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C. Audet, S. Belhaiza,
P. Hansen

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Charles Audet
Slim Belhaiza

GERAD and École Polytechnique de Montréal
C.P. 6079, Succ. Centre-ville
Montréal (Québec) Canada, H3C 3A7

charles.audet@gerad.ca
slim.belhaiza@gerad.ca

Pierre Hansen

GERAD and HEC Montréal
3000, chemin de la Côte-Sainte-Catherine
Montréal (Québec) Canada, H3T 2A7

pierre.hansen@gerad.ca

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Abstract

In previous work, bimatrix games were expressed as parametric linear mixed $0-1$ programs and the EχMIP algorithm was proposed to enumerate all their extreme Nash equilibria. In this paper, we use a maximal cliques enumeration algorithm to enumerate all Nash maximal subsets. We use a pair of linear programs in order to identify perfect extreme equilibria and enumerate all Selten maximal subsets. We also define a $0-1$ mixed quadratic program in order to identify non-proper extreme equilibria.

Key Words: enumeration, refinement, bimatrix game, extreme Nash equilibrium, perfect, proper, essential, quasi-strong, isolated, regular.

Résumé

Dans un travail précédent, les jeux bimatriciels ont été formulés comme des programmes mixtes $0-1$ paramétriques et l'algorithme EχMIP a été défini pour l'énumération complète de leurs équilibres de Nash extrêmes. Dans cet article, nous utilisons un algorithme d'énumération de cliques maximales afin d'identifier tous les ensembles Nash maximaux. Nous utilisons une paire de programmes linéaires afin de déterminer les équilibres extrêmes parfaits. Nous proposons aussi un programme quadratique mixte $0-1$ afin de déterminer les équilibres extrêmes non-propres.

Mots clés : énumération, raffinement, jeu bimatriciel, équilibre de Nash extrême, parfait, propre, quasi-strong, isolé, régulier.

1 Introduction

A *bimatrix game* is a strategic confrontation of two players I and II. Both players could be political, social or economic agents or institutions. Each player has a finite number of strategies, commonly called *pure strategies*. Player I has to choose between n pure strategies, while player II has to choose between m pure strategies. A bimatrix game $G(A, B)$ is described through a pair of $n \times m$ payoff matrices A and B. Elements a_{ij} and b_{ij} of matrix A and B are respectively the immediate payoffs of player I and player II when the first plays his i^{th} strategy while the second simultaneously plays his j^{th} strategy.

Each player attempts to maximize his own payoff by selecting a probability vector over his set of pure strategies. These vectors are combinations of pure strategies, called *mixed strategies*, and represented by probability vectors $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. Hence, player I's payoff is $x_1^t A x_2$ and player II's payoff is $x_1^t B x_2$.

A Nash *equilibrium* is defined as a situation where simultaneously player I maximizes his payoff given the strategic choice of player II and player II maximizes his payoff given the strategic choice of player I. An equilibrium point is thus a situation where neither player has an interest to unilaterally change his strategic choice unless the other player does so. As shown by Nash [18], a bimatrix game has at least one equilibrium. Formally a Nash equilibrium is a pair of strategies $(\hat{x}_1, \hat{x}_2)$ such that $\hat{x}_1 \in X_1(\hat{x}_2)$ and $\hat{x}_2 \in X_2(\hat{x}_1)$ where

$$X_1(\hat{x}_2) = \begin{array}{ll} \underset{x_1}{\text{argmax}} & x_1^t A \hat{x}_2 \\ \text{s.t.} & x_1^t e_n = 1, \\ & x_1 \geq 0, \end{array} \quad \text{and} \quad X_2(\hat{x}_1) = \begin{array}{ll} \underset{x_2}{\text{argmax}} & \hat{x}_1^t B x_2 \\ \text{s.t.} & e_m^t x_2 = 1, \\ & x_2 \geq 0, \end{array}$$

and where e_n and e_m are two $(n \times 1)$ and $(m \times 1)$ column vectors with all elements equal to 1. Clearly, $X_1(\hat{x}_1)$ for fixed $\hat{x}_2$, and $X_2(\hat{x}_1)$ for fixed $\hat{x}_1$ are polytopes.

Mills [15], then Mangasarian and Stone [13] studied the optimality conditions of the preceding system in order to establish necessary and sufficient conditions of equilibrium. Introducing real-valued variables α_1 and α_2, the duals of the above linear programs are

$$\begin{array}{ll} \underset{\alpha_1}{\min} & \alpha_1 \\ \text{s.t.} & e_n \alpha \geq A \hat{x}_2, \end{array} \qquad \begin{array}{ll} \underset{\alpha_2}{\min} & \alpha_2 \\ \text{s.t.} & \alpha_2 e_m^t \geq \hat{x}_1^t B. \end{array}$$

Primal and dual feasibility conditions yield that a pair of strategies $(\hat{x}_1, \hat{x}_2)$ is an equilibrium if there exist two scalars $\hat{\alpha_1}$ and $\hat{\alpha_2}$ satisfying

$$(\hat{x}_1, \hat{\alpha}_2) \in X_1 \equiv \{(x_1, \alpha_2) \in \mathbb{R}^{n+1} : x_1^t B \leq \alpha_2 e_m^t,\ x_1^t e_n = 1, x_1 \geq 0\},$$

and

$$(\hat{x}_2, \hat{\alpha}_1) \in X_2 \equiv \{(x_2, \alpha_1) \in \mathbb{R}^{m+1} : A x_2 \leq e_n \alpha_1\ ,\ e_m^t x_2 = 1,\ x_2 \geq 0\}.$$

Moreover, from duality theory of linear programming, the optimal dual objective values $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are respectively equal to the primal optimal payoffs of players I and II

$$\hat{x}_1^t A \hat{x}_2 = \hat{\alpha}_1 \qquad \text{and} \qquad \hat{x}_1^t B \hat{x}_2 = \hat{\alpha}_2.$$

An *extreme Nash equilibrium* is defined as a pair of strategies $(\hat{x}_1, \hat{x}_2)$ such that $\hat{x}_1$ is a vertex of the set $X_1(\hat{x}_2)$ of best responses to the strategy $\hat{x}_2$, and $\hat{x}_2$ is a vertex of the set $X_2(\hat{x}_1)$ of best responses to the strategy $\hat{x}_1$:

$$\hat{x}_1 \in ext(X_1(\hat{x}_2)) \qquad \text{and} \qquad \hat{x}_2 \in ext(X_2(\hat{x}_1)),$$

where *ext* denotes the set of extreme points.

The paper is divided as follows. Section 2 shows how the enumeration of maximal Nash subsets is equivalent to the enumeration of all maximal cliques of a graph. In Section 3, standards definitions of *essential, quasi-strong, isolated* and *regular* refinements are presented and illustrated. In Section 4, *perfect* refinement is studied through the definition of a pair of linear programs. Section 5 proposes a mixed $0-1$ quadratic program in order to identify *non-proper* extreme equilibria.

2 Enumeration of all extreme Nash equilibria

The set NE of all equilibrium points is the union of a finite number of polytopes called *maximal Nash subsets* (Millham [14]). A subset $T \subset E$ is a Nash subset if and only if every pair of elements in T is interchangeable, i.e. $(x_1, x_2) \in T$ and $(y_1, y_2) \in T$ implies that $(x_1, y_2) \in T$ and $(y_1, x_2) \in T$. A Nash subset T is called maximal if it is not properly contained in another Nash subset (Jansen [5]). Any extreme equilibrium is an extreme point of one of these maximal Nash subsets.

As each Nash equilibrium can be obtained by a convex combination of some extreme Nash equilibria (Mangasarian [12]), complete enumeration of extreme equilibria leads to a complete identification of the set NE (Vorob'ev [23]). Complete enumeration of all bimatrix game extreme equilibria is achieved in Audet et al. [2] for values of $n = m$ up to 29.

It is shown in [3] that the set of bimatrix game extreme Nash equilibria is the set of pairs of mixed strategies $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$, for which there exists vectors $(u_1, u_2) \in \{0,1\}^n \times \{0,1\}^m$, satisfying

$$\begin{cases} x_1^t e_n = 1, \\ e_m^t x_2 = 1, \\ x_1^t B - \alpha_2 e_n^t \ \leq \ 0, \\ A x_2 - \ \alpha_1 e_m \ \leq \ 0, \\ x_1 + u_1 \leq e_n, \\ x_2 + u_2 \leq e_m, \\ (e_n \alpha_1 - A x_2) \ - \ L_1 u_1 \leq \ 0, \\ (\alpha_2 e_m - B^t x_1) - \ L_2 u_2 \leq \ 0, \\ x_1 \geq 0, \quad x_2 \geq 0, \\ u_1 \in \{0,1\}^n \text{ and } u_2 \in \{0,1\}^m, \end{cases} \tag{2.1}$$

where L_1 and L_2 are fixed finite parameters that can easily be computed.

Any bimatrix game can then be expressed as a mixed $0-1$ linear program with $2+3(n+m)$ constraints, $2+n+m$ continuous variables and $n+m$ binary variables. The algorithm $E\chi$-MIP [3, 4] enumerates all extreme Nash equilibria through complete enumeration of extreme feasible solutions for feasible $0-1$ vectors.

The following example from Myerson [17] will be used in order detail the refinements studied in this paper.

Example 2.1 *Let A and B be the payoff matrices of a bimatrix game taken from Myerson [17]*

$$A = \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 6 & 3 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 4 \\ 4 & 4 \\ 6 & 0 \\ 0 & 2 \end{pmatrix}.$$

Both algorithms $E\chi MIP$ [3, 4] and EEE [2] enumerate all five extreme Nash equilibria of this bimatrix game presented in Table 1.

Table 1: Extreme Nash Equilibria for Myerson [17]

Eq.	x_1				x_2		α_1	α_2
1	0	0	1	0	1	0	6	6
2	0	1	0	0	0	1	4	4
3	0	1	0	0	1/3	2/3	4	4
4	1	0	0	0	0	1	4	4
5	1	0	0	0	1/3	2/3	4	4

2.1 Enumeration of all maximal Nash subsets

A Nash subset T is called maximal if it is not properly contained in another Nash subset (Jansen [5]). The game in the previous example has two maximal Nash subsets $N_1 = \{1\}$ and $N_2 = \{2, 3, 4, 5\}$.

von Stengel [22] mentions that enumeration of all maximal Nash subsets corresponding to bimatrix game can be achieved using an alogorithm for the enumeration of all maximal cliques of a graph G. We define $G = (V, E)$ as graph obtained from the analysis of the extreme Nash equilibria in NE, which was first defined as the of all Nash equilibria of the bimatrix game. The extreme points of NE corresponds define the set of nodes V of G. E is the set of edges of G. Any edge $e \in E$ is a connexion between two nodes (extreme Nash equilibria) of G, $X = (x_1, x_2) \in V$ and $Y = (y_1, y_2) \in V$, if and only if X and Y are interchangeable.

Every maximal clique of the graph G is a *complete* subgraph of G. Each maximal clique of G corresponds then to a set of extreme Nash equilibria, in which each extreme Nash equilibria is interchangeable with all others. Thus, a maximal Nash subset T is a maximal clique of G.

Several papers study the maximal cliques enumeration problem. A simplified C++ version of the widely used algorithm of Bron and Kerbosch [6] is available on this page *http://slim.premiersouffle.com* together with our *XGame1.0* package.

Example 2.2 *Let A and B be the payoff matrices of a bimatrix game taken from Jansen [5]*

$$A = \begin{pmatrix} -3 & -3 & -3 & -3 \\ 3 & 3 & 3 & -6 \\ -3 & -3 & -3 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{pmatrix}.$$

Using the $E\chi MIP$ [3] algorithm, ten extreme Nash equilibria are enumerated (Table 2). The graph G obtained from E is represented by Figure 1.

This game has six maximal Nash subsets $T_1 = \{1, 2, 3, 4, 5, 6\}$, $T_2 = \{4, 8\}$, $T_3 = \{6, 10\}$, $T_4 = \{7, 9\}$, $T_5 = \{7, 10\}$ and $T_6 = \{8, 9\}$.

3 Essential Equlibria

Perfect and proper equilibria are two refinements of the concept of Nash equilibrium based on the idea that a reasonable equilibrium should be stable against slight perturbations in the equilibrium strategies. In this section, we consider a refinement, *the essential equilibrium* concept, which is based on the concept of stability of an equilibrium against slight perturbations in the payoffs of the game.

Table 2: Extreme Nash Equilibria for Jansen [5]

Eq.	x_1			x_2				α_1	α_2
1	0	1	0	0	0	1	0	3	0
2	0	1	0	0	1	0	0	3	0
3	0	1	0	1	0	0	0	3	0
4	0	1	0	0	0	1/3	2/3	−3	0
5	0	1	0	0	1/3	0	2/3	−3	0
6	0	1	0	1/3	0	0	2/3	−3	0
7	2/3	0	1/3	0	0	0	1	−3	2
8	1/3	0	2/3	0	0	1/3	2/3	−3	2
9	1/3	0	2/3	0	0	0	1	−3	2
10	2/3	0	1/3	1/3	0	0	2/3	−3	2

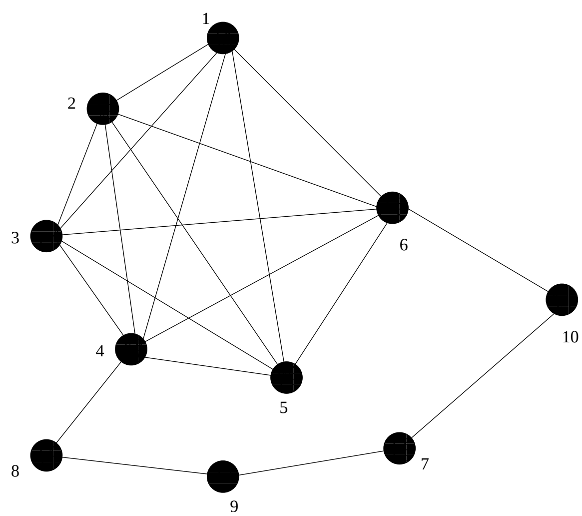

Figure 1: Interchangeablity graph $G = (C, E)$ for Jansen [5]

Definition 3.1 *Wu Wen-tsun and Jiang Jia-he [24] define* (x_1, x_2) *as an essential equilibria of a bimatrix game* $G(A, B)$ *if there exists, with every neighborhood* N_x *of* (x_1, x_2) *a neighborhood* N_G *of* (A, B) *such that* $G(A', B')$ *has no equilibria in* N_x *for all* $(A', B') \in N_G$.

It is known that every essential equilibrium is perfect [7]. Jansen [9] paid special attention to equilibria points that are quasi-strong and isolated at the same time. These equilibria where found to be *essential*.

3.1 Quasi-strong equilibria

For an equilibrium (x_1, x_2) of a bimatrix game $G(A, B)$, $M(A, x_2)$ is defined as the set of pure best replies of player I against x_2:

$$M(A, x_2) = \{i \in N = \{1, ..., n\}; e_i A x_2 = \max_{k \in N} e_k A x_2\}, \tag{3.2}$$

and similarly,

$$M(x_1, B) = \{j \in M = \{1, ..., m\}; x_1 B e_j = \max_{k \in M} x_1 B e_j\}, \tag{3.3}$$

is the set of pure best replies of player II against x_1.

The *carrier* of x_1, $C(x_1)$, is the set $\{i \in N; x_1 i > 0\}$ and carrier of x_2, $C(x_2)$, is the set $\{j \in M; x_2 j > 0\}$.

Definition 3.2 *Harsanyi [8] defined an equilibrium (x_1, x_2) as quasi-strong if*

$$C(x_1) = M(A, x_2) \text{ and } C(x_2) = M(x_1, B).$$

Example 3.3 *For the first extreme Nash equilibrium found for the bimatrix game taken from Myerson [17], $x_1 = (0, 1, 0, 0)$ and $x_2 = (0, 1)$. Then $C(x_1) = \{3\}$and $C(x_2) = \{1\}$. From Table 1, we obtain $M(A, x_2) = \{3\}$ and $M(x_1, B) = \{1\}$. This extreme Nash Equilibrium is then quasi-strong. The remaining four extreme Nash equilibria of this game are not quasi-strong.*

Jansen [9] showed that a *quasi-strong* and *isolated* equilibrium point is stable against slight perturbations of the payoffs of the game.

3.2 Isolated equilibria

An equilibrium (x_1, x_2) of a bimatrix game $G(A, B)$ is said to be *isolated* if there exists a neighborhood N_x of (x_1, x_2) such that it is the only equilibrium of $G(A, B)$ in this neighborhood N_x. In other words, any isolated equilibrium is an extreme equilibrium defining an isolated maximal Nash subset. Thus, the enumeration of all maximal Nash subsets can be used in order to automatically detect isolated equilibria. Moreover, Jansen [9] proposed the following definition.

Definition 3.4 *Let (x_1, x_2) be a quasi-strong equilibrium of a bimatrix game $G(A, B)$ with $A, B > 0$. Then (x_1, x_2) is isolated if and only if $|C(x_1)| = |C(x_2)|$ and the matrices $[a_{ij}]_{i \in C(x_1), j \in C(x_2)}$ and $[b_{ij}]_{i \in C(x_1), j \in C(x_2)}$ are nonsingular.*

Example 3.5 *The first extreme Nash equilibrium found for the bimatrix game taken from Myerson [17], defines an isolated maximal Nash subset. Therefore, this equilibrium is isolated.*

Jansen [9] points out that an isolated equilibrium is essential if and only if it is quasi-strong. Moreover, van Damme [7] showed that an isolated and quasi-strong equilibrium point is perfect and proper. This was also obtained by Okada [19] for bimatrix games.

3.3 Regular Equlibria

A *regular* equilibrium was first defined by Harsanyi [8] by requiring that the Jacobian of a mapping associated with the game evaluated at the equilibrium point is nonsingular. This definition was later revised by van Damme [7] for a two-person case. He proved that an equilibrium is regular if and only if it is *quasi-strong* and *isolated* and showed that such equilibria are *strongly stable* and proper. Regular equilibria have all kind of robustness properties. Jansen [10] showed that an equilibrium point of a bimatrix game is regular if and only if it is isolated and quasi-strong.

Example 3.6 *The first extreme Nash equilibrium found for the bimatrix game taken from Myerson [17], is quasi-strong and isolated. This equilibrium is then essential, strongly stable, regular, perfect and proper.*

4 Extreme Perfect Equilibria

When confronted to a situation where a huge number of equilibria can be considered to solve a game, decision makers would have to refine their choices using some other rational concepts in addition to the concept of Nash equilibrium. The following definition summarizes the definitions of Selten [21], van Damme [7] and Borm et al. [5] of a *perfect equilibrium*.

Definition 4.1 *Let $(\hat{x}_1, \hat{x}_2)$ be a Nash equilibrium of the bimatrix game. If there is a unit vector x_1 such that $x_1 A \geq \hat{x}_1 A$ and $x_1 A \neq \hat{x}_1 A$, or if there is a unit vector x_2 such that $Bx_2 \geq B\hat{x}_2$ and $Bx_2 \neq B\hat{x}_2$ then $(\hat{x}_1, \hat{x}_2)$ is not perfect. Otherwise, $(\hat{x}_1, \hat{x}_2)$ is said to be perfect.*

Selten [21] and Myerson [17] show that there is always at least one perfect equilibrium for any strategic form game. The following result shows that it is easy to verify if an equilibrium $(\hat{x}_1, \hat{x}_2)$ is perfect or not.

Proposition 4.2 *The equilibrium $(\hat{x}_1, \hat{x}_2)$ is perfect if and only if both of the optimal objective functions values of the following linear programs are equal to zero*

$$\begin{array}{ll} \max\limits_{(x_1,\epsilon_1)\in\mathbb{R}^n\times\mathbb{R}^m} & \mathbb{1}^t\epsilon_1 \\ s.t & \mathbb{1}x_1 = 1, \\ & x_1 A \geq \hat{x}_1 A + \epsilon_1, \\ & x_1, \epsilon_1 \geq 0, \end{array} \qquad \begin{array}{ll} \max\limits_{(x_2,\epsilon_2)\in\mathbb{R}^m\times\mathbb{R}^n} & \mathbb{1}^t\epsilon_2 \\ s.t & \mathbb{1}x_2 = 1, \\ & Bx_2 \geq B\hat{x}_2 + \epsilon_2, \\ & x_2, \epsilon_2 \geq 0. \end{array} \tag{4.1}$$

Proof. Let (x_1^*, ϵ_1^*) and (x_2^*, ϵ_2^*) be optimal solutions for these two programs (4.1). If one of the two optimal objective function values is strictly positive, then at least one of the ϵ variables is strictly positive. It means that there is at least one $\epsilon_1^i > 0, i \in \{1, 2, ..., n\}$, or $\epsilon_2^j > 0, j \in \{1, 2, ..., m\}$, such as $x_1^* A_i \geq \hat{x}_1 A_i + \epsilon_1^i$, or $B_j x_2^* \geq B_j \hat{x}_2 + \epsilon_2^j$. Thus, we either have $x_1^* A_i > \hat{x}_1 A_i$, or $B_j x_2^* > B_j \hat{x}_2$. It means that $x_1 A \neq \hat{x}_1 A$, or $Bx_2 \neq B\hat{x}_2$, while $x_1^* A \geq \hat{x}_1 A + \epsilon$ and $Bx_2^* \geq B\hat{x}_2 + \epsilon$ are both satisfied. Hence, the equilibrium $(\hat{x}_1, \hat{x}_2)$ is not perfect.

If the two optimal objective functions are equal to zero, then all of ϵ_1^* and ϵ_2^* vectors elements are equal to zero. These ϵ_1^* and ϵ_2^* vectors correspond to the maximum slack vectors between $x_1^* A$ and $\hat{x}_1 A$, and Bx_2^* and $B\hat{x}_2$. Thus, $x_1^* A = \hat{x}_1 A$ and $Bx_2^* = B\hat{x}_2$. Hence, if all the ϵ variables are equal to zero the equilibrium $(\hat{x}_1, \hat{x}_2)$ is perfect. ■

These two linear programs (4.1) are always feasible ($\epsilon_1 = 0$ and $x_1 = \hat{x}_1$ or $\epsilon_2 = 0$ and $x_2 = \hat{x}_2$). The same exact arithmetics libraries used in [4] are used in order to obtain exact solutions to these two linear programs.

Example 4.3 *For the first extreme Nash equilibrium found for the bimatrix game taken from Myerson [17], the corresponding pair of linear programs are expressed as follows*

$$\begin{array}{ll} \max\limits_{x_1,\epsilon_1} & \epsilon_{11} + \epsilon_{12} \\ s.t. & \\ x_{11} + x_{12} + x_{13} + x_{14} & = 1, \\ 4x_{11} + 4x_{12} + 6x_{13} + 0x_{14} & \geq 6 + \epsilon_{11}, \\ 4x_{11} + 4x_{12} + 3x_{13} + 2x_{14} & \geq 3 + \epsilon_{12}, \\ x_1, \epsilon_1 \geq 0, & \end{array} \qquad \begin{array}{ll} \max\limits_{x_2,\epsilon_2} & \epsilon_{21} + \epsilon_{22} + \epsilon_{23} + \epsilon_{24} \\ s.t. & \\ x_{21} + x_{22} & = 1, \\ 4x_{21} + 4x_{22} & \geq 4 + \epsilon_{21}, \\ 4x_{21} + 4x_{22} & \geq 4 + \epsilon_{22}, \\ 6x_{21} + 0x_{22} & \geq 6 + \epsilon_{23}, \\ 0x_{21} + 2x_{22} & \geq 0 + \epsilon_{24}, \\ x_2, \epsilon_2 \geq 0. & \end{array}$$

The five extreme Nash equilibria obtained for the bimatrix game taken from Myerson [17] are perfect. The corresponding linear programs defined have optimal objective functions equal to 0. The mixed strategies $\hat{x}_1$ or,$\hat{x}_2$ are not dominated for all of these extreme Nash equilibria.

As described by Borm et al. [5], a *maximal Selten subset* is a set of interchangeable perfect equilibria. Each maximal Selten subset is a subset of a Nash maximal subset and each extreme point of a maximal Selten subset corresponds to an extreme perfect equilibria.

The five extreme Nash equilibria found in Table 1 are also extreme perfect equilibria. These extreme perfect equilibria are not only the extreme points of the maximal Nash subsets but also the extreme points of the *maximal Selten subsets*. Each maximal Nash subset is then also a maximal Selten subset in this game.

Example 4.4 *For the second extreme Nash equilibrium found for the bimatrix game taken form Borm et al. [5], the corresponding pair of linear programs are expressed as follows*

$$
\begin{array}{llcll}
\max\limits_{x_1,\epsilon_1} & \epsilon_{11}+\epsilon_{12}+\epsilon_{13}+\epsilon_{14} & \quad & \max\limits_{x_2,\epsilon_2} & \epsilon_{21}+\epsilon_{23} \\
s.t. & & & s.t. & \\
x_{11}+x_{12}+x_{13} & =1, & & x_{21}+x_{22}+x_{23}+x_{24} & =1, \\
-3x_{11}+3x_{12}-3x_{13} & \geq 3+\epsilon_{11}, & & 3x_{21}+2x_{24} & \geq 0+\epsilon_{21}, \\
-3x_{11}+3x_{12}-3x_{13} & \geq 3+\epsilon_{12}, & & & \\
-3x_{11}+3x_{12}-3x_{13} & \geq 3+\epsilon_{13}, & & 3x_{23}+2x_{24} & \geq 0+\epsilon_{23}, \\
-3x_{11}-6x_{12}-3x_{13} & \geq -6+\epsilon_{14}, & & & \\
x_1,\epsilon_1 \geq 0, & & & x_2,\epsilon_2 \geq 0. &
\end{array}
$$

For this extreme Nash equilibrium, $\hat{x}_1=(0,1,0)$ *is undominated, while* $\hat{x}_2=(0,1,0,0)$ *is dominated by* $x_2=(0,0,0,1)$*, with* $\epsilon_{21}=2$ *and* $\epsilon_{23}=2$*. Thus, the second extreme Nash equilibrium is not perfect.*

Eight of the ten extreme Nash equilibria enumerated are perfect. These extreme perfect equilibria are 1, 3, 4, 6, 7, 8, 9 *and* 10*. Given the Nash maximal subsets identified earlier, and by eliminating the second and the fifth non-perfect extreme equilibria, the maximal Selten subsets of this game are:* $S_1=\{1,3,4,6\}$*,* $S_2=\{4,8\}$*,* $S_3=\{6,10\}$*,* $S_4=\{7,9\}$*,* $S_5=\{7,10\}$ *and* $S_6=\{8,9\}$*.*

This new definition provides a complete method to systematically enumerate all the extreme Nash equilibria using the $E\chi MIP$ [3] algorithm and then check which of them correspond to a maximal Selten subset extreme point, i.e. an extreme perfect equilibrium.

5 Extreme Proper Equilibria

The main idea behind the definition of the *proper* refinement of Nash equilibria is that a reasonable player would try harder to avoid important mistakes than he or she would try to avoid small ones.

Definition 5.1 *An equilibrium* (x_1^k,x_2^k) *of a bimatrix game is said to be* ϵ_k*-proper, for some* $\epsilon_k>0$*, if the following conditions are satisfied:*

$$ if \quad A_i x_2^k < A_h x_2^k, \quad then \quad x_{1i}^k \leq \epsilon_k x_{1h}^k, \quad \forall i,h \in \{1,2,...,n\}, \tag{5.1} $$

$$ if \quad x_1^k B_j < x_1^k B_l, \quad then \quad x_{2j}^k \leq \epsilon_k x_{2l}^k, \quad \forall j,l \in \{1,2,...,m\}, \tag{5.2} $$

$$ x_{1i}^k > 0, \quad \forall i \in \{1,2,...,n\}, \qquad x_{2j}^k > 0, \quad \forall j \in \{1,2,...,m\}. \tag{5.3} $$

In order to provide a practical tool to identify ϵ-proper equilibria and non-proper equilibria, for any $\epsilon \geq 0$ and $\sigma \geq 0$ we introduce the set

$$
\begin{array}{llll}
\Omega_\epsilon^\sigma = \{ \ (x_1, x_2) \ : & \exists u, v \text{ such that} & \mathbb{1}x_1 = 1, \ \mathbb{1}x_2 = 1, & \\
& \sigma \leq x_{1i}, & \forall i \in \{1, 2, ..., n\}, & \\
& \sigma \leq x_{2j}, & \forall j \in \{1, 2, ..., m\}, & \\
& & & \\
& A_h x_2 \leq A_i x_2 + L u_{ih}, & \forall i, h \in \{1, 2, ..., n\}, \ i \neq h, & \\
& x_{1i} + u_{ih} \leq \epsilon x_{1h} + 1, & \forall i, h \in \{1, 2, ..., n\}, \ i \neq h, & \\
& u_{ih} + u_{hi} \leq 1, & \forall i, h \in \{1, 2, ..., n\}, \ i < h, & \\
& u_{ih} \in \{0, 1\}, & \forall i, h \in \{1, 2, ..., n\}, \ i \neq h, & \\
& & & \\
& x_1 B_l \leq x_1 B_j + L v_{jl}, & \forall j, l \in \{1, 2, ..., m\}, \ j \neq l, & \\
& x_{2j} + v_{jl} \leq \epsilon x_{2l} + 1, & \forall j, l \in \{1, 2, ..., m\}, \ j \neq l, & \\
& v_{jl} + v_{lj} \leq 1, & \forall j, l \in \{1, 2, ..., m\}, \ j < l, & \\
& v_{jl} \in \{0, 1\}, & \forall j, l \in \{1, 2, ..., m\}, \ j \neq l & \}.
\end{array}
$$

The following proposition ensures that each element of Ω_ϵ^σ is an ϵ-proper equilibria.

Proposition 5.2 *If $(x_1, x_2) \in \Omega_\epsilon^\sigma$ for some $\epsilon > 0$ and $\sigma > 0$, then (x_1, x_2) is an ϵ-proper equilibria.*

Proof. Suppose that (x_1, x_2) belongs to Ω_ϵ^σ, for some $\epsilon > 0$ and $\sigma > 0$. Let i and h be indices in $\{1, 2, ..., n\}$ such that $i \neq h$. Then the inequality $u_{ih} + u_{hi} \leq 1$ ensures that the combination $u_{ih} = 1$ and $u_{hi} = 1$ is not possible. Furthermore,

- if $u_{ih} = 0$ and $u_{hi} = 0$ then $A_h x_2 = A_i x_2$.
- if $u_{ih} = 1$, then $\epsilon x_{1i} \leq x_{1i} \leq \epsilon x_{1h} \leq x_{1h}$ implies that $x_{1h} \geq \epsilon x_{1i}$ and $u_{hi} = 0$, thus $A_i x_2 \leq A_h x_2$,
- if $u_{hi} = 1$, then $\epsilon x_{1h} \leq x_{1h} \leq \epsilon x_{1i} \leq x_{1i}$ implies that $x_{1i} \geq \epsilon x_{1h}$ and $u_{ih} = 0$, thus $A_h x_2 \leq A_i x_2$.

It follows that conditions (5.1) are satisfied. In a similar way, conditions (5.2) are satisfied using binary variables v_{jl}, for all $j, l \in \{1, 2, ..., m\}$ with $j \neq l$.

Finally, with $0 < \sigma \leq x_{2j}$, for all $j \in \{1, 2, ..., m\}$, the conditions (5.3) are satisfied. ∎

Conversely, the following proposition ensures that any ϵ-proper equilibria belongs to Ω_ϵ^σ for all sufficiently small values of σ.

Proposition 5.3 *If (x_1, x_2) is an ϵ-proper equilibria for some $\epsilon > 0$, then there exists a $\bar{\sigma} > 0$ such that $(x_1, x_2) \in \Omega_\epsilon^\sigma$ for every $0 \leq \sigma \leq \bar{\sigma}$.*

Proof. If (x_1, x_2) is an ϵ-proper equilibria for some $\epsilon > 0$, conditions (5.1) can be reformulated using binary variables u_{ih}, for all $i, h \in \{1, 2, ..., n\}$, $i \neq h$:

$$\text{If } \begin{cases} A_i x_2 < A_h x_2, \\ x_{1i} \leq \epsilon x_{1h}, \end{cases} \text{then } \begin{cases} A_i x_2 \leq A_h x_2 + L u_{hi}, \\ x_{1i} + u_{ih} \leq \epsilon x_{1h} + 1, \\ u_{hi} = 0, \\ u_{ih} = 1. \end{cases}$$

$$\text{If } \begin{cases} A_h x_2 < A_i x_2, \\ x_{1h} \leq \epsilon x_{1i}, \end{cases} \text{then } \begin{cases} A_h x_2 \leq A_i x_2 + L u_{ih}, \\ x_{1h} + u_{hi} \leq \epsilon x_{1i} + 1, \\ u_{ih} = 0, \\ u_{hi} = 1. \end{cases}$$

$$\text{If } \begin{cases} A_i x_2 = A_h x_2, \\ x_{1i} \leq 1, \\ x_{1h} \leq 1, \end{cases} \text{then } \begin{cases} A_i x_2 \leq A_h x_2 + L u_{hi}, & x_{1i} + u_{ih} \leq \epsilon x_{1h} + 1, \\ A_h x_2 \leq A_i x_2 + L u_{ih}, & x_{1h} + u_{hi} \leq \epsilon x_{1i} + 1, \\ u_{hi} = 0, & u_{ih} = 0. \end{cases}$$

In a similar way, conditions (5.2) can be reformulated using binary variables v_{jl}, for all $j, l \in \{1, 2, ..., m\}$, $j \neq l$.

And finally, conditions (5.3) ensure that there exists a $\bar{\sigma} > 0$, such that $\bar{\sigma} \leq x_{1i}$, for all $i \in \{1, 2, ..., n\}$ and $\bar{\sigma} \leq x_{2j}$, for all $j \in \{1, 2, ..., m\}$.

Then, for every σ such that $0 \leq \sigma \leq \bar{\sigma}$ and $\sigma > 0$:

$$\begin{aligned} \sigma &\leq x_{1i}, \quad \text{for all } i \in \{1, 2, ..., n\}, \\ \sigma &\leq x_{2j}, \quad \text{for all } j \in \{1, 2, ..., m\}. \end{aligned}$$

Thus, $(x_1, x_2) \in \Omega_\epsilon^\sigma$ for every σ, such that $0 \leq \sigma \leq \bar{\sigma}$ and $\sigma > 0$. ■

Myerson [16] and Jansen [11] define a *proper* equilibrium to be limits of ϵ_k-proper equilibria, with ϵ_k converging to zero.

Definition 5.4 *An equilibrium $(\hat{x}_1, \hat{x}_2)$ is said to be proper if there is a sequence of ϵ_k-proper equilibria (x_1^k, x_2^k) such that*

$$\lim_{k \to \infty} \epsilon_k = 0 \quad \textit{and} \quad \lim_{k \to \infty} (x_1^k, x_2^k) = (\hat{x}_1, \hat{x}_2). \tag{5.4}$$

The main difficulty in applying this definition is to find a convergent sequence $\{\epsilon_k\}_{k \in \mathbb{N}}$ of positive real numbers making the sequence $\left\{(x_1^k, x_2^k)\right\}_{k \in \mathbb{N}}$ converge to $(\hat{x}_1, \hat{x}_2)$, where (x_1^k, x_2^k) are ϵ_k-proper for each $k \in \mathbb{N}$. Myerson [16] showed that every bimatrix game possesses at least one proper equilibrium.

Proposition (5.2) provides necessary and sufficient conditions for an equilibrium to be perfect. The conditions involved solving a pair of linear programs. The next theorem is analogous, but for proper equilibria. It involves solving parametrized mixed $0-1$ quadratic programs. The main idea behind these parametrized mixed $0-1$ quadratic programs is that there solutions, when the parameter σ converges to 0, define a sequence of ϵ-proper equilibria.

Proposition 5.5 *The perfect equilibrium $(\hat{x}_1, \hat{x}_2)$ is a proper equilibrium if and only if the following $0-1$-mixed quadratic program is feasible for some $\bar{\sigma} > 0$, and if $\lim_{\sigma \to 0^+} f(\sigma) = 0$.*

$$\begin{aligned} f(\sigma) \quad = \quad & \min_{(x_1,x_2)\in\Omega_\epsilon^\sigma,\epsilon} \quad \epsilon \\ & s.t. \quad \hat{x}_{1i} - \epsilon \le x_{1i} \le \hat{x}_{1i} + \epsilon, \quad \forall i \in \{1,2,...,n\}, \\ & \qquad\ \ \hat{x}_{2j} - \epsilon \le x_{2j} \le \hat{x}_{2j} + \epsilon, \quad \forall j \in \{1,2,...,m\}, \\ & \qquad\ \ 0 \le \epsilon \le 1. \end{aligned} \tag{5.5}$$

Proof. Let $(x_1(\sigma), x_2(\sigma), \epsilon(\sigma)$ be the optimal solution to (5.5) for some given perfect equilibrium $(\hat{x}_1, \hat{x}_2)$. Proposition (5.2) ensures that $(x_1(\sigma), x_2(\sigma))$, is an $\epsilon(\sigma)$-proper equilibrium. Conditions (5.4) were reformulated using the minimization of ϵ such that

$$\begin{aligned} \hat{x}_{1i} - \epsilon \le x_{1i} \le \hat{x}_{1i} + \epsilon, \quad \forall i \in \{1,2,...,n\}, \\ \hat{x}_{2j} - \epsilon \le x_{2j} \le \hat{x}_{2j} + \epsilon, \quad \forall j \in \{1,2,...,m\}, \end{aligned}$$

in order to make the ϵ-proper equilibria converge to $(\hat{x}_1, \hat{x}_2)$.

Hence, if the mixed $0-1$ quadratic program (5.5) is feasible for all σ, when $\sigma > 0$ converges to 0, we can conclude from Proposition (5.3) that there is always an ϵ-proper equilibrium $(x_1, x_2) \in \Omega_\epsilon^\sigma$.

Moreover, if the perfect equilibrium $(\hat{x}_1, \hat{x}_2)$ is proper then the optimal objective function value $f(\sigma) = \epsilon(\sigma)$ should necessarily converge to 0, when $\sigma > 0$ converges to 0, in order to make the solution $(x_1(\sigma), x_2(\sigma))$ converge to $(\hat{x}_1, \hat{x}_2)$ at the same time. One can also notice that $f(0) = 0$.

Else, if $f(\sigma)$ does not converge to 0, when $\sigma > 0$ converges to 0, then such a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper does not exist, when ϵ converges to 0. In this case, we can prove that the equilibrium point is not proper. ■

In conlusion, if $f(\sigma)$ converges to 0, when $\sigma > 0$ converges to 0, it is possible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to $(\hat{x}_1, \hat{x}_2)$, when $\epsilon(\sigma)$ converges to 0. Finding such sequences would be more difficult and the proof on properness of the equilibrium remains open due to the numerical noise which might appear.

In practice, the previous Proposition (5.5) is difficult to apply directly in order to show that an equilibria $(\hat{x}_1, \hat{x}_2)$ is proper or not. The way that we use it is to compute the value of $f(\sigma)$ for some small values of σ. The $0-1$-mixed quadratic program (5.5) is solved using the NEW-QP algorithm coded by Perron [20]. This algorithm is a new version of the QP algorithm [1]. The QP algorithm provides an ξ-optimal solution for feasible quadratic programs, where ξ is the precision parameter. In order to solve the $0-1$-mixed quadratic program (5.5) using NEW-QP, we have written the binary value constraints on the u and v variables using the quadratic constraints $u_{ih} - u_{ih}^2 = 0$ and $v_{jl} - v_{jl}^2 = 0$. Because of the discrete values taken by these binary variables, we can be sure that the NEW-QP algorithm provides the optimal solution to the mixed $0-1$ quadratic program (5.5). However, the numerical noise which might appear makes it more difficult to conclude that an equilibrium is proper.

Corollary 5.6 *Let $(x_1(\sigma), x_2(\sigma), \epsilon(\sigma))$ be an optimal solution to (5.5) for some $\sigma > 0$. Then $(x_1(\sigma), x_2(\sigma))$ is an $\epsilon(\sigma)$-proper equilibria, and if $\sigma" > \sigma' > 0$, then $\epsilon(\sigma") \geq \epsilon(\sigma') \geq 0$.*

Proof. If $\sigma" > \sigma' > 0$, the $0-1$ mixed quadratic program (5.5) for $\sigma' > 0$ is a relaxation of $0-1$ mixed quadratic program (5.5) for $\sigma'' > 0$. In fact the only difference between these two programs is in the constraints of $\Omega_\epsilon^{\sigma'}$ and $\Omega_\epsilon^{\sigma''}$:

$$\begin{array}{ll} \sigma' \leq x_{1i}, & \forall i \in \{1,2,...,n\}, \\ \sigma' \leq x_{2j}, & \forall j \in \{1,2,...,m\}, \end{array} \text{ and } \begin{array}{ll} \sigma'' \leq x_{1i}, & \forall i \in \{1,2,...,n\}, \\ \sigma'' \leq x_{2j}, & \forall j \in \{1,2,...,m\}. \end{array}$$

$$\Rightarrow \begin{array}{ll} \sigma' < \sigma'' \leq x_{1i}, & \forall i \in \{1,2,...,n\}, \\ \sigma' < \sigma'' \leq x_{2j}, & \forall j \in \{1,2,...,m\}. \end{array}$$

Thus $\Omega_\epsilon^{\sigma''} \subseteq \Omega_\epsilon^{\sigma'}$ and $\epsilon(\sigma") \geq \epsilon(\sigma') \geq 0$. ∎

There are two possible outcomes when evaluating $f(\sigma)$ for some small values of σ. One possibility is that $f(\sigma)$ appears to converge to zero. Unfortunately, this is not enough to allow us to conclude that the equilibria is proper. However, for any regular equilibrium we can conclude that it is proper.

The other possibility is that $f(\sigma)$ appears to be bounded below by some strictly positive value, say $\bar{\epsilon}$. This would imply that there are no ϵ-proper equilibrium near $(\hat{x}_1, \hat{x}_2)$ for values of ϵ less than $\bar{\epsilon}$, and therefore, $(\hat{x}_1, \hat{x}_2)$ would not be proper.

In (5.5), let us suppose that $f(\sigma)$ converges to $\bar{\epsilon} > 0$, when $\sigma > 0$ converges to 0. We define a $0-1$ mixed quadratic program with the same conditions as Ω, with $\epsilon \leq \bar{\epsilon}/2$ and maximizing σ. If the optimal objective function of this program is equal to *zero* we can conclude that it would be impossible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to this equilibrium. Therefore the equilibrium is not proper.

Theorem 5.7 *If the optimal objective value of the following* $0-1$ *mixed quadratic program*

$$\begin{array}{lll} \max\limits_{(x_1,x_2)\in\Omega^{\sigma}_{\epsilon},\epsilon,\sigma} & \sigma & \\ s.t. & \hat{x}_{1i}-\epsilon \leq x_{1i} \leq \hat{x}_{1i}+\epsilon, & \forall i \in \{1,2,...,n\}\,, \\ & \hat{x}_{2j}-\epsilon \leq x_{2j} \leq \hat{x}_{2j}+\epsilon, & \forall j \in \{1,2,...,m\}\,, \\ & 0 \leq \epsilon \leq \bar{\epsilon}/2. & \end{array} \tag{5.6}$$

is zero for some $\bar{\epsilon} > 0$*, then the equilibrium* $(\hat{x}_1, \hat{x}_2)$ *is not proper.*

Proof. If the optimal objective value is equal to 0, it is impossible to find a sequence of $(x_1(\sigma), x_2(\sigma))$ $\epsilon(\sigma)$-proper converging to $(\hat{x}_1, \hat{x}_2)$. The equilibrium $(\hat{x}_1, \hat{x}_2)$ is not proper. ∎

With this result, automatic detection of non-proper extreme Nash equilibria can be carried out over any set of extreme Nash equilibria of a bimatrix game.

Example 5.8 *As mentioned by Myerson [17], the first extreme Nash equilibrium is the only proper equilibrium of this game. For the four other extreme Nash and perfect equilibria, the optimal values of* ϵ *seem to converge to* $\bar{\epsilon} = 0.618$*, as* σ *approaches* 0 *(Figure* 2*). This suggests that these equilibria are not proper. In order to be sure that these four extreme equilibria are not proper, we define a* $0-1$ *mixed quadratic program with the same conditions as in (5.6), with* $\epsilon \leq \bar{\epsilon}/2$ *and maximizing* σ*. Such a* $0-1$ *mixed quadratic program has an optimal objective equal to zero. The extreme equilibria* 2*,* 3*,* 4 *and* 5 *of this game are all not proper. For the third extreme Nash and perfect equilibrium, Figure* 2 *shows how* $f(\sigma) = \epsilon$ *decreases when* σ *decreases from* $min(\frac{1}{4}, \frac{1}{2}) = \frac{1}{4}$ *to* 0*.*

The set of extreme proper Nash equilibria defines the set of extreme points of all *Maximal Myerson sets* (Jansen [11]). There is only one maximal Myerson subset for the bimatrix game taken from Myerson [17].

Example 5.9 *For the fifth extreme Nash equilibrium found for the bimatrix game taken from Jansen [5], the minimum value of* ϵ *converges to* 0.2843*, in the corresponding* $0-1$*-mixed quadratic program (5.5), when* σ *converges to* 0*. Therefore,* ϵ *does not converge to* 0 *when* σ *converges to* 0*. This equilibrium not perfect and not proper.*

All of the eight perfect extreme Nash equilibria 1*,* 3*,* 4*,* 6*,* 7*,* 8 *and* 10*, of this bimatrix game are found to be* ϵ*-proper for a value of* ϵ *very close to zero. Table 3 shows the computational experiments results over the corresponding* $0-1$ *mixed quadratic programs (5.5) defined for each extreme Nash equilibrium of this game.*

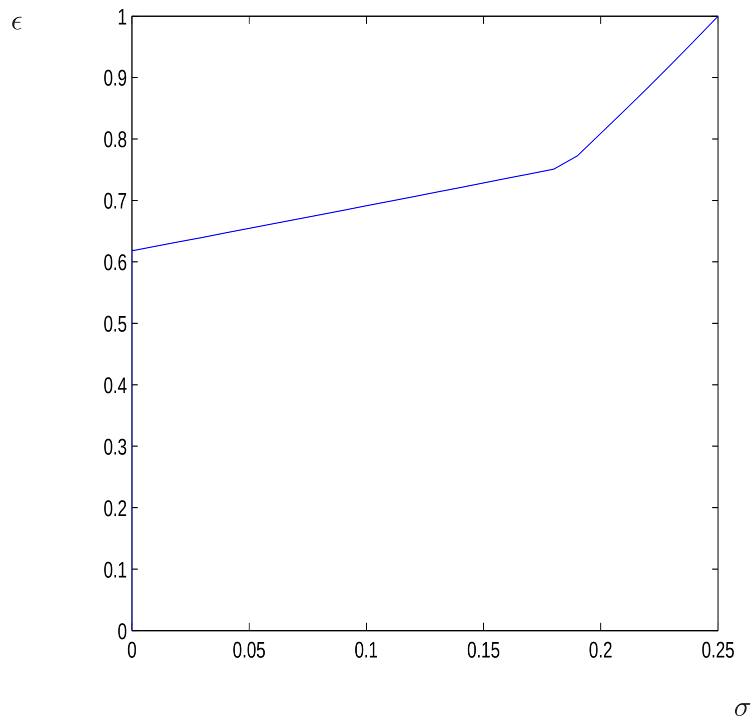

Figure 2: Plot of the minimal value that ϵ takes in function of σ – Proposition 5.5

Example 5.10 *The following* (5×5) *bimatrix game has* 7 *extreme Nash equilibria identified in Table 4.*

$$A = B = \begin{pmatrix} 2 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 8 & 1 \\ 2 & 5 & 6 & 0 & 1 \\ 0 & 2 & 5 & 4 & 7 \\ 2 & 3 & 6 & 5 & 7 \end{pmatrix}$$

This game has four maximal Nash subsets $T_1 = \{1, 2, 6\}$, $T_2 = \{3, 4\}$, $T_3 = \{5\}$ *and* $T_4 = \{7\}$. *Two regular and extreme Nash equilibria are found.*

Equilibrium 5

$C(x_1) = \{2\}$*and* $C(x_2) = \{4\}$, $M(A, x_2) = \{2\}$ *and* $M(x_1, B) = \{4\} \Rightarrow$ *quasi-strong. The determinant of* $\begin{pmatrix} 8 \end{pmatrix}$ *is equal to* $8 \neq 0 \Rightarrow$ *isolated. This equilibrium is regular, essential, perfect and proper.*

Table 3: Refinement Extreme Nash Equilibria for Jansen [5]

Eq.	Perfect	Proper	$\bar{\epsilon}$	$\bar{\sigma}$	Quasi-strong	Isolated	Regular
1	yes	yes	0.027	10^{-4}	no	no	no
2	no	no	0.8019	10^{-4}	no	no	no
3	yes	yes	0.014	10^{-4}	no	no	no
4	yes	yes	0.01	10^{-4}	no	no	no
5	no	no	0.2705	10^{-4}	no	no	no
6	yes	yes	0.052	10^{-3}	no	no	no
7	yes	yes	0.046	10^{-4}	no	no	no
8	yes	yes	0.056	10^{-3}	no	no	no
9	yes	yes	0.081	5×10^{-4}	no	no	no
10	yes	yes	0.0247	2×10^{-4}	no	no	no

Table 4: Extreme Nash Equilibria for (5×5) bimatrix game

Eq.	x_1					x_2					α_1	α_2
1	0	0	0	0	1	0	0	0	0	1	7	7
2	0	0	0	1	0	0	0	0	0	1	7	7
3	0	0	1	0	0	0	0	1	0	0	6	6
4	0	0	1/6	0	5/6	0	0	1	0	0	6	6
5	0	1	0	0	0	0	0	0	1	0	8	8
6	1	0	0	0	0	0	0	0	0	1	7	7
7	7/8	1/8	0	0	0	0	0	0	3/4	1/4	25/4	25/4

Table 5: Example (5×5)

Eq.	Perfect	Proper	$\bar{\epsilon}$	$\bar{\sigma}$	Quasi-strong	Isolated	Regular
1	yes	no	0.2894	5×10^{-3}	no	no	no
2	yes	no	0.7325	10^{-3}	no	no	no
3	yes	yes	0.05627	10^{-5}	no	no	no
4	yes	no	0.2000	10^{-6}	yes	no	no
5	yes	yes	0.0564	10^{-5}	yes	yes	yes
6	yes	yes	0.054	10^{-5}	no	no	no
7	yes	yes	0.0776	6×10^{-5}	yes	yes	yes

Equilibrium 7

$C(x_1) = \{1,2\}$*and* $C(x_2) = \{4,5\}$, $M(A,x_2) = \{1,2\}$ *and* $M(x_1,B) = \{4,5\} \Rightarrow$ *quasi-strong. The determinant of* $\begin{pmatrix} 6 & 7 \\ 8 & 1 \end{pmatrix}$ *is equal to* $-50 \neq 0 \Rightarrow$ *isolated. This equilibrium is regular, essential, perfect and proper.*

6 Conclusion

In this paper we presented a mathematical programming approach for the refinement of extreme Nash equilibria. After complete enumeration of all extreme Nash equilibria using the EχMIP algorithm [3], all extreme perfect Nash equilibria are detected using a pair of linear programs. Hence, ϵ-proper extreme Nash equilibria are found using the convergence numerical results of a $0-1$ mixed quadratic program. Regular refinement helps concluding on the properness of some of these equilibria. Finally, non-proper extreme Nash equilibria are found using the result of an other $0-1$ mixed quadratic program.

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