# Total Domination and the Caccetta-Häggkvist Conjecture

**Patrick St-Louis**
Patrick.StLouis@giro.ca
GIRO inc.
75, rue de Port-Royal Est
Montréal, Québec, H3L 3T1 Canada

**Bernard Gendron**
Bernard.Gendron@cirrelt.ca
Département d'informatique
et de recherche opérationnelle
and
CIRRELT
Université de Montréal
C.P. 6128, succ. Centre-ville
Montréal, Québec, H3C 3J7 Canada

**Alain Hertz**
Alain.Hertz@gerad.ca
Département de mathématiques et de génie industriel
École Polytechnique de Montréal
and
GERAD
C.P. 6079, succ. Centre-ville
Montréal, Québec, H3C 3A7 Canada

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**Abstract**

A total dominating set in a digraph $G$ is a subset $W$ of its vertices such that every vertex of $G$ has an immediate successor in $W$. The total domination number of $G$ is the size of the smallest total dominating set. We consider several lower bounds on the total domination number and conjecture that these bounds are strictly larger than $g(G) - 1$, where $g(G)$ is the number of vertices of the smallest directed cycle contained in $G$. We prove that these new conjectures are equivalent to the Caccetta-Häggkvist conjecture which asserts that $g(G) - 1 < \frac{n}{r}$ in every digraph on $n$ vertices with minimum outdegree at least $r > 0$.

# 1 Introduction

Throughout this paper, we consider only digraphs without multiple arcs and without directed cycles of length 1 or 2. Let $G = (V, A)$ be a digraph with vertex set $V$ and arc set $A$. The girth of $G$, denoted $g(G)$, is the number of vertices of the smallest directed cycle in $G$. Let $\delta^+(G)$ denote the minimum outdegree of $G$. In 1978, Caccetta and Häggkvist [1] proposed the following conjecture.

**Conjecture 1**

*Let $G$ be a digraph with $n$ vertices and $\delta^+(G) \geq r > 0$. Then $g(G) \leq \lceil \frac{n}{r} \rceil$.*

This conjecture has been verified for values of $r$ up to 5 [1, 4, 5] and for $n \geq 2r^2 - 3r + 1$ [7]. Another approach is to show that if $G$ is a digraph with $n$ vertices and $\delta^+(G) \geq r$, then there is a directed cycle in $G$ of length at most $\frac{n}{r} + c$ for some small $c$. This has been proved for $c = 2500$ [2], $c = 304$ [6] and $c = 73$ [8]. In 2006, a workshop was held in Palo Alto, California, with the Caccetta-Häggkvist conjecture as its central subject. A summary of the results (and much more) was published by Sullivan [9].

Let $N^+_G(v)$ denote the set of immediate successors of a vertex $v \in V$. A *total dominating set* in a digraph $G$ is a subset $W$ of its vertices such that $N^+_G(v) \cap W \neq \emptyset$ for every vertex $v \in V$. The total domination number of $G$, denoted $TD(G)$ is the size of the smallest total dominating set of $G$. We assume $\delta^+(G) > 0$, else $G$ does not contain any total dominating set. Finding a total dominating set of size $TD(G)$ can be modeled as the assignment of a weight $\omega_v \in \{0, 1\}$ to every vertex $v \in V$ so that $\sum_{u \in N^+_G(v)} \omega_u \geq 1$ for every $v \in V$ and $\sum_{v \in V} \omega_v$ is minimized. Note that $\sum_{u \in V} \omega_u \geq \sum_{u \in N^+_G(v)} \omega_u \geq 1$, for any $v \in V$, i.e., $TD(G)$ is at least 1. Better lower bounds on $TD(G)$ can be obtained by considering real values for the weights:

- We denote $TDF(G)$ the minimum total weight $\sum_{v\in V}\omega_v$ so that $\omega_v \in [0,1]$ and $\sum_{u\in N_G^+(v)}\omega_u \geq 1$ for every $v \in V$.
- By imposing $\sum_{u\in N_G^+(v)}\omega_u \geq 1$ only for vertices with a strictly positive weight and by requiring that $\sum_{v\in V}\omega_v \geq 1$, one gets a lower bound on $TDF(G)$. More precisely, we denote $TDFR(G)$ the minimum total weight $\sum_{v\in V}\omega_v$ so that $\omega_v \in [0,1]$ for every $v \in V$, $\sum_{u\in N_G^+(v)}\omega_u \geq 1$ for every $v$ with $\omega_v > 0$, and $\sum_{v\in V}\omega_v \geq 1$.

It follows from the above definitions that $TDFR(G) \leq TDF(G) \leq TD(G)$. We state the two following conjectures.

**Conjecture 2**
*The relation $g(G) - 1 < TDF(G)$ holds for all digraphs $G$ with $\delta^+(G) > 0$.*

**Conjecture 3**
*The relation $g(G) - 1 < TDFR(G)$ holds for all digraphs $G$ with $\delta^+(G) > 0$.*

We prove in this paper that the two new conjectures are equivalent to Conjecture 1 of Caccetta and Häggkvist. In Section 2, we present mathematical programming formulations that can be used to compute $TD(G)$ and its lower bounds $g(G)$, $TDF(G)$ and $TDFR(G)$. We use these formulations to prove the equivalence of the three conjectures. In Section 3, we show how to reformulate Conjecture 3 using Lagrangean relaxation techniques.

# 2 Mathematical Programming Formulations

The adjacency matrix $A$ of a digraph $G$ is the $n \times n$ matrix where $a_{ij} = 1$ if there is an arc from $i$ to $j$, and $a_{ij} = 0$ otherwise. We denote $e$ the vector with $n$ entries equal to 1. The problem of determining $TD(G)$ will be denoted $P_{TD}(G)$ and can be modeled as an integer programming model as follows:

$$\begin{aligned} TD(G) = \quad & Min \quad e^T\omega \\ & s.t. \quad A\omega \geq e, \\ & \qquad\quad \omega \in \{0,1\}^n. \end{aligned} \tag{1}$$

Determining $g(G)$ can be viewed as the selection of the smallest subset $W$ of vertices such that $N_G^+(v) \cap W \neq \emptyset$ for every vertex $v$ in $W$. This problem, denoted $P_g(G)$, can be modeled with the following integer programming model, where constraints (3) ensure that at least one vertex is selected in $W$:

$$
\begin{aligned}
g(G) = \quad & Min \quad e^T\omega \\
& s.t. \quad A\omega \geq \omega, & (2)\\
& \qquad\quad e^T\omega \geq 1, & (3)\\
& \qquad\quad \omega \in \{0,1\}^n.
\end{aligned}
$$

**Property 4** *The relation $g(G) \leq TD(G)$ holds for all digraphs $G$ with $\delta^+(G) > 0$.*

**Proof.** Since the inequalities $A\omega \geq e$ imply $e^T\omega \geq 1$, one can add constraints (3) to the computation of $TD(G)$ without modifying the optimal value of $P_{TD}(G)$. Since $\omega \leq e$, constraints (1) are stronger than (2), which proves that $g(G) \leq TD(G)$. ∎

To prove the validity of Conjecture 1 it would have been sufficient to show that $TD(G) < \frac{n}{r} + 1$, since this would imply $g(G) - 1 \leq TD(G) - 1 < \frac{n}{r}$, which is equivalent to $g(G) \leq \lceil \frac{n}{r} \rceil$. There are however digraphs for which $TD(G) \geq \frac{n}{r} + 1$. For example, it is not difficult to verify that the digraph in Figure 1 satisfies $n = 10$, $r = \delta^+(G) = 2$ and $6 = TD(G) = \frac{n}{r} + 1$, the black vertices corresponding to a total dominating set of minimum size.

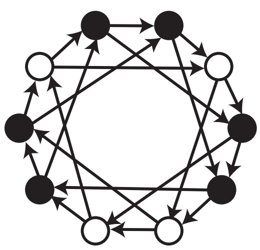


Figure 1. A digraph with $TD(G) = \frac{n}{r} + 1$

The problem of computing $TDF(G)$, denoted $P_{TDF}(G)$, can be modeled by relaxing the integrality constraints in $P_{TD}(G)$:

$$
\begin{aligned}
TDF(G) = Min \quad & e^T\omega \\
s.t. \quad & A\omega \geq e, & (1)\\
& \omega \geq 0.
\end{aligned}
$$

By imposing $A\omega \geq \lceil \omega \rceil$, we require $\sum_{u \in N_G^+(v)} \omega_u \geq 1$ for every vertex $v$ with $\omega_v > 0$. Hence, the problem $P_{TDFR}(G)$ of computing $TDFR(G)$ can be modeled as follows:

$$\begin{aligned} TDFR(G) = Min \quad & e^T\omega \\ s.t. \quad & A\omega \geq \lceil \omega \rceil, & (4) \\ & e^T\omega \geq 1, & (3) \\ & \omega \geq 0. \end{aligned}$$

Note that if Conjecture 3 is verified then $g(G)-1 < TDFR(G) \leq g(G)$ for all digraphs $G$ with $\delta^+(G) > 0$, since by setting $\omega_v = 1$ for all vertices $v$ in a smallest directed cycle in $G$ and $\omega_v = 0$ for the other vertices, one gets a feasible solution to $P_{TDFR}(G)$ of value $g(G)$.

**Theorem 5** *Conjectures 2 and 3 are equivalent.*

**Proof.** If Conjecture 3 is verified then $g(G) - 1 < TDFR(G) \leq TDF(G)$, which implies that Conjecture 2 is verified also.

So assume that Conjecture 3 is not verified and let $G$ be a smallest counter-example (in terms of number of vertices). It remains to prove that Conjecture 2 is also not verified. Let $\omega^*$ denote an optimal solution to $P_{TDFR}(G)$, and let $G'$ denote the sub-digraph of $G$ induced by all vertices $v$ with weight $\omega^*_v > 0$. Constraints (4) impose that each vertex in $G'$ has at least one successor in $G'$. Hence $G'$ contains at least one directed cycle and we obviously have $g(G') \geq g(G)$. Also, $TDFR(G') \leq TDFR(G)$ since the restriction of $\omega^*$ to $G'$ is a feasible solution to $P_{TDFR}(G')$. In summary, $g(G') - 1 \geq g(G) - 1 \geq TDFR(G) \geq TDFR(G')$. Since $G$ is the smallest counter-example to Conjecture 3, we necessarily have $G' = G$, which means that $\omega^*_v > 0$ for all vertices in $G$. Hence, $\omega^*$ is a feasible solution to $P_{TDF}(G)$, which means that $TDF(G) = TDFR(G)$ and $G$ is therefore also a counter-example to Conjecture 2. ■

**Theorem 6** *Conjectures 1 and 2 are equivalent.*

**Proof.** Let $G$ be a digraph with $n$ vertices and consider any real number $r$ such that $0 < r \leq \delta^+(G)$. The vector $\omega$ defined by $\omega_v = \frac{1}{r}$ for all $v \in V$ is a feasible solution to

$P_{TDF}(G)$, which means that $TDF(G) \leq e^T\omega = \frac{n}{r}$. Hence, if Conjecture 2 is verified, then $g(G) - 1 < TDF(G) \leq \frac{n}{r}$ for all digraphs $G$ with $n$ vertices and $\delta^+(G) \geq r > 0$, which implies that Conjecture 1 is verified also.

So assume that Conjecture 2 is not verified. It remains to show that Conjecture 1 is also not verified. Let $G$ be a smallest counter-example to Conjecture 2 (in terms of number of vertices), and let $\omega^*$ be any optimal basic solution to $P_{TDF}(G)$. We necessarily have $\omega^*_v > 0$ for all $v \in V$, otherwise by using the same arguments as in the proof of the previous theorem, we can show that the sub-digraph $G'$ induced by the vertices with $\omega^*_v > 0$ verifies $g(G') - 1 \geq g(G) - 1 \geq TDF(G) \geq TDF(G')$, which contradicts the minimality of $G$.

Constraints (1) can be rewritten as $A\omega - s = e$ by using slack variables $s \geq 0$. Consider the values $s^* = A\omega^* - e$ of the slack variables associated with $\omega^*$. Since $P_{TDF}(G)$ contains $n$ constraints, we necessarily have $s^* = 0$, else there would be at least one vertex $v \in V$ with $\omega^*_v = 0$. We therefore have $A\omega^* = e$. In other words, if we denote $P^{=}_{TDF}(G)$ the linear program obtained from $P_{TDF}(G)$ by replacing inequalities (1) by equalities, we have shown that $P^{=}_{TDF}(G)$ and $P_{TDF}(G)$ have the same set of optimal solutions.

We now show that the determinant $det(A)$ of matrix $A$ is not equal to 0. If $det(A) = 0$, then at least one of the $n$ constraints in $P^{=}_{TDF}(G)$ is redundant. By removing such a constraint, the optimal value remains unchanged, while there are now $n - 1$ constraints for $n$ variables. This means that $P^{=}_{TDF}(G)$ has an optimal solution $\omega^*$ (which is also optimal for $P_{TDF}(G)$) with at least one variable $\omega^*_v = 0$, a contradiction. We therefore have $det(A) \neq 0$.

Let $A_v$ denote the matrix obtained from $A$ by replacing the $v - th$ column by vector $e$. Cramer's rule [3] states that $\omega^*_v = \frac{\det(A_v)}{\det(A)}$. Since $\omega^*_v > 0$, we can write

$$\omega^*_v = \frac{|\det(A_v)|}{|\det(A)|}.$$

We now construct a new graph $\tilde{G}$ from $G$ by replacing every vertex $v$ by a set $S_v$ of $|\det(A_v)|$ non-adjacent vertices. We put an arc from a vertex in $S_u$ to a vertex in $S_v$ if and only if there is an arc from $u$ to $v$ in $G$. Let $\tilde{V} = \bigcup_{v \in V} S_v$ denote the vertex set of $\tilde{G}$ and define $\tilde{\omega}_{\tilde{v}} = \frac{1}{|\det(A)|}$ for all $\tilde{v} \in \tilde{V}$. In other words, $\tilde{G}$ is obtained from $G$ by replacing every vertex $v$ of weight $\omega^*_v$ by $|\det(A_v)|$ non-adjacent copies of $v$ of weight $\frac{1}{|\det(A)|}$. This means that the following equalities hold for every vertex $\tilde{v} \in S_v$:

$$\sum_{\tilde{u} \in N^+_{\tilde{G}}(\tilde{v})} \tilde{\omega}_{\tilde{u}} = \sum_{u \in N^+_G(v)} \sum_{\tilde{u} \in S_u} \frac{1}{|\det(A)|} = \sum_{u \in N^+_G(v)} \frac{|\det(A_u)|}{|\det(A)|} = \sum_{u \in N^+_G(v)} \omega^*_u = 1.$$

In addition, we have:

$$\sum_{\tilde{v} \in \tilde{V}} \tilde{\omega}_{\tilde{v}} = \sum_{v \in V} \sum_{\tilde{v} \in S_v} \tilde{\omega}_{\tilde{v}} = \sum_{v \in V} \frac{|\det(A_v)|}{|\det(A)|} = \sum_{v \in V} \omega^*_v = TDF(G).$$

Hence, $\tilde{\omega}$ is a feasible solution to $P_{TDF}(\tilde{G})$ of value $TDF(G)$, which means that $TDF(\tilde{G}) \leq TDF(G)$.

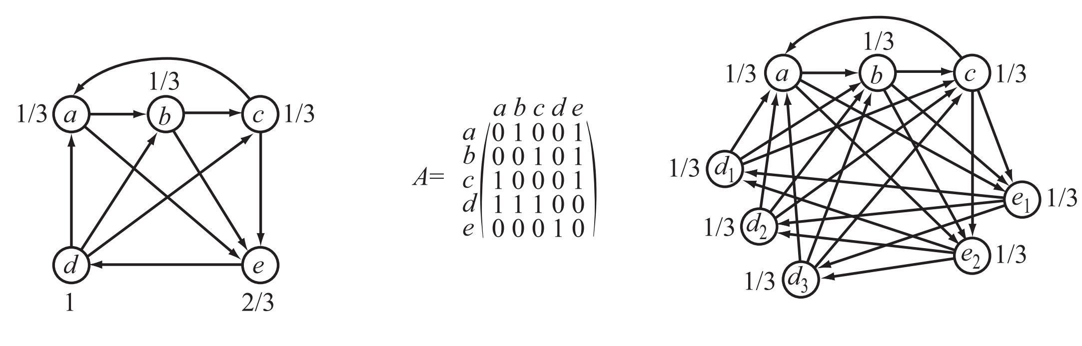
Since, for every $\tilde{v} \in S_v$, $\sum_{\tilde{u} \in N^+_{\tilde{G}}(\tilde{v})} \tilde{\omega}_{\tilde{u}} = \sum_{\tilde{u} \in N^+_{\tilde{G}}(\tilde{v})} \frac{1}{|\det(A)|} = 1$, we deduce that $|N^+_{\tilde{G}}(\tilde{v})| = |\det(A)|$, which means that $\delta^+(\tilde{G}) = |\det(A)|$. Also, since $\sum_{\tilde{v} \in \tilde{V}} \tilde{\omega}_{\tilde{v}} = \sum_{\tilde{v} \in \tilde{V}} \frac{1}{|\det(A)|} = TDF(G)$, we have $|\tilde{V}| = |\det(A)| \cdot TDF(G)$. Moreover, we clearly have $g(G) = g(\tilde{G})$.

To conclude, let $n' = |\tilde{V}|$ denote the number of vertices in $\tilde{G}$ and let $r' = \delta^+(\tilde{G}) = |\det(A)|$. We have

$$g(\tilde{G}) - 1 = g(G) - 1 \geq TDF(G) = \frac{|\det(A)| \cdot TDF(G)}{|\det(A)|} = \frac{n'}{r'}$$

which means that $\tilde{G}$ is a counter-example to Conjecture 1. ■

The construction described in the proof of Theorem 6 is illustrated in Figure 2 for a digraph $G$ with $TDF(G) = \frac{8}{3}$ and $g(G) = 3$.




A digraph $G$ with optimal weights $\omega^*_v$ on the vertices.

The adjacency matrix $A$ with *det(A)*=3.

The corresponding digraph $\tilde{G}$ with weights $\tilde{\omega}_{\tilde{v}}$ on the vertices.

Figure 2. A digraph $G$ and the corresponding digraph $\tilde{G}$.

## 3 Reformulation of Conjecture 3

Problem $P_{TDFR}(G)$ can be written as an integer linear programming model by replacing $A\omega \geq \lceil\omega\rceil$ with $A\omega \geq y \geq \omega$, $y \in \{0,1\}^n$. Hence, an equivalent model for $P_{TDFR}(G)$ reads as follows:

$$\begin{aligned} TDFR(G) = Min \quad & e^T\omega \\ s.t. \quad & A\omega \geq y, & (5) \\ & y \geq \omega, & (6) \\ & e^T\omega \geq 1, & (3) \\ & \omega \geq 0, y \in \{0,1\}^n. \end{aligned}$$

If we now replace constraints (6) by $y = \omega$, we obtain an equivalent model for $P_g(G)$ since

- constraints (5) are then equivalent to constraints (2);
- constraints $\omega \geq 0, y \in \{0,1\}^n$ are then equivalent to $\omega \in \{0,1\}^n$.

In other words, a model for $P_g(G)$ can be obtained from the above model for $P_{TDFR}(G)$ by adding the constraints $y \leq \omega$. Consider now the Lagrangian relaxation of this model for $P_g(G)$ obtained by relaxing constraints $y \leq \omega$ and by penalizing their violation in the objective function. More formally, given a penalty vector $\lambda \geq 0$ with $n$ entries, we consider the problem $P_{g_\lambda}(G)$ of computing $g_\lambda(G)$ defined as follows:

$$\begin{aligned} g_\lambda(G) = Min \quad & e^T\omega + \lambda^T(y-\omega) \\ s.t. \quad & A\omega \geq y, & (5) \\ & y \geq \omega, & (6) \\ & e^T\omega \geq 1, & (3) \\ & \omega \geq 0, y \in \{0,1\}^n. \end{aligned}$$

**Property 7**

*The relations $TDFR(G) \leq g_\lambda(G) \leq g(G)$, for any $\lambda \geq 0$ , $TDFR(G) = g_0(G)$ and $g(G) = g_\lambda(G)$, for any $\lambda \geq e$, hold for all digraphs $G$ with $\delta^+(G) > 0$.*

**Proof.** For any $\lambda \geq 0$, $P_{g_\lambda}(G)$ and $P_{TDFR}(G)$ have the same set of feasible solutions. Since $\lambda^T(y-\omega) \geq 0$ because of constraints (6), we have $TDFR(G) \leq g_\lambda(G)$. Let $\omega^*$ be an optimal solution to $P_g(G)$ and define $y^* = \omega^*$. Since $(\omega^*, y^*)$ is feasible for $P_{g_\lambda}(G)$ and $\lambda^T(y^* - \omega^*) = 0$, we have $g_\lambda(G) \leq g(G)$. For $\lambda = 0$, $P_{g_\lambda}(G)$ corresponds to $P_{TDFR}(G)$, which means that $TDFR(G) = g_0(G)$. For any $\lambda \geq e$, let $(\omega^*, y^*)$ be an optimal solution to $P_{g_\lambda}(G)$. If $\omega^* < y^*$, we can replace $\omega^*$ by $y^*$ and remain feasible, but also optimal, since the objective would then vary by the quantity $(e-\lambda)^T(y^* - \omega^*) \leq 0$. Therefore, for any $\lambda \geq e$, there exists an optimal solution to $P_{g_\lambda}(G)$ that satisfies $\omega^* = y^*$, which means that $g(G) = g_\lambda(G)$. ■

From this result, it follows directly that $TDFR(G) = \min_{\lambda \geq 0} g_\lambda(G)$ and $g(G) = \max_{\lambda \geq 0} g_\lambda(G)$. Conjecture 3 can therefore be rewritten in the following way:

**Reformulation of Conjecture 3**

*The relation* $\max_{\lambda \geq 0} g_\lambda(G) - \min_{\lambda \geq 0} g_\lambda(G) < 1$ *holds for all digraphs* $G$ *with* $\delta^+(G) > 0$.

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