarrow \{0, 1, \cdots, k-1\}$ that assigns a color $c(e)$ to every edge $e \in E$ such that $c(e) \neq c(e')$ whenever e and e' share a common endpoint. Let E_v denote the set of edges incident with vertex $v \in V$. The *degree* of a vertex v is the number of edges in E_v and the maximum degree in G is denoted $\Delta(G)$. Note that all k-edge-colorings of a graph G use at least $\Delta(G)$ different colors, which means that $\Delta(G) \leq k$.

A k-edge-coloring of a graph $G = (V, E)$ is *linear compact* if $\{c(e) : e \in E_v\}$ is a set of consecutive positive integers for each vertex $v \in V$. The terms *consecutive edge-colorings* [4, 7] and *interval edge-colorings* [1, 2, 8, 9, 12, 14] are also used by some authors. A graph is *linearly compactly colorable* if it admits a linear compact k-edge-coloring for some integer k. For a k-edge-coloring c of a graph $G = (V, E)$, let $c_{min}(v) = \min_{e \in E_v}\{c(e)\}$ and $c_{max}(v) = \max_{e \in E_v}\{c(e)\}$ denote, respectively, the smallest and the largest color assigned to an edge incident to v. It follows from the above definition that if c is linear compact, then $c_{max}(v) = c_{min}(v) + |E_v| - 1$ for all vertices $v \in V$.

A k-edge-coloring is *cyclic compact* if we can associate two positive integers $a_v, b_v < k$ to every vertex v so that $\{c(e) : e \in E_v\} = \{a_v, (a_v + 1) \text{mod } k, \cdots, (a_v + |E_v| - 1) \text{mod } k = b_v\}$ (i.e., color 0 is considered as consecutive to $k - 1$). A graph is *cyclically compactly colorable* if it admits a cyclic compact k-edge-coloring for some integer k. While linear compact k-edge-colorings are also cyclic compact (with $a_v = c_{min}(v)$ and $b_v = c_{max}(v)$), the reverse is not necessarily true. For example, the 3-edge-coloring of the triangle shown in Figure 1 is cyclic compact but not linear compact (since color 1 is missing in E_b). It is not difficult to observe that the triangle is not linearly compactly colorable.

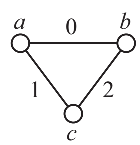

Figure 1: A cyclic compact 3-edge-coloring that is not linear compact

Cyclic compact k-edge-colorings are also studied in [11] and are closely related to the circular compact colorability defined in [10]. Given a positive real number r, let C^r denote the circle with circumference r. Taking an arbitrary point $0 \in C^r$ and orientation, we denote (a, b) the open arc starting in a and ending in b, where $a, b \in [0, r]$. The *length* of an arc (a, b) is $b - a$ if $b > a$ and $r - a + b$ if $b < a$. An r-circular edge-coloring is an assignment of an open arc $A(e) \subset C^r$ of length 1 to each edge e so that $A(e)$ and $A(e')$ are disjoint whenever e and e' share a common endpoint. Such a coloring is compact if the closure of the union of open arcs assigned to the edges of E_v is an arc on C^r for each vertex v. When r is integer, compact r-circular edge-colorings are equivalent to cyclic compact r-edge-colorings. For non-integer values of r, the situation is slightly different. For example, as shown in Figure 2, there exists a compact r-circular edge-coloring of the pentagon with $r = 2.5$, while it is easy to observe that all cyclic compact k-edge-colorings of the pentagon have $k \geq 3$, an example of a cyclic compact 3-edge-coloring being shown on the right of Figure 2.

The problem of determining a linear compact k-edge-coloring (if any) of a graph was introduced by Asratian and Kamalian [2]. It often arises in scheduling problems with compactness constraints [6]. For example, consider m processors $P_1, \cdots, P_m$ and n jobs $J_1, \cdots, J_n$. Each job J_i is a set of s_i tasks. Suppose that each task has to be processed in one time unit on a specific processor. No two tasks of the same job can be processed simultaneously and no processor can work on two tasks at the same time. Moreover, compactness requirements state that waiting periods are forbidden for every job and no idles are allowed on each processor. In other words, the time periods assigned to the tasks of a job must be consecutive, and each processor must be active during a set of consecutive periods. The existence of a feasible compact schedule with k time periods is equivalent to the existence of a linear compact k-edge-coloring of the graph G that contains one vertex for each job and each processor, and one edge for each task (i.e., a task of job J_i to be processed on P_j is represented by an edge between the vertices representing J_i and P_j). Each color used

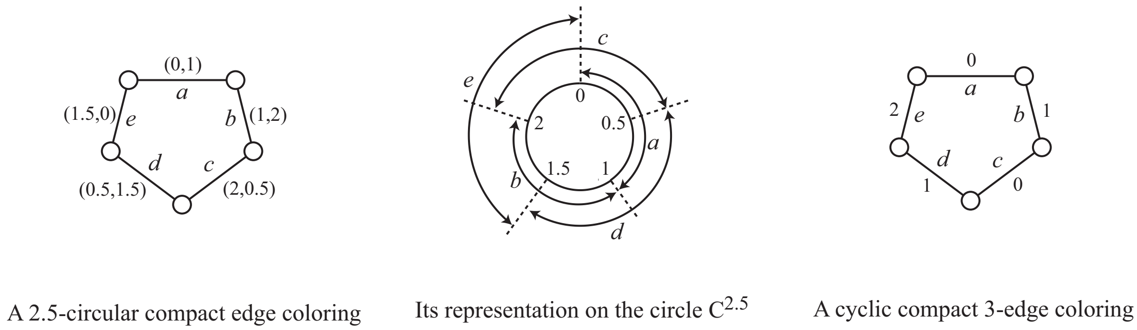

Figure 2: r-circular compact versus cyclic compact for non-integer r

in the k-edge-coloring corresponds to a time period. The compactness requirements for each job and each processor are equivalent to imposing that the colors appearing on the edges of E_v must be consecutive for every vertex v in G. In many automated production systems, the production is organized in a cyclic way, i.e., the same production schedule of length T is repeated continuously every T time units. Compactness requirements then impose that the time periods assigned to the tasks of each job and the active period of each processor form a cyclic interval in each production cycle. The existence of a feasible cyclic compact schedule is then equivalent to the existence of a cyclic compact T-edge-coloring of the same graph G.

The problem of determining whether or not a given graph is linearly compactly colorable is denoted $LCCP$ and is known to be $\mathcal{NP}$-complete [14], even for bipartite graphs. Given a k-edge-coloring c of a graph G, let $D_v(G,c)$ denote the minimum number of integers which must be added to $\{c(e) : e \in E_v\}$ to form an interval of consecutive integers. The *deficiency* of c is defined as the sum $D(G,c) = \sum_{v \in V} D_v(G,c)$. Hence, c is linear compact if and only if $D(G,c) = 0$. The *deficiency of a graph* G, denoted $Def(G)$, is the minimum deficiency $D(G,c)$ over all k-edge-colorings c of G (where k can take any positive integer value). This concept, which was introduced by Giaro et al. [5], provides a measure of how close G is to be linearly compactly colorable since the deficiency of G is the minimum number of pendant edges that must be added to G such that the resulting graph is linearly compactly colorable. The problem of determining the deficiency of a graph is $\mathcal{NP}$-hard [4]. This problem is also studied in [1, 5, 8, 9, 12, 13, 7, 3].

The problem of determining whether or not a given graph is cyclically compactly colorable is denoted $CCCP$. To demonstrate that it is $\mathcal{NP}$-complete, Kubale and Nadolski [10] build a graph H from a graph G so that G is linearly compactly colorable if and only if H is cyclically compactly colorable. Graph H is defined as $G \cup K_{1,m+1}$, where m is the number of edges in G and $K_{1,m+1}$ is the star with $m+1$ branches (i.e., the graph containing 1 vertex with degree $m+1$ and $m+1$ vertices with degree 1). In other words, H is obtained from G by adding a new connected component isomorphic to $K_{1,m+1}$. Assume that G is linearly compactly colorable and let k be the smallest integer such that G admits a linear compact k-edge-coloring c. We then have $k \leq m$ since all colors in $\{0, \cdots, k-1\}$ appear in c, which means that c can be extended to a cyclic compact $(m+1)$-edge-coloring of H by assigning colors $0, 1, \cdots, m$ to the edges of $K_{1,m+1}$. Also, if H admits a cyclic compact k-edge-coloring c, then $k \geq \Delta(H) = m+1$. Since G contains m edges, at least one of the k colors does not appear in G and, without loss of generality, we may assume that the missing color is k (otherwise, a cyclic permutation of the colors in c leads to such a coloring), which means that no vertex v in G has both colors 0 and k in E_v. The edge-coloring c restricted to G is therefore linear compact. In summary, G is linearly compactly colorable if and only if H is cyclically compactly colorable, and since H can be obtained from G in polynomial time (by adding $m+2$ vertices and $m+1$ edges), this proves that the $LCCP$ is polynomially reducible to the $CCCP$, which demonstrates the $\mathcal{NP}$-completeness of the $CCCP$. Note that H is bipartite if and only if G is bipartite, which proves that the $CCCP$ is $\mathcal{NP}$-complete even for bipartite graphs.

In this paper, we are interested in determining whether or not a given graph admits a linear (cyclic) compact k-edge-coloring for a fixed integer k.

Definition 1 *Let G be a graph and $k > 0$ an integer.*

- *G is k-linearly compactly colorable if it admits a linear compact k-edge-coloring. We denote k-LCCP the problem of determining whether or not G is k-linearly compactly colorable.*
- *G is k-cyclically compactly colorable if it admits a cyclic compact k-edge-coloring. We denote k-CCCP the problem of determining whether or not G is k-cyclically compactly colorable.*

Note that if $k' > k$ and G is k-linearly compactly colorable, then G is also k'-linearly compactly colorable. However, a k-cyclically compactly colorable graph is not necessarily k'-cyclically compactly colorable for $k' > k$. For example, the triangle and the pentagon are 3-cyclically compactly colorable while they are not 4-cyclically compactly colorable. The reduction proposed in [10] shows that a given graph G with m edges is m-linearly compactly colorable if and only if H is $(m+1)$-cyclically compactly colorable. We note however that the linear compact m-edge-coloring of G derived from the cyclic compact $(m+1)$-edge-coloring of H uses possibly up to m different colors while less colors may be sufficient. For example, consider the graph G containing a chain on 4 vertices a, b, c, d, with edges between a and b, b and c, and c and d. As shown in Figure 3, H is 4-cyclically compactly colorable, and we therefore know that G is 3-linearly compactly colorable. However, the reduction does not provide any information about the 2-linearly compactly colorability of G which is demonstrated by the linear compact 2-edge-coloring on the right of Figure 3.

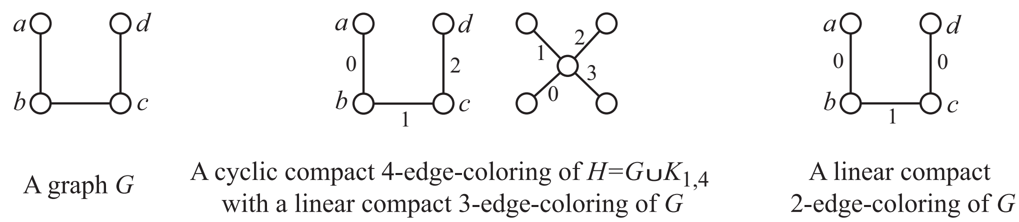

Figure 3: Reduction of the $LCCP$ to the $CCCP$

In this paper, we prove that there is a polynomial time reduction of the $k-LCCP$ to the $k-CCCP$ for every graph G and every integer $k \geq 12$. For smaller values of k, we show that the $k-LCCP$ can be reduced to the $12-CCCP$. These reductions are described in detail in Section 2. A basic tool in our reductions is a graph transformation which makes it possible to impose a same color on two non-adjacent edges. As a corollary, as shown in Section 3, we can impose or forbid a given color on any edge of the graph. This may be helpful when solving production scheduling problems in which some tasks can only be processed at specific time periods. We also show in Section 4 that finding a non-preemptive cyclic production schedule of length k (with possibly non-uniform processing times) is equivalent to solving a $k-CCCP$ in an appropriate graph.

2 Reduction of the $k-LCCP$ to the $k-CCCP$

Note that we always assume $k \geq \Delta(G)$ else G is obviously not k-linearly or k-cyclically compactly colorable. For $k \geq 12$, we describe in this section a construction of a graph H from a graph G so that G is k-linearly compactly colorable if and only if H is k-cyclically compactly colorable. For $k < 12$, we slightly modify our construction of H so that G is k-linearly compactly colorable if and only if H is 12-cyclically compactly colorable.

Given two positive integers $x, y < k$, we denote $[x, y]_k$ the set $\{x \bmod k, (x+1) \bmod k, \cdots, y \bmod k\}$. Hence, a k-edge-coloring c of a graph G is cyclic compact if we can associate two integers $0 \leq a_v, b_v < k$ to every vertex v so that $\{c(e) : e \in E_v\} = [a_v, b_v]_k$.

For an integer $q \geq 1$, we call q-bundle a set of q parallel edges linking two vertices u and v. When drawing a graph, q-bundles are represented as in Figure 4. Given a graph G, we build a new graph, denoted G^B by adding a new vertex v' for each vertex v of degree strictly smaller than k in G and by linking each pair v, v'

of vertices with a $(k - |E_v|)$-bundle. We denote E^B_u the set of edges incident to a vertex u in G^B. The next theorem shows how to link the $k - LCCP$ in G to the $k - CCCP$ in G^B.

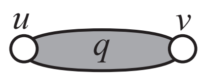

Figure 4: A q-bundle linking vertices u and v

Theorem 1 *G is k-linearly compactly colorable if and only if G^B admits a cyclic compact k-edge-coloring such that every $(k - |E_v|)$-bundle added to G contains color 0 and/or $k - 1$.*

Proof. ($\Rightarrow$) Let c be a linear compact k-edge-coloring of G and let $[a_v, b_v]_k$ denote the set of colors on the edges of E_v incident to v in G. For vertices v with $|E_v| < k$, let us assign all colors in $[b_v + 1, a_v - 1]_k$ to the edges of the $(k - |E_v|)$-bundle linking v to v'. The edge-coloring c is thus extended to an edge-coloring c^B of G^B where $\{c^B(e) : e \in E^B_v\} = [0, k - 1]_k$ for every v in G, and $\{c^B(e) : e \in E^B_{v'}\} = [b_v + 1, a_v - 1]_k$ for every new vertex v'. Hence, c^B is a cyclic compact k-edge-coloring of G^B. Since c is linear compact, $|[a_v, b_v]_k \cap \{0, k - 1\}| \leq 1$ for every vertex v with $|E_v| < k$. Hence, $|[b_v + 1, a_v - 1]_k \cap \{0, k - 1\}| \geq 1$ for these vertices, which means that every $(k - |E_v|)$-bundle added to G contains color 0 and/or $k - 1$.

($\Leftarrow$) Let c^B be a cyclic compact k-edge-coloring of G^B such that every $(k - |E_v|)$-bundle added to G contains color 0 and/or $k - 1$. For every vertex u in G^B, let $[a_u, b_u]_k = \{c^B(e) : e \in E^B_u\}$. If u is a vertex in G with $|E_u| = k$, then $\{c^B(e) : e \in E_u\} = [0, k - 1]_k$. If u is a vertex in G with $|E_u| < k$ then E^B_u contains all edges of E_u as well as those of the $(k - |E_u|)$-bundle linking u to u'. Since $|E^B_u| = k$ and $\{c^B(e) : e \in E^B_{u'}\} = [a_{u'}, b_{u'}]_k$, we have $\{c^B(e) : e \in E_u\} = [b_{u'} + 1, a_{u'} - 1]_k$. Moreover, $|[a_{u'}, b_{u'}]_k \cap \{0, k - 1\}| \geq 1$ implies $|[b_{u'} + 1, a_{u'} - 1]_k \cap \{0, k - 1\}| \leq 1$, which means that the edge-coloring c^B restricted to G is a linear compact k-edge-coloring of G. □

We now have to show how color 0 and/or $k - 1$ can be imposed on every $(k - |E_v|)$-bundle added to G. Let t denote the number of vertices in G with $|E_v| < k$ and let H_t denote the graph with $2t$ vertices $u_1, \cdots, u_t, w_1, \cdots, w_t$ and such that every u_i is linked to w_i $(1 \leq i \leq t)$ with two parallel edges (i.e., a 2-bundle) and every w_i is linked to u_{i+1} $(1 \leq i < t)$ with a $(k - 2)$-bundle (see Figure 5). Also, let $G^{B+} = G^B \cup H_t$. In other words, G^{B+} is obtained from G^B by adding a new connected component isomorphic to H_t. Let us label $v_1, \cdots, v_t$ the vertices of G with $|E_{v_i}| < k$, let e^B_i be one of the edges of the $(k - |E_{v_i}|)$-bundle linking v_i to v'_i, and let e^H_i be one of the two edges linking u_i to w_i. The next theorem provides a link between cyclic compact k-edge-colorings of G^B and G^{B+}.

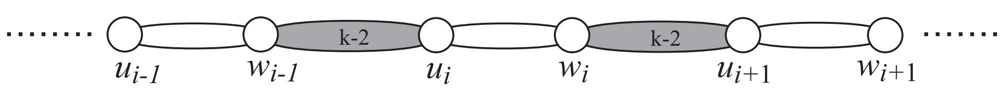

Figure 5: Part of the graph H_t

Theorem 2 *G^B admits a cyclic compact k-edge-coloring such that every $(k - |E_{v_i}|)$-bundle added to G contains color 0 and/or $k - 1$ if and only if G^{B+} admits a cyclic compact k-edge-coloring c with $c(e^B_i) = c(e^H_i)$ for $i = 1, \cdots, t$.*

Proof. ($\Rightarrow$) Let c be a cyclic compact k-edge-coloring of G^B such that every $(k - |E_v|)$-bundle added to G contains color 0 and/or $k - 1$. The colors on every $(k - |E_{v_i}|)$-bundle can be permuted so that e^B_i gets color 0 or $k - 1$. Now, assign color $c(e^B_i)$ to e^H_i, color $k - 1 - c(e^B_i)$ (i.e., the other color in $\{0, k - 1\}$) to the second edge linking u_i to w_i, and colors $1, \cdots, k - 2$ to the $(k - 2)$-bundle linking w_i to u_{i+1} (if $i < t$). By construction, the extension of c to G^{B+} is a cyclic compact k-edge-coloring with $c(e^B_i) = c(e^H_i)$ for $i = 1, \cdots, t$.

(⇐) Assume now that c is a cyclic compact k-edge-coloring of G^{B+} with $c(e_i^B) = c(e_i^H)$ for $i = 1, \cdots, t$. Since u_1 has degree 2, we can permute the colors in c in a cyclic way so that the two edges incident to u_1 get colors 0 and $k-1$. The $(k-2)$-bundle linking w_1 to u_2 then necessarily contains all colors in $\{1, \cdots, k-2\}$, which means that the 2 edges linking u_2 to w_2 also have color 0 and $k-1$. By repeating the same reasoning, we conclude that the two edges linking u_i to w_i $(1 \le i \le t)$ have color 0 and $k-1$ and the edges of the $(k-2)$-bundle linking w_i to u_{i+1} $(1 \le i < t)$ have colors $1, \cdots, k-2$. Hence, $c(e_i^H) = 0$ or $k-1$ for $i = 1, \cdots, t$. Since $c(e_i^B) = c(e_i^H)$, we conclude that the restriction of c to G^B is a cyclic compact k-edge-coloring such that each $(k - |E_{v_i}|)$-bundle added to G contains color 0 and/or $k-1$. □

It remains to show how to impose the same color on e_i^B and e_i^H $(1 \le i \le t)$. More generally, given two non-adjacent edges e and e' in a graph G, we are interested in building a new graph G' so that G' is k-cyclically compactly colorable if and only if G admits a cyclic compact k-edge-coloring c with $c(e) = c(e')$. For illustration, Figure 6 contains two cyclic compact 4-edge-colorings of the same graph. In the left graph, the two edges e and e' have different colors while $c(e) = c(e') = 2$ in the right graph. We are interested in imposing the same color on e and e', which means that the first 4-edge-coloring would not be acceptable.

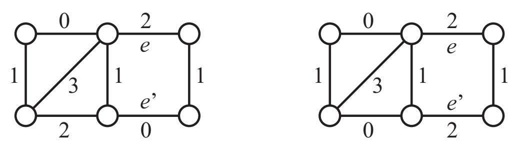

Figure 6: Two cyclic compact 4-edge-colorings of a graph

The next lemma illustrates how q-bundles can be used to impose restrictions on two edges.

Lemma 1 *Let G be a graph and q an integer with $1 \le q \le k-3$. Consider three vertices u, v, w in G such that there exist exactly one edge e_{uv} linking u to v, exactly one edge e_{uw} linking u to w, and exactly q edges (i.e. a q-bundle) linking v to w. Let c be a cyclic compact k-cedge-coloring of G. Then $c(e_{uw}) = (c(e_{uv}) \pm (q+1))$mod k.*

Proof. Since c is a cyclic compact k-edge-coloring of G, there are four integers a_v, b_v, a_w, b_w such that $\{c(e) : e \in E_v\} = [a_v, b_v]_k$ and $\{c(e) : e \in E_w\} = [a_w, b_w]_k$. Note that $b_v = (a_v + q)$mod k, $b_w = (a_w + q)$mod k, and $[a_w, b_w]_k$ is a subset of $q+1$ consecutive integers (modulo k) chosen in $[a_v, b_v]_k \cup \{c(e_{uw})\}$. Since $q \le k-3$, we have $a_v - 1 \ne b_v + 1$ (modulo k), which means that $c(e_{uw}) = (a_v - 1)$mod k or $(b_v + 1)$mod k.

- if $c(e_{uw}) = (a_v - 1)$mod k then $c(e_{uv}) = b_v$ and $[a_w, b_w]_k = [a_v - 1, b_v - 1]_k$; we therefore have $c(e_{uw}) = (b_v - q - 1)$mod $k = (c(e_{uv}) - (q+1))$mod k.
- if $c(e_{uw}) = (b_v + 1)$mod k then $c(e_{uv}) = a_v$ and $[a_w, b_w]_k = [a_v + 1, b_v + 1]_k$; we therefore have $c(e_{uw}) = (a_v + q + 1)$mod $k = (c(e_{uv}) + (q+1))$mod k.

□

We now consider a more complex structure called *s-shift* that also imposes a restriction on two adjacent edges. Given three vertices u, v, w with an edge e_{uv} between u and v and an edge e_{uw} between u and w, the s-shift structure restricts the color of e_{uw} to $c(e_{uv}) \pm s$, but without imposing any restriction on the other edges incident to v and w. We first give a precise definition of an s-shift.

Definition 2 *Let s be an integer so that $2 \le s \le k-2$. Let u, v, w, u', v', w' be six distinct vertices in a graph G such that there is exactly one edge between u and v, one between u and w, one between u' and v' and one between u' and w'. Assume also that there is a $(k-2)$-bundle between u and u' and an $(s-1)$-bundle between v' and w'. Suppose finally that no other edge in G is incident to u, u', v' and w' (while there are possibly other edges incident to v and to w, including one between these two vertices). Let e denote the edge linking u to v and e' the edge linking u to w. Such a structure is called an s-shift between e and e'. It is illustrated in Figure 7, with also a simplified representation.*

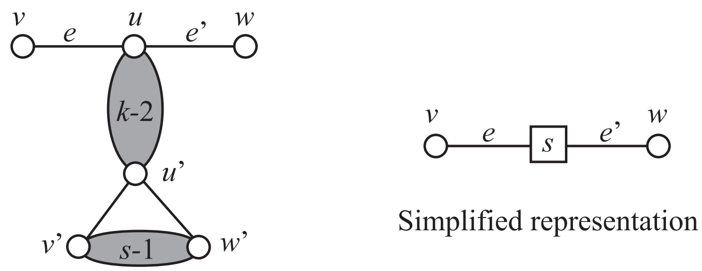

Figure 7: An s-shift

Lemma 2 *Let c be a cyclic compact k-edge-coloring of a graph G and assume there is an s-shift between two edges e and e' in G. Then $c(e') = (c(e) \pm s) mod\ k$.*

Proof. Let c be a cyclic compact k-edge-coloring of G and let u, v, w, u', v', w' denote the six vertices of the s-shift between e and e', as in Figure 7. By applying Lemma 1 with $q = s - 1$ and the three vertices u', v' and w', and by denoting $e_{u'v'}$ and $e_{u'w'}$ the edges linking u' with v' and u' with w', we get $c(e_{u'w'}) = (c(e_{u'v'}) \pm s) \text{mod } k$. It follows that the edges of the $(k-2)$-bundle linking u with u' use all colors in $\{0, \cdots, k-1\} - \{c(e_{u'v'}), c(e_{u'w'})\}$, which means that $\{c(e), c(e')\} = \{c(e_{u'v'}), c(e_{u'w'})\}$. In other words, $c(e') = (c(e) \pm s) \text{mod } k$. □

The next structure imposes the same color on four edges. It is called an *equalizer* and is defined as follows.

Definition 3 *Let u, v, u', v' be four distinct distinct vertices linked together according to the structure depicted in Figure 8. We assume that all vertices except u, v, u', v' in this structure have no additional edges incident to them. Such a structure is called an equalizer for u, v, u', v'. A simplified representation is also given in Figure 8.*

Note that an equalizer contains 28 vertices and $39 + 3k$ edges (since each 6-shift contains 6 vertices and $k + 7$ edges). We now prove that if $k \geq 12$ then the four edges incident to u, v, u' and v' necessarily have the same color in every cyclic compact k-edge-coloring of an equalizer.

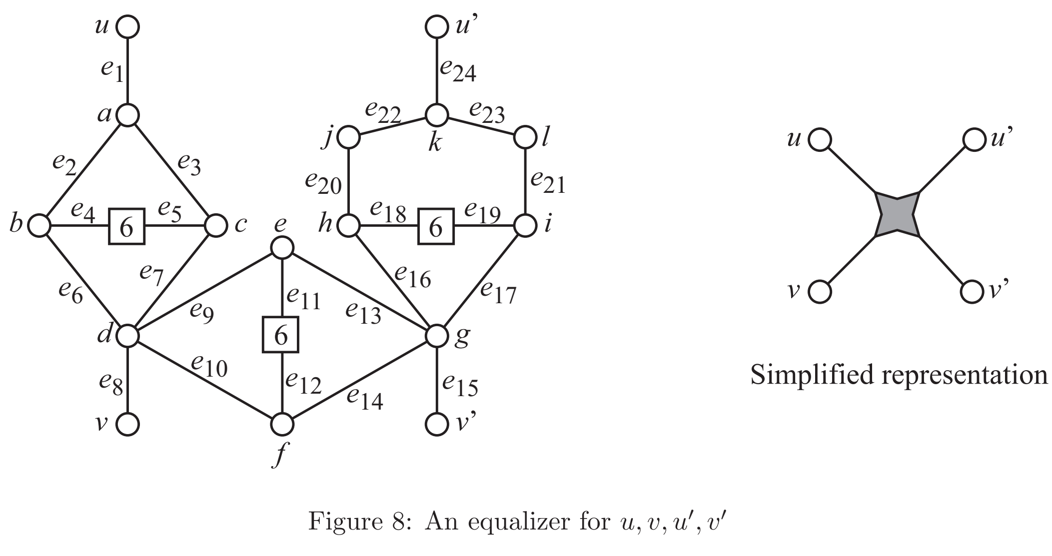

Figure 8: An equalizer for u, v, u', v'

Lemma 3 *Assume $k \geq 12$ and let c be a cyclic compact k-edge-coloring of an equalizer for u, v, u', v'. Then the four edges incident to u, v, u' and v' (i.e, edges e_1, e_8, e_{15}, e_{24} in Figure 8) have the same color which can be any integer in $\{0, \cdots, k-1\}$.*

Proof. Let us label the vertices and the edges of the equalizer as in Figure 8 and let r be any integer in $\{0, \cdots, k-1\}$. We know from Lemma 2 that $c(e_4) = (c(e_5) \pm 6)$mod k. Without loss of generality, we can assume that $c(e_4) = r - 3$ and $c(e_5) = r + 3$ (all values are taken modulo k).

We first concentrate on vertices u, a, b, c and the 6-shift between b and c. Since b and c are of degree 3, we have $c(e_2) \in [c(e_4) - 2, c(e_4) + 2]_k$ and $c(e_3) \in [c(e_5) - 2, c(e_5) + 2]_k$. In other words, $c(e_2) \in [r-5, r-1]_k$ and $c(e_3) \in [r+1, r+5]_k$. Since the edges incident to a must have consecutive colors, we necessarily have $c(e_2) = r - 1$, $c(e_1) = r$ and $c(e_3) = r + 1$ if $k > 12$, while the unique other possibility with $k = 12$ is $c(e_2) = r{-}5 = r{+}7$, $c(e_1) = r{+}6$ and $c(e_3) = r{+}5$. In both cases, we have $\{c(e_2), c(e_3)\} = \{c(e_1){-}1, c(e_1){+}1\}$ and $\{c(e_4), c(e_5)\} = \{c(e_1) - 3, c(e_1) + 3)\}$.

Consider now the five edges incident to d. If $k > 12$, we have $c(e_6) = r - 2$, $c(e_7) = r + 2$ and $\{c(e_8), c(e_9), c(e_{10})\} = \{r - 1, r, r + 1\}$. If $k = 12$, a second possibility is $c(e_6) = r - 4$, $c(e_7) = r + 4$ and $\{c(e_8), c(e_9), c(e_{10})\} = \{r + 5, r + 6, r + 7\}$. In both cases, the colors on e_8, e_9 and e_{10} are consecutive and equal to those on e_1, e_2, e_3.

We can therefore repeat the same reasoning with the subgraph of the equalizer containing vertices v, d, e, f and the 6-shift between e and f. We get $\{c(e_9), c(e_{10})\} = \{c(e_8) - 1, c(e_8) + 1\}$ and $\{c(e_{11}), c(e_{12})\} = \{c(e_8) - 3, c(e_8) + 3\}$, which means that $c(e_8) = c(e_1)$ and $\{c(e_{11}), c(e_{12})\} = \{r - 3, r + 3\}$. Also, by considering the five edges incident to g, we get the same conclusion as for those incident to d : the colors on e_{15}, e_{16} and e_{17} are consecutive and equal to those on e_8, e_9, e_{10}. By repeating the same reasoning a third and last time on the subgraph of the equalizer containing vertices v', g, h, i and the 6-shift between h and i, we get $\{c(e_{16}), c(e_{17})\} = \{c(e_{15}) - 1, c(e_{15}) + 1\}$, and $\{c(e_{18}), c(e_{19})\} = \{c(e_{15}) - 3, c(e_{15}) + 3)\}$ which means that $c(e_{15}) = c(e_8) = c(e_1)$ and $\{c(e_{18}), c(e_{19})\} = \{r - 3, r + 3\}$.

From $\{c(e_{18}), c(e_{19})\} = \{c(e_{15}) - 3, c(e_{15}) + 3)\}$ and $\{c(e_{16}), c(e_{17})\} = \{c(e_{15}) - 1, c(e_{15}) + 1)\}$, we deduce $\{c(e_{20}), c(e_{21})\} = \{c(e_{15}) - 2, c(e_{15}) + 2)\}$. Since vertex k is of degree 3, we therefore necessarily have $\{c(e_{22}), c(e_{23})\} = \{c(e_{15}) - 1, c(e_{15}) + 1)\}$, which means that $c(e_{24}) = c(e_{15})$. □

Two cyclic compact k-edge-colorings of the equalizer are depicted in Figure 9. The first coloring is valid for all $k \geq 12$ while the second coloring demonstrates that the assumption $k \geq 12$ is important. Indeed, a cyclic compact 11-edge-coloring is shown with three different colors on e_1, e_8, e_{15}, e_{24}.

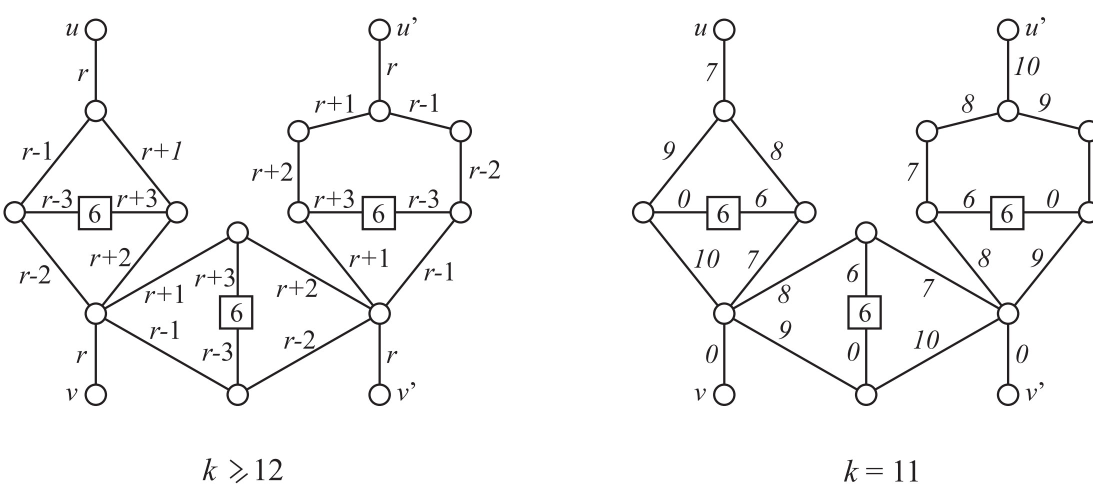

Figure 9: Two k-cyclic compact edge-colorings of an equalizer for u, v, u', v'

Lemma 4 *Assume $k \geq 12$, let e and e' be two non adjacent edges in a graph G, and let G' be the graph obtained from G by removing e and e' and adding an equalizer for the four endpoints of these two edges. Then G' is k-cyclically compactly colorable if and only if G admits a cyclic compact k-edge-coloring c with $c(e) = c(e')$.*

Proof. Let c be a cyclic compact k-edge-coloring of G' and let r be the color on the four edges of the equalizer incident to the endpoints of e and e'. By coloring e and e' with color r and every other edge of G as in G', one gets a cyclic compact k-edge-coloring of G with $c(e) = c(e')$.

On the opposite, let c be a cyclic compact k-edge-coloring of G with $c(e) = c(e')$. By coloring the edges of the equalizer as in the left graph of Figure 9, with $r = c(e)$, and every other edge of G' as in G, one gets a cyclic compact k-edge-coloring of G'. □

We now complete the description of the polynomial time reduction from the $k - LCCP$ to a $k - CCCP$ for $k \geq 12$. We denote $T(G)$ the graph obtained from G^{B+} by removing the edges e_i^B and e_i^H $(i = 1, \cdots, t)$ and replacing them by an equalizer for v_i, v_i', u_i, w_i . A summary of the transformation is shown in Figure 10.

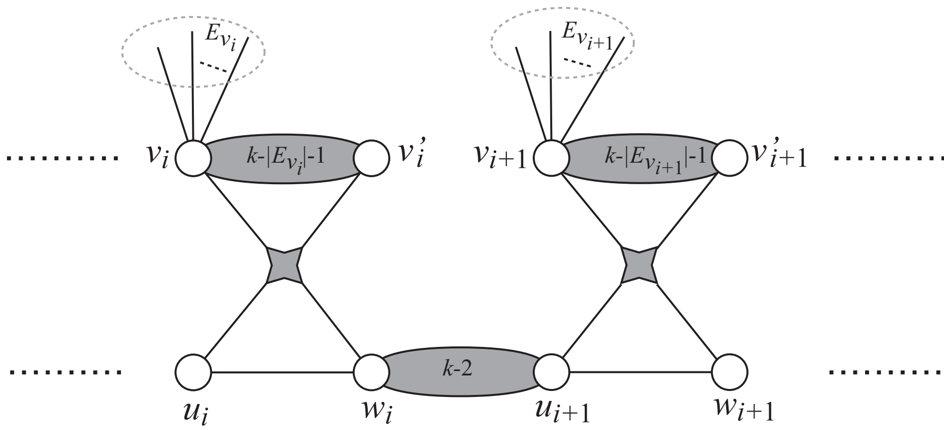

Figure 10: Illustration of $T(G)$

Theorem 3 *Let G be a graph with n vertices and let $k \geq 12$. Then G is k-linearly compactly colorable if and only if $T(G)$ is k-cyclically compactly colorable. Moreover, the time needed to build $T(G)$ from G is in $O(nk)$.*

Proof. It follows from Theorems 1 and 2 that G is k-linearly compactly colorable if and only if G^{B+} admits a cyclic compact k-edge-coloring c with $c(e_i^B) = c(e_i^H)$ for $i = 1, \cdots, t$. According to Lemma 4 (applied t times, with $e = e_i^B$ and $e' = e_i^H$ $(i = 1, \cdots, t)$), such a coloring exists if and only if $T(G)$ is k-cyclically compactly colorable.

The graph G^B contains at most $2n$ vertices and $m + \sum_{i=1}^{n}(k - |E_{v_i}|) = kn - m$ edges, where m is the number of edges in G, and the graph H_t contains $2t$ vertices and $kt - (k - 2) = k(t - 1) + 2$ edges. Since $t \leq n$ we conclude that G^{B+} contains at most $4n$ vertices and $k(2n-1)-m+2$ edges. As already mentioned, each equalizer contains 24 vertices in addition to u, u', v, v' and $39 + 3k$ edges. Hence, $T(G)$ contains at most $4n + 24n = 28n$ vertices and $k(2n-1)-m+2+n(39+3k-2) = k(5n-1)+37n-m+2$ edges. In summary, the number of vertices in $T(G)$ is in $O(n)$ and its number of edges is in $O(nk)$. □

We now show how to transform the $k - LCCP$ to a $12 - CCCP$ for $k < 12$. The construction we use is similar to that of $T(G)$ with small variations. First of all, instead of G^B, we build a graph $\tilde{G}^B$ by adding a new vertex v' for each vertex v in G and by linking each pair v, v' of vertices with a $(12 - |E_v|)$-bundle. We then consider the graph $\tilde{H}_n$ (instead of H_t) with $2n$ vertices $u_1, \cdots, u_n, w_1, \cdots, w_n$ and such that each u_i is linked to w_i $(1 \leq i \leq n)$ with a $(12 - k)$-bundle and each w_i is linked to u_{i+1} $(1 \leq i < n)$ with a k-bundle. We then define $\tilde{G}^{B+} = \tilde{G}^B \cup \tilde{H}_n$. Finally, let us label $v_1, \cdots, v_n$ the vertices of G, let $e_{i,1}^{\tilde{B}}, \cdots, e_{i,12-k}^{\tilde{B}}$ denote $12-k$ edges of the $(12-|E_{v_i}|)$-bundle linking v_i to v_i' in $\tilde{G}^B$, and let $e_{i,1}^{\tilde{H}}, \cdots, e_{i,12-k}^{\tilde{H}}$ denote the edges linking u_i to w_i in $\tilde{H}_n$. We construct $\tilde{T}(G)$ from $\tilde{G}^{B+}$ by replacing every pair $(e_{i,j}^{\tilde{B}}, e_{i,j}^{\tilde{H}})$ of edges ($i = 1, \cdots, n$;

$j = 1, \cdots, 12-k$) by an equalizer for v_i, v'_i, u_i, w_i. The construction of $\tilde{T}(G)$ is illustrated on Figure 11 for $k = 10$.

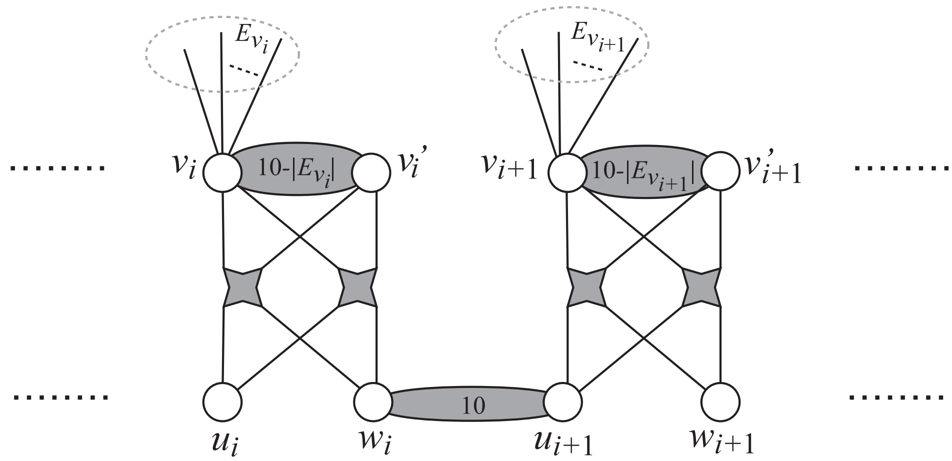

Figure 11: Illustration of $\tilde{T}(G)$ for $k = 10$

Theorem 4 *Let G be a graph with n vertices and let $k < 12$. Then G is k-linearly compactly colorable if and only if $\tilde{T}(G)$ is 12-cyclically compactly colorable. Moreover, the time needed to build $\tilde{T}(G)$ from G is in $O(n)$.*

Proof. The proof is similar as for $k \geq 12$. We only mention the variations. First of all, following the proof of Theorem 1, we can prove in a similar way that G is k-linearly colorable if and only if $\tilde{G}^B$ admits a cyclic compact 12-edge-coloring such that every $(12 - |E_v|)$-bundle added to G contains colors $k, k+1, \cdots, 11$. Then following the proof of Theorem 2, we get that such a coloring exists in $\tilde{G}^B$ if and only if $\tilde{G}^{B+}$ admits a cyclic compact 12-edge-coloring with $c(e^{\tilde{B}}_{i,j}) = c(e^{\tilde{H}}_{i,j})$ for $i = 1, \cdots, n$ and $j = 1, \cdots, 12-k$. Finally, using Lemma 4, we get that such a coloring exists in $\tilde{G}^{B+}$ if and only if $\tilde{T}(G)$ is 12-cyclically compactly colorable.

The graph $\tilde{G}^B$ contains $2n$ vertices and $m + \sum_{i=1}^{n}(12 - |E_{v_i}|) = 12n - m$ edges, where m is the number of edges in G. Since $\tilde{H}_n$ contains $2n$ vertices and $12n - k$ edges, $\tilde{G}^{B+}$ contains $4n$ vertices and $24n - k - m$ edges. Hence, $\tilde{T}(G)$ contains $4n + 24(12-k)n$ vertices and $24n - k - m + (12-k)n(39 + 3k - 2)$ edges, which means that its number of vertices and its number of edges are in $O(n)$. □

3 Imposing and forbidding colors

The equalizer described in the previous section is very helpful for imposing or forbidding a color on a given edge. More precisely, suppose $k \geq 12$ and assume that every edge e in G has a list F_e of forbidden colors. We only consider such sets with $|F_e| < k - 1$. If $k - 1$ colors are forbidden for e, then the unique color r not in F_e is imposed on e and we instead introduce the pair (e, r) in a set I. If there is no restriction for an edge e, then F_e is empty and I does not contain any (e, r) with $0 \leq r < k$.

For a color $r \in \{0, \cdots, k-1\}$, let f_r denote the number of edges e with $r \in F_e$ and let i_r denote the number of pairs (e, r) in I (i.e., the number of edges having r as imposed color). Also, for convenience, we denote $\ell_r = 2(f_r + i_r) + 1$.

We now construct a graph that only depends on the sets F_e and I. Consider vertices x^r_j for $r = 0, \cdots, k-1$ and $j = 0, \cdots, \ell_r$, as well as two vertices y and z. Link x^r_{2j} to x^r_{2j+1} $(0 \leq j \leq f_r + i_r)$ with a single edge denoted e^r_j and x^r_{2j-1} to x^r_{2j} $(1 \leq j \leq f_r + i_r)$ with a $(k-1)$-bundle. Link also y to z with a k-bundle. Finally, identify every $x^r_{\ell_r}$ with x^{r+1}_0, $r = 0, \cdots, k-2$. The resulting graph is denoted P^F_I and is illustrated on Figure 12.

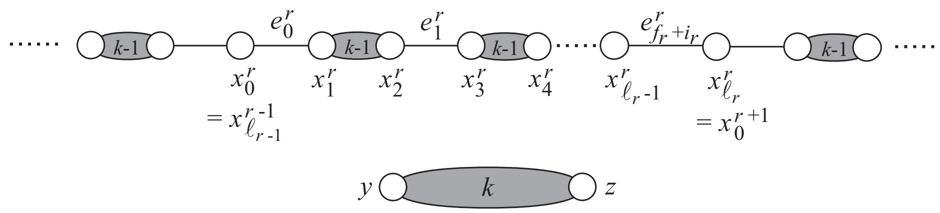

Figure 12: Part of the graph P_I^F

From P_I^F we then construct Q_I^F by replacing k pairs of edges by equalizers. More precisely, for every $r = 0, \cdots, k-1$, we replace one edge of the k-bundle linking y to z and the edge e_0^r linking x_0^r to x_1^r by an equalizer for y, z, x_0^r, x_1^r. Q_I^F is k-cyclically compactly colorable. Indeed, one can for example assign color r $(0 \le r < k)$ to every edge e_j^r $(1 \le j \le f_r + i_r)$ and all colors in $\{0, \cdots, r-1, r+1, \cdots, k-1\}$ to every $(k-1)$-bundle linking x_{2j-1}^r to x_{2j}^r $(j = 1, \cdots, f_r + i_r)$. Also, every equalizer for y, z, x_0^r, x_1^r can be colored so that the four pendant edges get color r. It is then an easy exercise to check that such a k-edge-coloring is cyclic compact. We now prove that Q_I^F admits a cyclic compact k-edge-coloring with a special property.

Lemma 5 *Assume $k \ge 12$. For every cyclic compact k-edge-coloring c of Q_I^F there exist $\alpha \in \{-1, 1\}$ and $\beta \in \{0, \cdots, k-1\}$ such that the k-edge-coloring c' defined by $c'(e) = (\alpha c(e) + \beta)\text{mod } k$ is cyclic compact with $c'(e_j^r) = r$ for all $r = 0, \cdots, k-1$, $j = 1, \cdots, f_r + i_r$.*

Proof. Let c be a cyclic compact k-edge-coloring of Q_I^F and let c^P be the cyclic compact k-edge-coloring of P_I^F obtained from c by coloring e_0^r $(r = 0, \cdots, k-1)$ as the four pendant edges of the equalizer for y, z, x_0^r, x_1^r and by assigning colors $0, \cdots, k-1$ to the edges of the k-bundle linking y to z. Observe first that $c^P(e_0^r) = c^P(e_j^r)$ for $j = 1, \cdots, f_r + i_r$ since x_{2j-1}^r is linked to x_j^r by a $(k-1)$-bundle. Since $x_{\ell_r}^r = x_0^{r+1}$ is incident to only two edges, we have (modulo k)

$$c^P(e_0^r) = c^P(e_{f_r+i_r}^r) = c^P(e_0^{r+1}) \pm 1 \text{ for all } r = 0, \cdots, k-2. \tag{1}$$

Note then that $\{c^P(e_0^0), \cdots, c^P(e_0^{k-1})\} = \{0, \cdots, k-1\}$ since the same set of colors appears on the k-bundle linking y to z. In other words,

$$c^P(e_0^r) \neq c^P(e_0^{r'}) \text{ for all } r \neq r' \tag{2}$$

Putting (1) and (2) together, we get $c^P(e_0^r) = c^P(e_0^0) \pm r$ for all $r = 1, \cdots, k-1$. Consider $\alpha = 1$, $\beta = k - c^P(e_0^0)$ if $c^P(e_0^1) = c^P(e_0^0) + 1$ and $\alpha = -1$, $\beta = c^P(e_0^0)$ if $c^P(e_0^1) = c^P(e_0^0) - 1$. By defining $c'^P(e) = (\alpha c^P(e) + \beta)\text{mod } k$ we get $c'^P(e_j^r) = r$ for $r = 0, \cdots, k-1$, $j = 0, \cdots, f_r + i_r$. Observe that c'^P is cyclic compact since consecutive colors for c^P remain consecutive for c'^P. Hence, by defining $c'(e) = (\alpha c(e) + \beta)\text{mod } k$ for all edges in Q_I^F we get a cyclic compact k-edge-coloring with $c'(e_j^r) = r$ for all $r = 0, \cdots, k-1$, $j = 1, \cdots, f_r + i_r$. $\square$

For every $r \in \{0, \cdots, k-1\}$ we now use the edges e_j^r $(1 \le j \le f_r)$ in Q_I^F to forbid color r on edges of G and the edges e_j^r $(f_r + 1 \le j \le f_r + i_r)$ in Q_I^F to impose color r on edges of G. This is done as follows. Let $a_1^r, \cdots, a_{f_r}^r$ denote the set of edges in G having r as forbidden color (i.e., with $r \in F_{a_j^r}$) and let $b_1^r, \cdots, b_{i_r}^r$ denote the set of edges in G with imposed color r. For every edge e in G with $F_e \neq \emptyset$ we consider a graph A_e containing three vertices u_e, v_e, w_e, a single edge linking v_e to u_e and a $(k-1)$-bundle linking v_e to w_e. We then define $R = G \cup Q_I^F \cup \bigcup_{e:F_e \neq \emptyset} A_e$ and we finally transform R into a graph $S(G)$ as follows:

(i) for every edge e in G with $F_e \neq \emptyset$, we replace e and the edge linking u_e to v_e by an equalizer for u_e, v_e and the two endpoints of e;

(ii) for every $r = 0, \cdots, k-1$, $j = 1, \cdots, f_r$, we replace e_j^r and one edge linking $v_{a_j^r}$ to $w_{a_j^r}$ by an equalizer for $x_{2j}^r, x_{2j+1}^r, v_{a_j^r}, w_{a_j^r}$;

(iii) for every $r = 0, \cdots, k-1$, $j = f_r + 1, \cdots, f_r + i_r$, we replace the two edges b_j^r and e_j^r by an equalizer for x_{2j}^r, x_{2j+1}^r and the two endpoints of b_j^r.

It is not difficult to observe that $S(G)$ contains $O(n + mk)$ vertices and $O(mk^2)$ edges, where n is the number of vertices and m the number of edges in G. Part of this construction is illustrated in Figure 13. Points (i) and (ii) are used to forbid a color on an edge while point (iii) is used to impose a color.

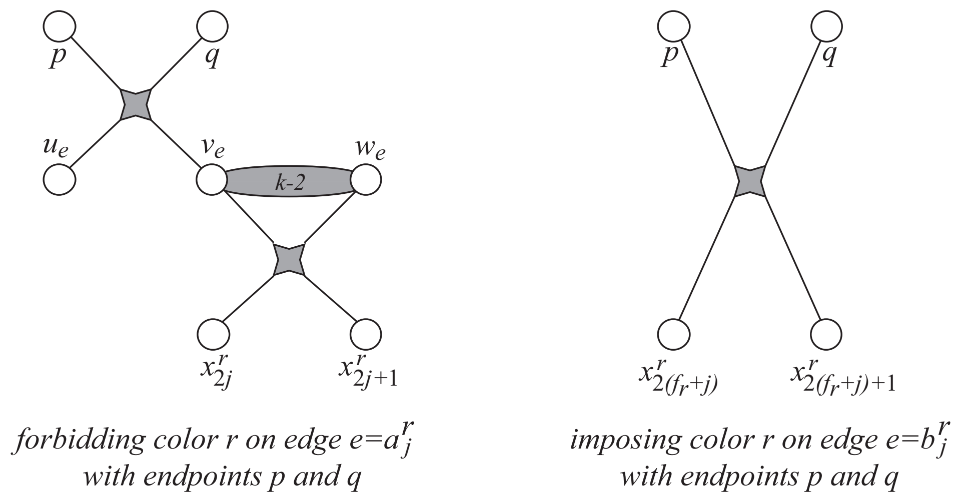

Figure 13: Equalizers for forbidding or imposing colors

Theorem 5 *Assume $k \geq 12$. Then $S(G)$ is k-cyclically compactly colorable if and only if G admits a cyclic compact k-edge-coloring c such that $c(e) \neq r$ for all edges e with $r \in F_e$ and $c(e) = r$ for all $(e, r) \in I$.*

Proof. ($\Leftarrow$) Let c be a cyclic compact k-edge-coloring of G such that $c(e) \neq r$ for all edges e with $r \in F_e$ and $c(e) = r$ for all $(e, r) \in I$. We extend c to a k-edge-coloring c^R of R as follows:

- for every edge e with $F_e \neq \emptyset$, we color A_e by assigning color $c(e)$ to the edge linking u_e to v_e and the colors in $\{0, \cdots, c(e)-1, c(e)+1, \cdots, k-1\}$ to the $(k-1)$-bundle linking v_e to w_e;
- we color Q_I^F by assigning color r ($0 \leq r < k$) to every edge e_j^r ($1 \leq j \leq f_r + i_r$) and the colors in $\{0, \cdots, r-1, r+1, \cdots, k-1\}$ to every $(k-1)$-bundle linking x_{2j-1}^r to x_{2j}^r ($j = 1, \cdots, f_r + i_r$). Also, we color every equalizer for y, z, x_0^r, x_1^r so that the four pendant edges get color r.

We finally define c^S from c^R by coloring every equalizer so that their pendant edges have the same color as the edges that they replace in R. It is easy to check that c^S is a cyclic compact k-edge-coloring of $S(G)$.

($\Rightarrow$) Let c^S be a cyclic compact k-edge-coloring of $S(G)$ and let c^R be the corresponding cyclic compact k-edge-coloring of R obtained by coloring the removed edges $a_1^r, \cdots, a_{f_r}^r, b_1^r, \cdots, b_{i_r}^r, e_1^r, \cdots, e_{f_r+i_r}^r$ ($r = 0, \cdots, k-1$) as the pendant edges of the corresponding equalizers that replace them in $S(G)$. Then,

(a) the edge linking u_e to v_e in R has color $c^R(e)$ for all e with $F_e \neq \emptyset$;

(b) one edge linking $v_{a_j^r}$ to $w_{a_j^r}$ in R has color $c^R(e_j^r)$ (for $r = 0, \cdots, k-1$, $j = 1, \cdots, f_r$);

(c) $c^R(b_j^r) = c^R(e_j^r)$ (for $r = 0, \cdots, k-1$, $j = f_r + 1, \cdots, f_r + i_r$).

Consider the restriction c^Q of c^R to Q_I^F. We know from Lemma 5 that there exist $\alpha \in \{-1, 1\}$ and $\beta \in \{0, \cdots, k-1\}$ such that the k-edge-coloring c'^Q defined by $c'^Q(e) = (\alpha c^Q(e) + \beta) \bmod k$ is cyclic compact with $c'^Q(e_j^r) = r$ for all $r = 0, \cdots, k-1$, $j = 1, \cdots, f_r + i_r$.

By performing the same transformation on c^R we get a new cyclic compact k-edge-coloring $c'^R(e) = (\alpha c^R(e) + \beta) \bmod k$ of R with the following properties. Let e be any edge in G :

- if color r is forbidden for e, there is an index j $(1 \leq j \leq f_r)$ with $a_j^r = e$. We know from (a) and (b) that $c^R(e) \neq c^R(e_j^r)$, which implies $c'^R(e) \neq r$.
- if color r is imposed on e, there is an index j $(f_r + 1 \leq j \leq f_r + i_r)$ with $b_j^r = e$. We know from (c) that $c^R(b_j^r) = c^R(e_j^r)$, which implies $c'^R(e) = c'^R(e_j^r) = r$.

In summary, the restriction of c'^R to G is the desired cyclic compact k-edge-coloring of G. □

The same kind of theorem can be stated for k-linear compact edge-colorings of G with imposed and forbidden colors. Indeed, as a direct consequence of Theorems 3 and 5 we obtain the following corollary.

Corollary 1 *Assume $k \geq 12$. Then $S(T(G))$ is k-cyclically compactly colorable if and only if G admits a linear compact k-edge-coloring c with $c(e) = r$ for all $(i, r) \in I$ and $c(e) \neq r$ for all edges e with $r \in F_e$.*

4 Non-preemptive cyclic production scheduling

Consider m processors $\mathcal{P}_1, \cdots, \mathcal{P}_m$ and n jobs $J_1, \cdots, J_n$. Each job J_i is a set of s_i tasks. Each task T_{ij} $(i = 1, \cdots, s_i)$ of J_i has to be processed on a specific processor $P_{ij} \in \{\mathcal{P}_1, \cdots, \mathcal{P}_m\}$ and has a fixed integer processing time p_{ij}. No two tasks of the same job can be processed simultaneously and no processor can work on two tasks at the same time. Moreover, compactness requirements state that waiting periods are forbidden for every job and no idles are allowed on each processor. Assume that the production must be organized in a cyclic way, i.e., the same production schedule of length k must be repeated continuously every k time units. It is then imposed that the time periods assigned to each job and the active period of each processor form a cyclic interval in each production cycle. The problem is to determine whether or not such a cyclic schedule exists.

To solve this problem, let us associate a bipartite graph G with vertex set $\{J_1, \cdots, J_n, \mathcal{P}_1, \cdots, \mathcal{P}_m\}$. For each task T_{ij}, we link J_i to P_{ij} with p_{ij} parallel edges $e^G_{ij,1}, \cdots, e^G_{ij,p_{ij}}$. If preemption is allowed (i.e., the tasks can be interrupted during their execution), the existence of a feasible cyclic compact schedule of length k is equivalent to the existence of a cyclic compact k-edge-coloring of G. If no preemption is allowed, we have to impose consecutive colors on the p_{ij} parallel edges representing each task T_{ij} (color 0 being considered as consecutive to $k-1$). This can be done easily as follows. For each task T_{ij} with $2 \leq p_{ij} \leq k-1$, we add a star S_{ij} to G with p_{ij} branches $e^S_{ij,1}, \cdots, e^S_{ij,p_{ij}}$ to obtain a new graph G^S. We then create $\tilde{G}^S$ from G^S by replacing each pair $(e^G_{ij,r}, e^S_{ij,r})$ of edges $(r = 1, \cdots, p_{ij})$ by an equalizer. By Lemma 4, $\tilde{G}^S$ is k-cyclically compactly colorable if and only if G^S admits a cyclic compact k-edge-coloring c with $c(e^G_{ij,r}) = c(e^S_{ij,r})$ for every task T_{ij} with $2 \leq p_{ij} \leq k-1$ and every $r = 1, \cdots, p_{ij}$. Since every center of a star S_{ij} is incident to its p_{ij} branches and to no other edge in G^S, we deduce that the colors $c(e^S_{ij,1}), \cdots, c(e^S_{ij,p_{ij}})$ are consecutive (modulo k), which means that the colors on the p_{ij} parallel edges representing task T_{ij} are consecutive also. We can therefore state the following Property.

Property 1 *Assuming $k \geq 12$, there exists a non-preemptive cyclic production schedule of length k if and only if $\tilde{G}^S$ is k-cyclically compactly colorable.*

5 Conclusion

We have proved that for $k \geq 12$, the problem of finding a k-linear compact edge-coloring of a graph G with forbidden or imposed colors on some edges is polynomially reducible to the problem of finding a k-cyclic compact edge coloring of another graph. For $k < 12$, we have proposed a reduction to the $12-CCCP$. As a consequence, production scheduling problems with compactness requirements and in which some tasks can only be processed at specific time periods may be modeled as a $k - CCCP$. We have also shown how the problem of finding a non-preemptive cyclic production schedule of length k can be modeled as a $k - CCCP$.

The restriction $k \geq 12$ comes from the equalizer which cannot impose the same color on the four pendant edges when $k < 12$. It would be interesting to find a different structure that imposes the same color on several edges without assuming $k \geq 12$. Note also that many production scheduling problems can be modeled as edge-coloring problems in bipartite graphs. The transformations $T(G)$ and $S(G)$ proposed in this paper do not preserve the bipartiteness of the original graph G. While testing linearly (cyclically) compactly colorability of bipartite graphs is $\mathcal{NP}$-complete ([14, 10]), it might be interesting to reduce the $k - LCCP$ in G to the $k - CCCP$ in H so that H is bipartite whenever G is.

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