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Abstract

The main goal of this paper is to bring a contribution in order to facilitate automatic refinement of Bimatrix Game Nash extreme equilibria. We show how maximal cliques enumeration algorithms can be used to enumerate all Nash maximal subsets of a Bimatrix Game. We also use a pair of linear programs to identify perfect extreme equilibria and enumerate all Selten Maximal Subsets. These two results are used by the *XGame-Solver* software to automatically generate all Selten Maximal Subsets corresponding to a Bimatrix game.

Key Words: Enumeration, Bimatrix Game, Extreme Nash Equilibrium, Maximal Selten Subset, XGame-Solver.

Résumé

L'objectif de cet article est de faire une contribution dans le but de faciliter le raffinement automatique des équilibres en théorie des jeux. Nous montrons comment le problème d'énumération des sous-ensembles Nash maximaux peut être résolu comme un problème d'énumération des cliques maximales dans un graphe. Nous utilisons aussi une paire de programmes linéaires afin d'identifier les équilibres extrêmes parfaits et énumérer tous les sous-ensemble de Selten maximaux. Ces deux résultats sont utilisés par le logiciel *XGame-Solver* afin de générer automatiquement tous les sous-ensembles de Selten maximaux correspondant à un jeu bimatriciel.

Mots clés : énumération, jeu bimatriciel, équilibre de Nash extrême, sous-ensembles de Selten maximaux, XGame-Solver.

1 Introduction

A *bimatrix game* is a strategic confrontation of two players I and II. Both players could be political, social or economic agents or institutions. Each player has a finite number of actions to choose from. The deterministic choice of an action is called *pure strategy.* Player I has to choose between n pure strategies, while player II has to choose between m pure strategies. A bimatrix game $G(A, B)$ is described through a pair of $n \times m$ payoff matrices A and B. Elements a_{ij} and b_{ij} of matrix A and B are respectively the immediate payoffs of player I and player II when the first plays his i^{th} strategy while the second simultaneously plays his j^{th} strategy.

Each player attempts to maximize his own payoff by selecting a probability vector over his set of pure strategies. These vectors are combinations of pure strategies, called *mixed strategies*, and represented by probability vectors $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. Hence, player I's payoff is $x_1^t A x_2$ and player II's payoff is $x_1^t B x_2$.

A Nash *equilibrium* is defined as a profile of strategies such that simultaneously player I maximizes his payoff given the strategic choice of player II and player II maximizes his payoff given the strategic choice of player I. An equilibrium point is a profile of strategies where neither player has an interest to unilaterally change his strategic choice unless the other player does so. As shown by Nash [12], a bimatrix game has at least one equilibrium. Formally a Nash equilibrium is a pair of strategies $(\hat{x}_1, \hat{x}_2)$ such that $\hat{x}_1 \in X_1(\hat{x}_2)$ and $\hat{x}_2 \in X_2(\hat{x}_1)$ where

$$X_1(\hat{x}_2) = \begin{array}{ll} \underset{x_1}{\text{argmax}} & x_1^t A \hat{x}_2 \\ \text{s.t.} & x_1^t e_n = 1, \\ & x_1 \geq 0, \end{array} \qquad \text{and} \qquad X_2(\hat{x}_1) = \begin{array}{ll} \underset{x_2}{\text{argmax}} & \hat{x}_1^t B x_2 \\ \text{s.t.} & e_m^t x_2 = 1, \\ & x_2 \geq 0, \end{array}$$

and where e_n and e_m are two $(n \times 1)$ and $(m \times 1)$ column vectors with all elements equal to 1. Clearly, $X_1(\hat{x}_2)$ for fixed $\hat{x}_2$, and $X_2(\hat{x}_1)$ for fixed $\hat{x}_1$ are polytopes.

Mills [10], then Mangasarian and Stone [8] studied the optimality conditions of the preceding system in order to establish necessary and sufficient conditions of equilibrium. Introducing real-valued variables α_1 and α_2, the duals of the above linear programs are

$$\begin{array}{ll} \underset{\alpha_1}{\min} & \alpha_1 \\ \text{s.t.} & e_n \alpha \geq A \hat{x}_2, \end{array} \qquad \qquad \begin{array}{ll} \underset{\alpha_2}{\min} & \alpha_2 \\ \text{s.t.} & \alpha_2 e_m^t \geq \hat{x}_1^t B. \end{array}$$

Primal and dual feasibility conditions yield that a pair of strategies $(\hat{x}_1, \hat{x}_2)$ is a Nash equilibrium if there exist two optimal scalars $\hat{\alpha_1}$ and $\hat{\alpha_2}$ satisfying

$$(\hat{x}_1, \hat{\alpha}_2) \in X_1 \equiv \{(x_1, \alpha_2) \in \mathbb{R}^{n+1} : x_1^t B \leq \alpha_2 e_m^t,\ x_1^t e_n = 1, x_1 \geq 0\},$$

and

$$(\hat{x}_2, \hat{\alpha}_1) \in X_2 \equiv \{(x_2, \alpha_1) \in \mathbb{R}^{m+1} : A x_2 \leq e_n \alpha_1 \ ,\ e_m^t x_2 = 1,\ x_2 \geq 0\}.$$

Moreover, from duality theory of linear programming, the optimal dual objective values $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are respectively equal to the primal optimal payoffs of players I and II

$$\hat{x}_1^t A \hat{x}_2 = \hat{\alpha}_1 \qquad \text{and} \qquad \hat{x}_1^t B \hat{x}_2 = \hat{\alpha}_2.$$

An *extreme Nash equilibrium* is defined as a pair of strategies $(\hat{x}_1, \hat{x}_2)$ such that $\hat{x}_1$ is a vertex of the set $X_1(\hat{x}_2)$ of best responses to the strategy $\hat{x}_2$, and $\hat{x}_2$ is a vertex of the set $X_2(\hat{x}_1)$ of best responses to the strategy $\hat{x}_1$:

$$\hat{x}_1 \in ext(X_1(\hat{x}_2)) \qquad \text{and} \qquad \hat{x}_2 \in ext(X_2(\hat{x}_1)),$$

where *ext* denotes the set of extreme points.

As the number of extreme solutions to this problem could be huge, Game theorists introduced many refinements of the concept of Nash equilibria.

The Perfectness concept is a refinement based on the idea that a reasonable equilibrium should be stable against slight perturbations in the equilibrium strategies [13].

In order to bring a contribution to automatic refinement of Nash equilibria, we first show in Section 2 how Bimatrix Game maximal Nash subsets enumeration is equivalent to maximal cliques enumeration for a graph. In Section 3, we show that the*Perfectness* certificate of a Nash equilibrium is obtained through the optimization of a pair of linear programs. These two results make it possible to define all Bimatrix Game maximal Selten subsets.

2 Enumeration of all extreme Nash equilibria

The set NE of all equilibrium points is the union of a finite number of polytopes called *maximal Nash subsets* (Millham [9]). A subset $T \subset NE$ is a Nash subset if and only if every pair of elements in T is interchangeable, i.e. $(x_1, x_2) \in T$ and $(y_1, y_2) \in T$ implies that $(x_1, y_2) \in T$ and $(y_1, x_2) \in T$. A Nash subset T is called maximal if it is not properly contained in another Nash subset (Jansen [5]). Any extreme equilibrium is an extreme point of one of these maximal Nash subsets.

As each Nash equilibrium can be obtained by a convex combination of some extreme Nash equilibria (Mangasarian [7]), complete enumeration of extreme equilibria leads to a complete identification of the set NE (Vorob'ev [15]). Complete enumeration of all bimatrix game extreme equilibria is achieved in Audet et al. [1] for values of $n = m$ up to 29.

It is shown in [2] that the set of bimatrix game extreme Nash equilibria is the set of pairs of mixed strategies $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$, for which there exist vectors $(u_1, u_2) \in \{0,1\}^n \times \{0,1\}^m$, satisfying

$$\left\{\begin{array}{l}
x_1^t e_n = 1, \\
e_m^t x_2 = 1, \\
x_1^t B - \alpha_2 e_m^t \ \leq \ 0, \\
A x_2 - \ \alpha_1 e_n \ \leq \ 0, \\
x_1 + u_1 \leq e_n, \\
x_2 + u_2 \leq e_m, \\
(e_n \alpha_1 - A x_2) \ - \ L_1 u_1 \leq \ 0, \\
(\alpha_2 e_m - B^t x_1) - \ L_2 u_2 \leq \ 0, \\
x_1 \geq 0, \quad x_2 \geq 0, \\
u_1 \in \{0,1\}^n \text{ and } u_2 \in \{0,1\}^m,
\end{array}\right. \tag{2.1}$$

where L_1 and L_2 are fixed finite parameters such that

$$L_1 = \max_{i,j} \ a_{ij} \quad \text{and} \quad L_2 = \max_{i,j} \ b_{ij}.$$

Any bimatrix game can then be expressed as a mixed $0-1$ linear program with $2+3(n+m)$ constraints, $2+n+m$ continuous variables and $n+m$ binary variables. The $E\chi$-MIP [2] algorithm enumerates all extreme Nash equilibria through complete enumeration of extreme feasible solutions for feasible $0-1$ vectors.

2.1 Enumeration of all maximal Nash subsets

In Game Theory literature no specific algorithm is found for the enumeration of all maximal Nash subsets corresponding to bimatrix game. However, an unpublished paper of von Stengel [14] mentions that enumeration of all maximal Nash subsets corresponding to a bimatrix game can be achieved using an algorithm for the enumeration of all maximal cliques of a graph G.

We define $G = (V, E)$ as the graph obtained from the analysis of the extreme Nash equilibria in NE, which was first defined as the set of all Nash equilibria of the bimatrix game. The extreme points of NE define the set of nodes V of G while E is the set of edges of G. Any edge $e \in E$ is a connection between two nodes (extreme Nash equilibria) of G, $X = (x_1, x_2) \in V$ and $Y = (y_1, y_2) \in V$, if and only if X and Y are interchangeable.

Every maximal clique of the graph G is a *complete* subgraph of G. Each maximal clique of G corresponds then to a set of extreme Nash equilibria, in which each extreme Nash equilibria is interchangeable with all others. Thus, a maximal Nash subset T is a maximal clique of G.

Several papers study the maximal cliques enumeration problem. We have coded a C++ version of the widely used algorithm of Bron and Kerbosch [6]. Our implementation of this algorithm is included in the *XGame1.0* package available in [4].

Example 2.1 *Let A and B be the payoff matrices of a bimatrix game taken from Borm et al. [5]*

$$A = \begin{pmatrix} -3 & -3 & -3 & -3 \\ 3 & 3 & 3 & -6 \\ -3 & -3 & -3 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}.$$

Using the $E\chi MIP$ [2] algorithm, ten extreme Nash equilibria are enumerated (Table 1). The graph G obtained from NE is represented by Figure 1.

This game has six maximal Nash subsets $T_1 = \{1,2,3,4,5,6\}$, $T_2 = \{4,8\}$, $T_3 = \{6,10\}$, $T_4 = \{7,9\}$, $T_5 = \{7,10\}$ and $T_6 = \{8,9\}$.

Table 1: Extreme Nash Equilibria for Borm et al. [5]

Eq.	x_1			x_2				α_1	α_2
1	0	1	0	0	0	1	0	3	0
2	0	1	0	0	1	0	0	3	0
3	0	1	0	1	0	0	0	3	0
4	0	1	0	0	0	1/3	2/3	−3	0
5	0	1	0	0	1/3	0	2/3	−3	0
6	0	1	0	1/3	0	0	2/3	−3	0
7	2/3	0	1/3	0	0	0	1	−3	2
8	1/3	0	2/3	0	0	1/3	2/3	−3	2
9	1/3	0	2/3	0	0	0	1	−3	2
10	2/3	0	1/3	1/3	0	0	2/3	−3	2

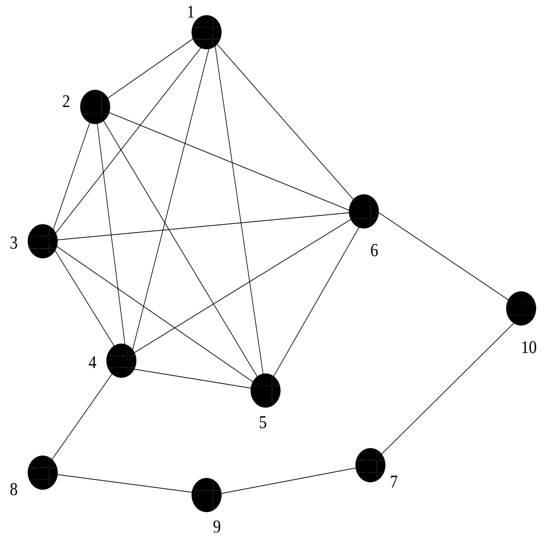

Figure 1: Interchangeablity graph $G = (C,E)$ for Borm et al. [5]

3 Extreme Perfect Equilibria

When confronted to a situation where a huge number of equilibria can be considered to solve a game, decision makers would have to refine their choices using some other rational concepts in addition to the concept of Nash equilibrium. The following definition summarizes the definition of a *perfect equilibrium* given by Selten [13].

Definition 3.1 *Let $(\hat{x}_1, \hat{x}_2)$ be a Nash equilibrium of the bimatrix game. If there is a unit vector x_1 such that $x_1 A \geq \hat{x}_1 A$ and $x_1 A \neq \hat{x}_1 A$, or if there is a unit vector x_2 such that $Bx_2 \geq B\hat{x}_2$ and $Bx_2 \neq B\hat{x}_2$ then $(\hat{x}_1, \hat{x}_2)$ is not perfect. Otherwise, $(\hat{x}_1, \hat{x}_2)$ is said to be perfect.*

Selten [13] and Myerson [11] show that there is always at least one perfect equilibrium for any strategic form game. The following result shows that it is easy to verify if an equilibrium $(\hat{x}_1, \hat{x}_2)$ is perfect or not.

Proposition 3.2 *The equilibrium $(\hat{x}_1, \hat{x}_2)$ is perfect if and only if both of the optimal objective functions values of the following linear programs are equal to zero*

$$\begin{array}{ll} \max\limits_{(x_1,\epsilon_1)\in\mathbb{R}^n\times\mathbb{R}^m} & e_n^t \epsilon_1 \\ s.t & e_n^t x_1 = 1, \\ & x_1 A \geq \hat{x}_1 A + \epsilon_1, \\ & x_1, \epsilon_1 \geq 0, \end{array} \qquad \begin{array}{ll} \max\limits_{(x_2,\epsilon_2)\in\mathbb{R}^m\times\mathbb{R}^n} & e_m^t \epsilon_2 \\ s.t & e_m^t x_2 = 1, \\ & Bx_2 \geq B\hat{x}_2 + \epsilon_2, \\ & x_2, \epsilon_2 \geq 0. \end{array} \tag{3.1}$$

Proof. Let (x_1^*, ϵ_1^*) and (x_2^*, ϵ_2^*) be optimal solutions for these two programs (3.1). If one of the two optimal objective function values is strictly positive, then at least one of the ϵ variables is strictly positive. It means that there is at least one $\epsilon_1^i > 0, i \in \{1, 2, ..., n\}$, or $\epsilon_2^j > 0, j \in \{1, 2, ..., m\}$, such as $x_1^* A_i \geq \hat{x}_1 A_i + \epsilon_1^i$, or $B_j x_2^* \geq B_j \hat{x}_2 + \epsilon_2^j$. Thus, we either have $x_1^* A_i > \hat{x}_1 A_i$, or $B_j x_2^* > B_j \hat{x}_2$. It means that $x_1 A \neq \hat{x}_1 A$, or $Bx_2 \neq B\hat{x}_2$, while $x_1^* A \geq \hat{x}_1 A + \epsilon_1$ and $Bx_2^* \geq B\hat{x}_2 + \epsilon_2$ are both satisfied. Hence, the equilibrium $(\hat{x}_1, \hat{x}_2)$ is not perfect.

If the two optimal objective functions are equal to zero, then all of ϵ_1^* and ϵ_2^* vectors elements are equal to zero. These ϵ_1^* and ϵ_2^* vectors correspond to the maximum slack vectors between $x_1^* A$ and $\hat{x}_1 A$, and Bx_2^* and $B\hat{x}_2$. Thus, $x_1^* A = \hat{x}_1 A$ and $Bx_2^* = B\hat{x}_2$. Hence, if all the ϵ variables are equal to zero the equilibrium $(\hat{x}_1, \hat{x}_2)$ is perfect. □

These two linear programs (3.1) are always feasible ($\epsilon_1 = 0$ and $x_1 = \hat{x}_1$ or $\epsilon_2 = 0$ and $x_2 = \hat{x}_2$). One should note that same exact arithmetics libraries used in [3] were used in order to obtain exact solutions for these two linear programs.

Example 3.3 *For the second extreme Nash equilibrium found for the bimatrix game taken form Borm et al. [5], the corresponding pair of linear programs are expressed as follows*

$$\begin{array}{lll} \max\limits_{x_1,\epsilon_1} & \epsilon_{11} + \epsilon_{12} + \epsilon_{13} + \epsilon_{14} & \\ s.t. & & \\ x_{11} + x_{12} + x_{13} & & = 1, \\ -3x_{11} + 3x_{12} - 3x_{13} & & \geq 3 + \epsilon_{11}, \\ -3x_{11} + 3x_{12} - 3x_{13} & & \geq 3 + \epsilon_{12}, \\ -3x_{11} + 3x_{12} - 3x_{13} & & \geq 3 + \epsilon_{13}, \\ -3x_{11} - 6x_{12} - 3x_{13} & & \geq -6 + \epsilon_{14}, \\ x_1, \epsilon_1 \geq 0, & & \end{array} \qquad \begin{array}{lll} \max\limits_{x_2,\epsilon_2} & \epsilon_{21} + \epsilon_{23} & \\ s.t. & & \\ x_{21} + x_{22} + x_{23} + x_{24} & & = 1, \\ 3x_{21} + 2x_{24} & & \geq 0 + \epsilon_{21}, \\ & & \\ 3x_{23} + 2x_{24} & & \geq 0 + \epsilon_{23}, \\ & & \\ x_2, \epsilon_2 \geq 0. & & \end{array}$$

For this extreme Nash equilibrium, $\hat{x}_1 = (0, 1, 0)$ is undominated, while $\hat{x}_2 = (0, 1, 0, 0)$ is dominated by $x_2 = (0, 0, 0, 1)$, with $\epsilon_{21} = 2$ and $\epsilon_{23} = 2$. Thus, the second extreme Nash equilibrium is not perfect.

Eight of the ten extreme Nash equilibria enumerated are perfect. These extreme perfect equilibria are 1, 3, 4, 6, 7, 8, 9 and 10. Given the Nash maximal subsets identified earlier, and by eliminating the second and the fifth non-perfect extreme equilibria, the maximal Selten subsets of this game are: $S_1 = \{1, 3, 4, 6\}$, $S_2 = \{4, 8\}$, $S_3 = \{6, 10\}$, $S_4 = \{7, 9\}$, $S_5 = \{7, 10\}$ and $S_6 = \{8, 9\}$.

This new definition provides a complete method to systematically enmuerate all the extreme Nash equilibria using the $E\chi MIP$ [2] algorithm and then check which of them correspond to a maximal Selten subset extreme point, *i.e.* an extreme perfect equilibrium. Using this approach, automatic identification of all perfect extreme Nash equilibria is made possible.

As described by Borm et al. [5], a *maximal Selten subset* is a set of interchangeable perfect equilibria. Each maximal Selten subset is a subset of a maximal Nash subset and each extreme point of a maximal Selten subset corresponds to an extreme perfect equilibrium.

The $XGame - Solver$ Software [4] uses such pairs of linear programs to generate all maximal Selten Subsets corresponding to a Bimatrix Game.

4 Conclusion

In this paper we used maximal cliques enumeration in order to find all maximal Nash subsets corresponding to a bimatrix game. After defining a pair of linear programs, we were able to enumerate all maximal subsets of extreme perfect equilibra. Because any proper equilibrium is also perfect, automatic certification of perfect equilibria will possibly make identification of all Bimatrix Game extreme proper equilibria easier.

References

[1] Audet C., Hansen P., Jaumard B., Savard G., Enumeration of all extreme equilibrium strategies of bimatrix games, *SIAM Journal on Scientific Computing*, 23(1), 2001, 323–338.

[2] Audet C., Belhaiza S., Hansen P., Enumeration of all Extreme Equilibria in Game Theory: Bimatrix and Polymatrix Games, *Journal Of Optimization Theory and Applications*, 129(3), 2006.

[3] Audet C., Belhaiza S., Hansen P., A new sequence form approach for the enumeration and refinement of all extreme Nash equilibria for extensive form games, *International Game Theory Review*, December, 2009.

[4] Belhaiza S., Dzimi A., XGame-Solver User Manual, *XGame-Solver Home Site*, *http://www.XGame-Solver.net*, August 2008.

[5] Borm P.E.M., Jansen M.J.M, Potters J.A.M., Tijs S.H., On the structure of the set of perfect equilibria in bimatrix games, *O-R Spektrum*, 15, 1993, 17–20.

[6] Bron C., Kerbosch J., Algorithm 457 : Finding All Cliques of an Undirected Graph, *Communications of the ACM* , 16(9), 1973, 575–577.

[7] Mangasarian O.L., Equilibrium points of bimatrix games, *Journal of the Society for Industrial and Applied Mathematics*, 12, 1964, 778–780.

[8] Mangasarian O.L., Stone H., Two person nonzero-sum games and quadratic programming, *Journal of Mathematical Analysis and Applications*, 9, 1964, 348–355.

[9] Millham C.B., On Nash subsets of bimatrix games, *Naval Research Logistics Quarterly*, 74, 1974, 307–317.

[10] Mills H., Equilibrium points in finite games, *Journal of the Society for Industrial and Applied Mathematics*, 8, 1960, 397–402.

[11] Myerson R.B., *Game Theory : Analysis of Conflict.* Harvard University Press, Cambridge-Massachusetts, London-England, 1997.

[12] Nash J.F., Equilibrium points in n-person games, *Proceedings of the National Academy of Sciences*, 36, 1950, 48–49.

[13] Selten R., Reexamination of the perfectness concept for equilibrium points in extensive games, *International Journal of Game Theory*, 4, 22–55.

[14] von Stengel B., Improved equilibrium enumeration for bimatrix games. Extended Abstract of talk at OR'98, *International Conference on Operations Research*, ETH Zurich, Aug 31-Sept 3, 1998 (unpublished).

[15] Vorob'ev N.N., Equilibrium points in bimatrix games, *Theoriya Veroyatnostej i ee Primwneniya*, 3, 1958, 318–331. [English version : *Theory of Probability and its Applications* 3, 297–309.]