Start-up Prices and Shadow Prices for Resource Allocation Models with Indivisibilities

Suvrajeet Sen[a] & Talat S. Genc[b]

October 21, 2008

Abstract: In the presence of indivisible goods, resource allocation models often result in mixed-integer linear programs (MILP). Unlike linear programming duality however, MILP problems present duality gaps and dual variables (as part of the price system) are not unique and not as conveniently interpreted. These issues have been visited for almost fifty years starting with Gomory and Baumol (1960) and subsequently by a number of other authors. However, finding a price system in resource allocation models with indivisibilities that has attributes of shadow prices has remained a long-standing unresolved problem in economic theory. In this paper, we resolve this issue for binary MILP problems. We provide an important step in allocating charges of indivisible goods, and recover the total costs of inputs. Moreover, we characterize unique (two-sided) shadow prices for resources in binary MILP problems. We also provide an economic interpretation of implied constraints in the form of productivity requirements that must be satisfied for integer programming problems.

Key words: resource allocation; mixed-integer linear programming; shadow prices.

[a] Data-Driven Decisions Lab, Industrial and Systems Engineering, The Ohio State University, Columbus, OH, 43210, USA. Email: sen.22@osu.edu
[b] Department of Economics, University of Guelph, Guelph, Ontario, N1G 2W1, Canada. Email: tgenc@uoguelph.ca.

1. Introduction

In economic theory, cases of increasing returns to scale may be formulated by integer programming problems in which the price system, including pricing integral activities, runs into difficulties (see, for example, Gomory and Baumol (1960), Scarf (1990, 1994)). Pricing integral activities or discrete decisions (as well as continuous decisions) is vital for industries in which the scale of the operation and large fixed-costs play a key role. These discrete levels of activities may involve selling or buying a discrete quantity of goods or involve "yes/no" type (binary) decisions such as entry-exit decisions, shutting down or starting up some experimental technology, and investment on competing projects. One may model these choices using mixed-integer linear programming (MILP).

We study a class of MILP problems, specifically binary MILP problems, in which some decisions are "yes or no" type (i.e. assume values 1 or 0, respectively) and others are continuous valued, and the objective function and inequality constraints are linear. It is known that in general, MILP problems present duality gaps and dual variables (as part of the price system) are not unique and not as conveniently interpreted. These issues have been visited for almost fifty years starting with Gomory and Baumol (1960) and subsequently by a number of other authors; however, these issues remained unresolved. Yet, shadow prices continue to play a prominent role in economic theory and practice. For instance, starting in 2008, economists for the British government have agreed to a certain shadow price schedule for pricing carbon emissions in evaluating new environmental projects until the year 2050[1]. Without access to tools that provide justifiable estimates of shadow prices, the social value of such programs will remain questionable. Another practical instance where shadow prices for binary MILP can play a very important role is in forecasting efficient nodal electricity prices from unit-commitment models which are typically binary MILP problems. Shadow prices, also known as nodal prices, are calculated at the nodes (i.e., energy delivery points) on the transmission system by independent system operators (such as the IESO of Ontario, PJM Interconnection, NY ISO, and ERCOT). They reflect the values of generation constraints, and are published along with energy and operating reserve prices for every five minutes. Although, shadow prices are not used for payments (to either generators or distributors),

[1] See http://www.defra.gov.uk/environment/climatechange/research/carboncost/pdf/HowtouseSPC.pdf

they are utilized for calculation of generators' dispatchable quantities. They are also useful for forming and/or fine tuning of supply offer functions.[2]

This paper provides an important step in allocating charges of indivisible goods, and moreover, characterizes shadow prices for resources in binary MILP problems. Our approach is guided by two ideas:

a) Balas (1979) and Sherali and Adams (1990) have shown that for binary MILP problems, the convex hull of feasible points can be generated by using certain linear programs in which the contributions of each technology can be traced.

b) We propose a new measure of two-sided shadow prices for binary MILP problems in which the optimal objective value function is both non-convex and discontinuous. This measure also generalizes the two-sided shadow prices used in the context of linear programming (LP) (Gal (1997)).

For the case of binary MILP problems, we show that these two ideas lead to appropriate properties of "start-up" prices, as well as unique shadow prices for binary MILP problems. In the process of developing this framework, we also provide an interpretation of implied constraints in the form of productivity requirements that must be satisfied for integer programming problems.

The structure of this paper is as follows. In section 2, we summarize the long-standing questions and approaches that have appeared in the literature. Section 3 provides the setting of our analysis. As in O'Neill et al (2005), we explain the optimal cost of a primal MILP as the value of outputs plus the total "start-up" price of technologies. However, in contrast to the prices by O'Neill et al (2005), we identify conditions under which our prices are guaranteed to be non-negative. In order to help interpret the condition required by our result, section 3.1 presents an economic interpretation of cutting planes as goal setting mechanisms for MILP problems. In section 3.2, we present the conditions for non-negative start-up prices. We also demonstrate how one obtains "start-up" prices that are zero, in cases where a technology is not used. In section 4, we

[2] See, for instance, http://www.sygration.com/UtilizingShadowPrices.pdf, for the utilization of shadow prices in Ontario.

proceed to study the value function of MILP problems with a view towards obtaining accurate shadow prices. In order to do so, we define a new class of two-sided shadow prices. This section also suggests an LP-based methodology for calculating them. While there is a modest computational cost due to the solution of LPs, the shadow price estimates can be expected to be stronger in general. These prices also have the shadow price interpretations similar to those in classic linear programming. Section 5 concludes the paper.

2. The Literature

This section provides a brief review of the literature related to *economic* interpretations of dual multipliers for MILP. Gomory and Baumol (1960) were the first to explore the economic meaning of dual prices in an integer programming (IP) problem and studied the price system (or dual prices) associated with Gomory's method of solving integer programming problems (Gomory (1958)). They showed that the dual prices obtained in this manner possess *some* of the shadow price properties of linear programming. However, these prices are not unique and may assign zero prices to non-free goods. Moreover, the dual prices of the resources may not be equal to the marginal revenue of products.

Alcaly and Klevorick (1966) extend the approach of Gomory-Baumol (1960) so that the new methodology ensures that non-free goods do not take zero prices. It should be noted that while Gomory-Baumol prices in IP satisfy the property that total value of the output goods is equal to the total imputed value of the *used* capacities, the total value does not equal the total imputed value of all *original* capacities. Alcaly and Klevorick (1966) interpret the difference between total imputed value of all original capacities in LP and total imputed value of the used capacities in IP as "subsidy" that should be paid to the industry.

Kim and Cho (1988) argue that shadow prices in IP should not be based on marginal analysis, since marginal analysis is not reasonable in the presence of discontinuous decisions. Instead, they suggest a methodology which they refer to as the "average analysis" to calculate and give an economic meaning for shadow prices. Using this methodology they show the existence of (shadow) prices for IP, but these prices are non-unique.

Scarf (1990, 1994) review resource allocation decisions in the presence of indivisibilities and argue that when a competitive firm's production model involves indivisibilities the optimality conditions may not support competitive prices. The competitive prices verify optimality at the constant returns to scale production technology. He notes that LP problems and Walrasian models, which do not allow economies of scale, assume that production possibility sets exhibit constant or decreasing returns to scale. This assumption is often not realistic, because it denies the existence of firms that operate at a very large-scale. Scarf also emphasizes that IP models neither provide meaningful economic interpretations nor facilitate clear understanding of "prices" and/or "dual variables".

O'Neill et al. (2005) discuss how one might recover "multi-part" prices (involving a price for starting up, and a marginal price of the commodity) that support a Walrasian competitive equilibrium in a binary MILP problem that involves an auction-based market with significant fixed costs. These prices announced by a Walrasian auctioneer will clear the perfectly competitive market. To find these prices they solve the original binary MILP problem and then solve a linear program that is augmented with equality constraints that restrict the integral activities to their optimal integer values. They use the dual prices from this linear program as the prices supporting a competitive equilibrium. They illustrate their methodology with the example presented by Scarf (1994), in which the planner minimizes total cost of meeting inelastic electric load while allowing lumpy capacity investments in the electric generation sector. However, it is important to note that these dual prices are not unique since different sets of prices also support the values of these additional constraints.

The above review has focused on the literature dealing with economic interpretations of dual variables for MILP models. A related line of research is concerned with algorithmic tools that rely on identifying prices for integer programming problems. For instance, Wolsey (1981) shows how one might compute sub-additive price functions from branch-and-bound methods, as well as Gomory cuts. These price functions, while difficult to compute, can provide an optimal solution for the sub-additive dual of an integer program (see Nemhauser and Wolsey (1988)). For a survey of these results, we refer to Williams (1996) who surveys the duality in linear and integer programming, especially Lagrangian and surrogate duality. He notes that in general dual problems in

MILP do not satisfy mathematical and economic properties of the duals in LP. These properties involve equality of objectives, complementarity, and reflexivity conditions as the mathematical properties, and pricing imputations and sensitivity analysis as the economic properties of dual. He concludes that in general duality in MILP is an ill-defined concept. However, he admits some economic use of well defined duals for some types of MILP models; for example, dual variables of some of the constraints in the IP model may guide how to reallocate fixed costs of production and investment costs in the model. More recently, Klabjan (2006) not only provides an up-to-date review of duality in integer programming, but also discusses a new Benders'-type decomposition algorithm based on sub-additive duality. Incidentally, Caroe and Tind (1998) also discuss integer programming duality in the context of similar decomposition algorithms for stochastic mixed-integer programming (SMIP). For a survey of value function approximations for SMIP see Sen (2005). Fuller (2008), in a recent paper, also provides some algorithms for solutions of binary MILPs for competitive and imperfectly competitive game-theoretic settings.

3. Implied Constraints and Non-negative start-up prices for Binary MILP

As discussed above, integer programming problems may exhibit a duality gap when the prices are computed using LP or Lagrangian duals. While this duality gap can be closed by using sub-additive duality, prices associated with individual technologies are not always obvious. The approach used by O'Neill et al (2005) suggests that appropriate prices can be recovered by creating an LP in which the original constraints are retained, and in addition, the integer variables are constrained to the optimal values (obtained via a branch-and-bound method to solve the original binary MILP). The dual variables associated with the integrality constraints are then interpreted by O'Neill et al (2005) as the "start-up" price for the associated technology. Generally speaking however, the augmented LP has alternative dual optima, and given that the augmented constraints are equalities, they may even lead to the possibility of negative prices for "start-up." Indeed, the estimated prices for the outputs (commodity prices in the terminology of O'Neill et al (2005)) can therefore be lower than the actual shadow price for a commodity. In order to calculate shadow prices, we recommend studying the *value function* $v(b)$ defined by the optimal value of the following MILP problem (P)

$$
\text{(P)} \qquad \begin{aligned} v(b) = \min\, & c^T x \\ s.t. \quad & Ax \geq b \\ & x \in B^{n_1} \times \Re_+^{n_2} \end{aligned}
$$

where *x*, *b*, *c*, *A* may represent vector of input goods, vector of outputs, unit costs of inputs, and a technology matrix, respectively. The decision variables (activities/inputs) *x* are allowed to include binary (0-1) decisions. Above B^{n_1} denotes the set of binary lattice with n_1 denoting the number of indivisible inputs, and n_2 is the number of infinitely divisible inputs. We will use index sets J_1 and J_2 to denote the two types of variables. For example, *x* may be comprised of binary decisions which determine whether investments in certain technologies are to be made, and the continuous variables may denote inputs such as labor and capital. The above MILP implicitly assumes additively separable production function, $f: B^{n_1} \times \Re_+^{n_2} \to \Re_+^m$, where *m* is the number of output goods. In particular, we let $f(x) = Ax$, where *A* specifies technology matrix, which in turn specifies how to mix inputs for producing outputs. The constraint $Ax \geq b$ bounds the levels of outputs to *b*. The last constraint ensures nonnegative use of available inputs.

Next we focus on expressing problem (P) via a polyhedral approximation through which one can estimate non-negative "start-up" prices (to recover fixed costs of operation) for individual technologies modeled by the binary variables in MILP problems. Our approach relies on the characterization of binary MILP problems as facial disjunctive programs. For such problems, Balas (1979) has shown that the closure of the convex hull of binary feasible solutions can be generated sequentially by creating facets of sets that are defined recursively. Let $Z_0(b) = \{x : Ax \geq b, 0 \leq x_j \leq 1, j \in J_1, x_j \geq 0, j \in J_2\}$ and for $j = 1, \ldots, n_1$ define

$$
Z_j(b) = \text{clconv}\left[\left(Z_{j-1}(b) \cap \{x_j \leq 0\}\right) \cup \left(Z_{j-1}(b) \cap \{x_j \geq 1\}\right)\right]. \qquad (0)
$$

One of the important results of this theory is that $Z_{n_1}(b)$ provides the convex hull of feasible binary points of the binary MILP. Accordingly, the value *v*(*b*) can be explained by augmenting the inequalities of (P) with inequalities that describe $Z_{n_1}(b)$. In the integer programming literature, these inequalities are referred to as cutting planes (or simply cuts). We refer to these constraints as the *constraints implied by* indivisibility.

From a computational point of view, such constraints have become indispensable for large scale MILP, and disjunctive cuts have proved to be very effective for both deterministic MILP (see Balas et al (1993)) as well as stochastic MILP (Ntaimo and Sen (2005)).

Another important result from disjunctive programming is that the facets of any set $Z_j(b)$ have a one-one correspondence with extreme points of another polyhedron referred to as the reverse polar polyhedron, and may be denoted by $Z_j^{\#}(b)$. As shown by Balas (1979), each extreme point of $Z_j^{\#}(b)$ provides one facet inequality defining $Z_j(b)$. Hence the collection of extreme points define a system of inequalities of the form $\Pi_j(b)x \geq \Pi_{0j}(b)$. Any arbitrary inequality in this system of inequalities will be denoted $\pi_j(b)x \geq \pi_{0j}(b)$, and when the dependence on the index *j* is inconsequential, we will simply write $\pi(b)x \geq \pi_0(b)$.[3]

3.1. Interpretation of the Implied Constraints in Disjunctive Programming

The purpose of this sub-section is to provide an economic interpretation of implied inequalities in integer programming. We argue that they should be interpreted as productivity requirements, and in this sense, the unit cost of inputs should reflect the sum of a (per unit) imputed value of inputs and an imputed cost of productivity metrics associated with the indivisible technologies. In order to examine this interpretation, we will focus on implied inequalities resulting from disjunctive programming (Balas (1979), Sherali and Shetty (1980)), and the following discussion begins with the process of forming the inequalities.

Consider a set $S = \bigcup_{h \in H} S_h$, where $S_h = \{x : A_h x \geq b_h,\ x \geq 0\}$ is a convex polyhedron and $H = \{0,1\}$ is an index set. The set *S* is the union of polyhedra, hence it is typically non-convex. S_h is formed in the algorithmic process when one encounters non-integer points (that are supposed to be integers) in the MILP problem. For example, when a

[3] We explicitly indicate the dependence of the inequalities on the vector *b* so as to emphasize that the inequalities may change with changes in *b*. Nevertheless, see Proposition 3.

binary MILP problem is being solved, and one of the variables, say x_j, $j \in J_1$, is a fractional solution of the LP relaxation then we construct the conditions: $S_0 = \left\{x : Ax \ge b, 1 \ge x \ge 0,\ x_j \le 0\right\}$, $S_1 = \left\{x : Ax \ge b, 1 \ge x \ge 0,\ x_j \ge 1\right\}$, and $S = \bigcup_{h=0}^{1} S_h$. In what follows we interpret the formation of implied constraints in the form of productivity requirements.

Consider the union of polyhedra $S = \bigcup_{h \in H} S_h$. Let $\operatorname{clconv}(S)$ denote the closure of the convex hull of *S*. Choose $\bar{x} \notin \operatorname{clconv}(S)$.[4] Then by the disjunctive cut principle of disjunctive programming one can find the implied constraints coefficients (π, π_0) such that $\forall x \in S \quad \pi x \ge \pi_0$, and $\pi \bar{x} < \pi_0$. These coefficients satisfy the properties in (1.1-1.3), and an effective choice of implied constraints coefficients involves the solution of the following LP problem:

$$
\begin{aligned}
&\max\ \pi_0 - \pi\bar{x} && (1.0)\\
&s.t.\quad \pi_0 - b_h^T \lambda_h \le 0\ :(\alpha_h) \quad \forall h \in H && (1.1)\\
&\qquad -\pi + A_h \lambda_h \le 0\ :(y_h) \quad \forall h \in H && (1.2)\\
&\qquad \lambda_h \ge 0 && (1.3)\\
&\qquad \pi \le 1\ :(\delta^+) && (1.4)\\
&\qquad -\pi \le 1\ :(\delta^-) && (1.5)
\end{aligned}
$$

The above LP finds inequalities that separate infeasible points ($\bar{x}$) from those satisfying a subset of binary requirements. An interpretation of the above LP provides the pathway to the interpretation of “start-up” prices.

We look upon H as an index set of production strategies, and *h* is a specific strategy in *H*. For instance, the index *h* may indicate the adoption of a specific new technology (like ethanol), and b_h denotes a vector of output requirements under this strategy *h*. The vector λ_h denotes a vector of non-negative weights whose elements provide conversions to equivalent units of output. Hence $b_h^T \lambda_h$ provides the “effective” cumulative output associated with strategy *h*. Moreover, since λ_h provides conversions to

[4] $\bar{x}$ could be optimal fractional solution of LP relaxation of problem (P).

equivalent units of outputs for strategy *h*, it follows that they must be non-negative, and hence constraint (1.3) must be satisfied.

Suppose that we set a goal π_0 such that all feasible strategies (for a particular technology) provide effective outputs larger than π_0. Then we must have $\pi_0 \le b_h^T \lambda_h$, which is the constraint (1.1). Next consider the vector A_{hj} which represents column *j* of the technology matrix *A* for strategy *h*. If the units of elements of A_{hj} are (output units)/(input units) for technology *j* under strategy *h*, then for technology *j*, $\lambda_h^T A_{hj}$ specifies an effective productivity index. Since technology *j* is included in vectors A_{hj} for all *h*, the productivity metric of technology *j*, denoted π_j, must be at least as good as the effective productivity index under each strategy. Hence we obtain the constraint (1.2) $\pi_j \ge \lambda_h^T A_{hj}$ for all *j* and *h*.

Note that the interpretation of π_j as the productivity metric of technology *j* leads to the notion of effective output resulting from input x_j. Hence for any vector of inputs *x*, the *effective output* measured as a result of π is πx. We can now finally interpret the role of the objective function (1.0). Since $\bar{x}$ is infeasible, we seek to set a goal π_0 and a productivity metric π such that the effective output associated with $\bar{x}$ does not meet the goal set by π_0. In other words, $\pi\bar{x} < \pi_0$ must hold, and the objective function (1.0) maximizes the amount by which the effective output associated with $\bar{x}$ falls short of the goal π_0. Note that, by requiring π to be at least as high as the effective productivity index for each strategy *h*, one guarantees that $\pi x \ge \pi_0$ for all *x* that satisfies some strategy. Hence, the constraints added to convexify a non-convex resource allocation problem (such as MILP) imposes productivity goals π_0 and productivity metrics π that enforce the integer restrictions.

We should also interpret the dual to problem (1.0-1.5) which may be represented as follows:

$$
\begin{aligned}
\min\ & \textstyle\sum_j (\delta_j^+ + \delta_j^-) \\
s.t.\ & \textstyle\sum_h \alpha_h = 1 \\
& \delta^+ - \delta^- - \textstyle\sum_h y^h = -\bar{x} \\
& -\alpha_h b_h + A_h y^h \geq 0 \\
& \delta^+, \delta^-, \alpha_h, y^h \geq 0,\ \forall h \in H.
\end{aligned}
$$

In order to facilitate the interpretation, we recommend a reformulation using a change of variables. Assume $\forall h \quad \alpha_h \neq 0$, and let $z^h = y^h / \alpha_h$. If there exists a strategy h such that α_h is equal to zero, then we discard this strategy in the production planning process. By this change of variables we reformulate the dual problem as,

$$
\begin{aligned}
\min\ & \textstyle\sum_j (\delta_j^+ + \delta_j^-) \\
s.t.\ & \textstyle\sum_h \alpha_h = 1 \\
& -\delta^+ + \delta^- + \textstyle\sum_h \alpha_h z^h = \bar{x} \\
& A_h z^h \geq b_h \\
& \delta^+, \delta^-, \alpha_h, z^h \geq 0,\ \forall h \in H.
\end{aligned}
$$

Note that this dual is a generalized linear programming problem, which specifies finding feasible resource allocation points $z^h \in S_h$ such that infeasible allocation ($\bar{x}$) plus a deviation amount (i.e., $\bar{x} + (\delta^+ - \delta^-)$) can be written as a convex combination of the feasible points $z^h \in S_h$ for all h in H. The deviation vector $\delta = \delta^+ - \delta^-$ must be chosen so that it has the smallest 1-norm. Clearly, units of the deviation vector δ are the same as the units as an allocation z^h. The objective function corresponds to minimizing the level of error between a partially feasible solution (which is a vector of both discrete and continuous variables) and the infeasible solution (which is a vector of continuous variables). Also observe that the Lagrange multiplier of the constraints $A_h z^h \geq b_h$, denoted by μ^h, must be measured in units of inputs per unit of output, because $\sum_h \mu^h b_h$ must obtain the same units as the objective function (which in this case is units of inputs). Therefore, μ^h measures inputs necessary per unit of output.

3.2. Non-negative start-up prices for Binary MILP

We now proceed to use the valid inequalities discussed above to propose a class of start-up prices. Suppose now that problem (P) is replaced by its linear relaxation (i.e. replace the binary variables by continuous variables $0 \le x_j \le 1, \forall j \in J_1$), and this LP is augmented with implied constraints $\Pi_j(b)x \ge \Pi_{0j}(b) \quad \forall j \in J_1$. Since these constraints provide the convex hull of feasible points, the augmented LP provides a solution to (P). Consequently, standard LP duality implies that there exist non-negative multipliers $(y^*, \{\theta_j^*\}_{j\in J_1}, \sigma_1^*, \sigma_0^*)$ such that $v(b) = b^T y^* + \sum_{j\in J_1}(\Pi_{0j}^T(b)\theta_j^* - \sigma_{1j}^*)$, where the right hand side is the objective value of the dual to the *augmented* LP. Here y^* is the optimal dual vector for the original constraints $Ax \ge b$, θ_j^* denotes the vector of optimal dual multipliers of the implied constraints $\Pi_j(b)x \ge \Pi_{0j}(b)$, σ_1^* is the vector of optimal dual multipliers associated with the bounds $x \le 1$, and σ_0^* denotes the vector of multipliers associated with the lower bounds $x \ge 0$. Let E_0 (resp. E_1) denote a matrix that encodes the presence or absence of these dual variables σ_0 (resp. σ_1) associated with the lower (resp. upper) bounding constraints for each j in $J_1 \cup J_2$. In general, dual feasible multipliers will satisfy the following constraints

$$A^T y + \sum_{j\in J_1}\Pi_j^T\theta_j + E_0\sigma_0 - E_1\sigma_1 = c$$

where all multipliers are non-negative.

An interpretation of the variables θ_j is that θ_j denotes the price to be paid for setting goals Π_{0j} associated with technology *j*. Moreover, the dual feasibility condition should be looked upon as allocating the unit cost of an input according to a (per unit) imputed value of the inputs, together with a cost associated with productivity metrics embodied in the implied inequalities.

Definition 1: For each technology $j \in J_1$, we define the "start-up" price of technology as the expression $\Pi_{0j}^T(b)\theta_j^* - \sigma_{1j}^*$.

Note that as with O'Neill et al (2005), the total cost of the primal problem ($v(b)$) is explained in the dual to the *augmented* LP by the total value of the output ($b^T y^*$) and the sum of start-up prices denoted $\sum_{j \in J_1} (\Pi_{0j}^T(b)\theta_j^* - \sigma_{1j}^*)$.

We also note that ISOs in some wholesale electricity markets such as New York Independent System Operator and the Pennsylvania-New Jersey-Maryland Interconnection (PJM) ask generators to submit their marginal cost functions as supply offers along with their fixed costs associated with start-up and shut down costs (see O'Neill et al. (2005)). Indeed, the PJM Market Monitoring Unit even asks generators (for each plant and for each generating unit) to submit specific cost data such as actual fuel consumption quantities and fuel prices, and fuel types and qualities.[5] In that environment, it is important for ISOs to determine what the "fair prices" for integral activities are.

Proposition 1: Suppose that the LP approximation of the binary MILP is constructed using facets $\Pi_j(b)x \geq \Pi_{0j}(b)$ of $Z_j(b)$, $j \in J_1$, and assume that this approximation has a finite optimum. Let θ_j^* denote an optimal dual vector associated with the above implied inequalities, and assume that for all $j \in J_1$, we have $\Pi_{0j}^T(b)\theta_j^* \geq 0$. Then the start-up price for technology $j \in J_1$ must be non-negative.

Proof: By assumption, we have $\Pi_{0j}^T(b)\theta_j^* \geq 0$. Now suppose that there is an index *j* for which the start-up price satisfies $\Pi_{0j}^T(b)\theta_j^* - \sigma_{1j}^* < 0$. For this index then, we must have $\sigma_{1j}^* > 0$. Consequently, we can choose $\varepsilon > 0$ such that $\hat{\sigma}_{1j} = \sigma_{1j}^* - \varepsilon > 0$. Now let $\hat{\sigma}_{0j} = \sigma_{0j}^* + \varepsilon$. We can now define vectors $\hat{\sigma}_1$ and $\hat{\sigma}_0$ whose j^{th} elements are as defined above, and the remaining elements are the same as in σ_1^* and σ_0^* (satisfying dual optimality). It follows that the vector $(y^*, \{\theta_j^*\}_{j \in J_1}, \hat{\sigma}_0, \hat{\sigma}_1)$ satisfies the dual feasibility constraints. However, the dual objective value associated with this choice of multipliers

[5] See http://www.pjm.com/markets/marketmonitor/downloads/20020614_fuel_cost_data_request.pdf

yields the value $b^T y^* + \sum_{j\in J_1} (\Pi_{0j}^T(b)\theta_j^* - \hat{\sigma}_{1j}) = v(b) + \varepsilon$, which contradicts the dual optimality of $(y^*, \{\theta_j^*\}_{j\in J_1}, \sigma_1^*, \sigma_0^*)$. Hence the result. QED

We should note that the condition that $\Pi_{0j}^T(b)\theta_j^* \geq 0$ is the requirement that the total value (reflected by the price vector θ_j^*) of goals set by Π_{0j} must be non-negative in order to obtain non-negative start-up prices.

It is important to discuss the implication of the assumptions of this proposition. At first sight, computing the facets of $Z_j(b)$ may seem like a daunting task, but several authors have solved MILPs using such facets (Balas et al (1993), Ntaimo and Sen (2005), and Yuan and Sen (2007)). As for the assumption $\theta_{jk}^* > 0 \Rightarrow \pi_{0jk}(b) \geq 0$, we observe that this condition can be satisfied under the hypotheses of the following theorem, and these conditions are natural in economic models.

Theorem 1: Let $x^* = (x_1^*, x_2^*)$ where $x_1^* \in B^{n_1}$ and $x_2^* \in \Re_+^{n_2}$ denote an optimum solution and let $\Pi_j = (\Pi_{1j}, \Pi_{2j})$ $\forall j \in J_1$ denote the cuts in the LP representation solving the MILP. Suppose that these cuts satisfy $\Pi_{1j}x_1^* \geq 0$ and $\Pi_{2j}x_2^* \geq 0$ $\forall j \in J_1$. Then there exists multipliers $\{\theta_j^*\}_{j\varepsilon J_1}$ such that $(\theta_j^*)^T \Pi_{0j} \geq 0$ $\forall j \in J_1$, and consequently, such start-up prices are non-negative.

Proof: See the Appendix.

We should comment about the assumptions $\Pi_{1j}x_1^* \geq 0$ and $\Pi_{2j}x_2^* \geq 0$. As discussed in Section 3.1, $\Pi_j x$ measures the "effective output" of input vector *x*. Thus what we conclude is that we can ensure non-negative start-up prices when the effective commodity output $\Pi_{2j}x_2^*$ is non-negative. As for the quantity $\Pi_{1j}x_1^*$, it is usually necessary to use positive amounts of capacity for production, thus implying that ordinarily, A_1 is non-negative (where A_1 is a partition matrix, $A = [A_1, A_2]$, as defined in the proof of the theorem). When such a condition is satisfied, (1.2) implies that $\Pi_{1j} \geq 0$, which of course implies that $\Pi_{1j} x_1^* \geq 0$. Also note that if the multipliers y^* and $\{\theta_j^*\}_{j \varepsilon J_1}$ are calculated after the primal solution x^* has been obtained (using branch-and-bound, say), then the cut formation process can use $x^* = (x_1^*, x_2^*)$ in the convexification process, and the conditions of Theorem 1 can be imposed via linear inequality constraints in a cut generation LP of the form given in Section 3.1.

From an economic point of view, the presence of negative start-up prices has been suggested as a way to identify technologies that are not efficient. However, we caution that such an interpretation may be misleading in those cases where there exist start-up prices which are non-negative. The presence of alternative dual optima may lead to the existence of some negative start-up prices, although one may still deploy such technology in an efficient way, leading to non-negative start-up prices.

We should also comment on the possibility of obtaining a positive "start-up" price for any binary variable that has a value of zero at optimality. First note that such a binary variable can be *eliminated* from the problem, without any loss of optimality. In doing so, one may also need to *eliminate* tight cuts that may have caused the "start-up" price to be positive. By eliminating these cuts, and the corresponding variable, one arrives at an alternative dual optimum which will provide "start-up" prices that are zero for variables whose optimal values are zero. It is these prices that are most appropriate for economic models.

4. Shadow Prices for Binary MILP

In the previous section, we have suggested a methodology for computing start-up prices with reasonable properties. However, allocation decisions are also guided by shadow prices which reflect the change in the optimal objective value associated with infinitesimal changes in the level of output b_i. As a result, it is necessary to investigate the manner in which $v(b)$ changes with respect to perturbations in b_i. Previously suggested methods are not well suited to characterizing the variation of the function $v(b)$, and as a result, shadow price estimates may be weak.

It is well known that $v(b)$ is non-decreasing and lower semi-continuous on its effective domain (see Nemhauser and Wolsey (1988)). The difficulty with defining appropriate shadow prices for a binary MILP stems from the fact that $v(b)$ can be both non-convex and discontinuous. So we begin this discussion by proposing a new measure of the shadow price for the binary MILP. Let a particular right hand side vector $\bar{b}$ be given, and define

$$v_i^+(\bar{b};\varepsilon) = \begin{cases} +\infty & \textit{if} \lim_{\tau\to 0^+} \dfrac{v(\bar{b}+\tau\, e_i)-\varepsilon-v(\bar{b})}{\tau} \textit{ does not exist} \\ \lim_{\tau\to 0^+} \dfrac{v(\bar{b}+\tau\, e_i)-\varepsilon-v(\bar{b})}{\tau} & \textit{otherwise} \end{cases}$$

Here e_i denotes a vector whose i^{th} component is +1 and other components are 0. Also, here ε represents amount of jump in the value function at point $\bar{b}$. It can be interpreted as a fixed opportunity cost of operation. Similarly, we define $v_i^-(\bar{b};\varepsilon)$ by replacing the vector e_i by $-e_i$ in the above definition. Then we define a one-sided shadow price in direction e_i and $-e_i$ as

$$\rho_i^+(\bar{b}) = \inf_{\varepsilon\ge 0} v_i^+(\bar{b};\varepsilon) \text{ and } \rho_i^-(\bar{b}) = \inf_{\varepsilon\ge 0} v_i^-(\bar{b};\varepsilon)\,, \tag{2}$$

respectively. One could borrow more general definitions of directional derivatives from non-smooth optimization (e.g. Clarke (1983)). However we prefer to work with more standard quotients and limits so as to focus our attention on the discontinuity of $v(b)$. We illustrate the shadow prices of (2), as well as our notation in Figure 1.

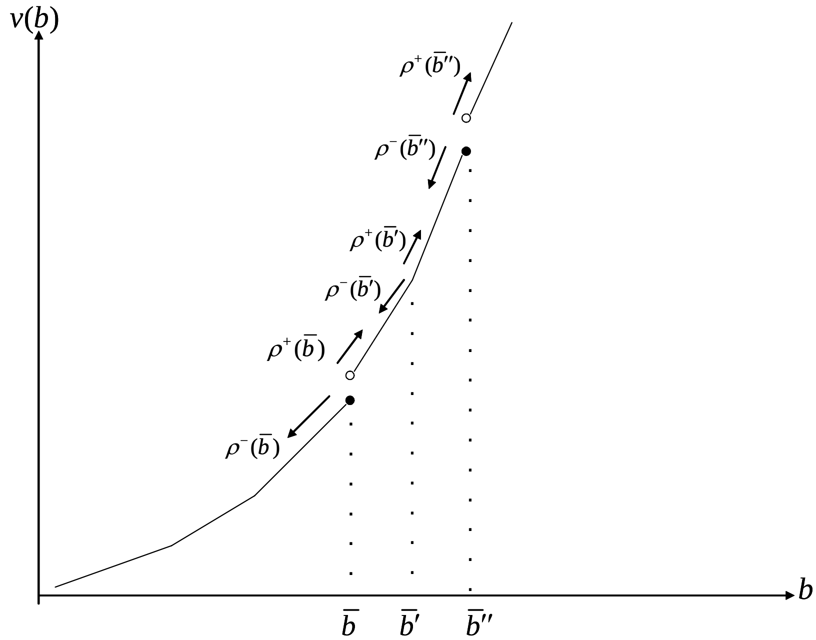

Figure 1: Two-sided prices as directional derivatives of a MILP value function.

The use of two-sided shadow prices was first discussed in the context of LP shadow prices when an optimal basic feasible solution of a LP is degenerate (see Gal (1997)). In essence, degeneracy leads to alternative dual optima, which in turn, suggests that the shadow prices be defined using directional derivatives of the LP value function, which is convex. More generally, if $v(b)$ possesses directional derivatives (as in convex programming), then the infima above are attained for $\varepsilon = 0$, and hence the shadow prices defined here subsume the convex programming setting. For IP and MILP value functions, we will, in general, need to allow $\rho_i^+(\bar{b})$ to be defined using $\varepsilon \neq 0$. However, we show below that the infimum in the definition of $\rho_i^-(\bar{b})$ is attained for $\varepsilon = 0$.

Proposition 2: Assuming that the optimal value of the MILP for a given right hand side $\bar{b}$ is finite, the value of the infimum defining $\rho_i^-(\bar{b})$ is attained for $\varepsilon = 0$.

Proof: Because $v(b)$ is lower semi-continuous, $\liminf_{b^k \to b} v(b^k) \geq v(b)$. With $b = \bar{b}$, consider $b^k = \bar{b} - \tau^k e_i$ where $\tau^k \to 0^+$. Since v is non-decreasing, $v(\bar{b}) \geq v(b^k)$. Hence $\liminf_{b^k \to \bar{b}} v(b^k) \geq v(\bar{b}) \geq \limsup_{b^k \to \bar{b}} v(b^k)$ proving that $\varepsilon = 0$ for $\rho_i^-(\bar{b})$. QED

For a given vector $\bar{b}$, let $\bar{\Pi}_j x \geq \bar{\Pi}_{0j}$, $j \in J_1$ denote the collection of implied inequalities used to solve the MILP. Having obtained $v(\bar{b})$ we now wish to calculate the shadow prices. Define

$$D(\bar{b}) = \left\{ (y, \{\theta_j\}_{j \in J_1}, \sigma_1, \sigma_0) \geq (0,0,0,0) \,|\, \left(\bar{b}\right)^T y + \sum_{j \in J_1} \left(\bar{\Pi}_{0j}^T \theta_j - \sigma_{1j} \right) = v(\bar{b}), A^T y + \sum_{j \in J_1} \bar{\Pi}_j^T \theta_j + E_0 \sigma_0 - E_1 \sigma_1 = c \right\}$$

where one may assume that the necessary extreme points of $Z_j^{\#}(b)$ are represented in the equations defining $D(\bar{b})$. This set provides the set of all dual vectors associated with expressing c as a non-negative combination of inequalities defining the convexification of $X(\bar{b})$, the set of (binary) mixed-integer points.

In $D(\bar{b})$ the first equality (dual optimality) indicates how the total cost of inputs is allocated among the value of the outputs created and the total start-up prices. The second equality (dual feasibility) in $D(\bar{b})$ suggests an allocation scheme for the unit cost of input. Following the standard calculus of non-smooth functions (e.g. Clarke (1983)), the left shadow price can then be calculated by solving the following linear program:

$$\rho_i^-(\bar{b}) = Max \left\{ -y_i \,|\, \left(y, \{\theta_j\}_{j \in J_1}, \sigma_1, \sigma_0 \right) \in D(\bar{b}) \right\}.$$

The above objective function represents the directional derivative of $v(\bar{b})$ in the direction $-e_i$.

There are several observations to make about this LP. First, the number of columns defining the LP can be extremely large, and one should not list all of them before solving the LP. Second, it is customary to use a column generation approach (see, for example, Lasdon (1970), Martin (1998)) to generate only those columns that are necessary. While all columns (inequalities in the primal) need not be generated, one may need more than the columns used to solve the original binary MILP.

As for calculating the right shadow price $\rho_i^+(\bar{b})$, we first check whether the optimal integer solution, denoted $x(\bar{b})$ remains feasible for some $\tau > 0$, when the i^{th} output is increased to $\bar{b}_i + \tau$. This is true if the i^{th} row of the original constraints satisfies $a_i^T x(\bar{b}) > \bar{b}_i$. In this case we calculate

$$\rho_i^+(\bar{b}) = Max\left\{ y_i \mid \left(y, \left\{ \theta_j \right\}_{j \in J_1} \sigma_1, \sigma_0 \right) \in D(\bar{b}) \right\}.$$

4.1 Approximation of Shadow Prices when $\varepsilon > 0$

In the above development, we verify whether $\varepsilon = 0$ is legitimate for calculating $\rho_i^+(\bar{b})$ for the case in which the constraint is non-binding. On the other hand, if $a_i^T x(\bar{b}) = \bar{b}_i$ then $x(\bar{b})$ is not feasible for any $\tau > 0,$ and we now solve a different LP whose solution will provide us an *approximation* of the right shadow price $\rho_i^+(\bar{b}_i)$.

Proposition 3: Let $\overline{\Pi}_j x \geq \overline{\Pi}_{0j}$, $j \in J_1$ denote any collection of implied constraints (including the disjunctive constraints of Balas (1979)) for a given output vector $\bar{b}$. Consider the perturbed output vector $\bar{b} + \tau e_i$, with $\tau > 0.$ The implied inequalities $\overline{\Pi}_j x \geq \overline{\Pi}_{0j}$, $j \in J_1$ remain valid (i.e. do not delete mixed-integer feasible points) for the perturbed right hand side for all $\tau > 0.$

Proof: If the perturbation leads to an infeasible problem, then the implied inequalities cannot delete any feasible point. On the other hand, if the resulting problem is feasible, then the truth of the proposition is obvious because the perturbed output vector $\bar{b} + \tau e_i$ yields a mixed-integer feasible set that is a subset of $X(\bar{b})$. QED

The above proposition thus allows us to use the inequalities $\overline{\Pi}_j x \geq \overline{\Pi}_{0j}$ even when $\bar{b}$ is replaced by $\bar{b} + \tau e_i$. It also justifies the illustration in Figure 1 where the slopes (when they exist) increase as any right-hand side element is increased. However, the estimate of objective value cannot be improved without further refinement of the primal polyhedron. We accomplish this by generating another inequality that deletes the point $x(\bar{b})$ without eliminating any of the integer feasible points associated with $\bar{b} + \tau e_i$, for $\tau > 0.$

Proposition 4: Let *i* denote a row index used in the above development. Suppose that b_i is infinitesimally larger than $\bar{b}_i$. With the exception of the point $x(\bar{b})$, all mixed-integer solutions that satisfy $a_i x \geq b_i$ must also satisfy

$$\sum_{j\in I^+(\bar{b})} a_{ij}x_j - \sum_{j\in I^-(\bar{b})} a_{ij}x_j - \sum_{j\in N^+(\bar{b})} a_{ij}x_j + \sum_{j\in N^-(\bar{b})} a_{ij}x_j \leq \sum_{j\in I^+(\bar{b})} a_{ij} - \sum_{j\in I^-(\bar{b})} a_{ij} - \zeta \qquad (3)$$

where $\zeta > 0$ is defined in the Appendix (and so are the sets used for defining (3)).

Proof: See the Appendix.

Proposition 4 helps improve the objective value by adding (3) to the primal LP, hence we can obtain the directional derivative in the direction +e_i. Denoting (3) as the constraint $\gamma x \leq \gamma_0$, we obtain the following dual feasible set which is a strengthened version of $D(\bar{b})$, and this new set is denoted by $D^+(\bar{b})$

$$D^+(\bar{b}) = \left\{ (y, \{\theta_j\}_{j\in J_1}, \sigma_1, \sigma_0, \vartheta) \geq (0,0,0,0,0) \,|\, \left(\bar{b}\right)^T y + \sum_{j\in J_1}\left(\Pi_{0j}^T\theta_j - \sigma_{1j}\right) - \vartheta\gamma_0 \geq v(\bar{b}), \right.$$
$$\left. A^T y + \sum_{j\in J_1} \Pi_j^T \theta_j + E_0\sigma_0 - E_1\sigma_1 - \gamma^T\vartheta = c \right\}.$$

Note that the above set requires the dual objective function to be at least $v(\bar{b})$. We now approximate the right shadow price by solving the following LP which yields a lower bound; that is,

$$\rho_i^+(\bar{b}) \geq Max\left\{ y_i \,|\, \left(y, \{\theta_j\}_{j\in J_1} \sigma_1, \sigma_0, \vartheta\right) \in D^+(\bar{b}) \right\}$$

A lower bound on the value of ε is also obtained as a by-product of solving the above LP. Letting $\left(\bar{y}, \{\bar{\theta}_j\}_{j\in J_1}, \bar{\sigma}_{1j}, \bar{\sigma}_{0j}\right)$ denote the optimal values obtained from the above LP, the right hand side below gives a lower bound on ε, that is

$$\varepsilon \geq \bar{\varepsilon} = \bar{b}^T\bar{y} + \sum_{j\in J_1}(\Pi_{0j}^T\bar{\theta}_j - \bar{\sigma}_{1j}) - \bar{\vartheta}\gamma_0 - v(\bar{b}).$$

Thus, for an arbitrarily small increase in output (i.e. changing output $\bar{b}_i$ to a value $\bar{b}_i + \tau$), one expects the optimal cost to increase by at least $\bar{\varepsilon} + \bar{y}_i\tau$. One obtains the correct value of ε by solving a primal MILP in which the original problem is augmented with constraint (3) (in addition to all the other implied inequalities), *provided there are no alternative mixed-integer optima that produce the same output* $\bar{b}_i$. However if the latter (*italicized*) situation is encountered, then one may require the solution of a series of binary MILP problems until a non-zero value of ε is revealed. Each MILP in this series includes one additional constraint of the form (2) to eliminate one of the alternative optima which also produce the output $\bar{b}_i$. Thereafter, the corresponding shadow price $\rho_i^+(\bar{b})$ may be calculated.

In general, these shadow price estimates can be expected to be stronger than those obtained via previously suggested methods because we focus on estimates derived from approximating directional "derivatives" of the value function. Previous approaches have typically restricted computations to those that are obtained directly from LPs used in branch-and-bound (see O'Neill et al. (2005)). Note that the shadow prices defined here are obtained by using the value function $v(b)$, and the calculation is defined using a scalar optimization problem.

Proposition 5: For any instance of the problem (P), suppose that the tight valid inequalities are facets of $Z_{n_1}(b)$. Then the shadow prices $\rho_i^+(b)$ and $\rho_i^-(b)$ are unique.

Proof: First observe that the facets of $Z_{n_1}(b)$ define the convex hull of mixed-integer points, and are independent of any algorithmic choices (e.g. the sequence in which inequalities are generated during the solution scheme). Moreover, the values $\rho_i^+(b)$ and $\rho_i^-(b)$ are generated using scalar optimization problems, whose value must be unique. It follows that the shadow prices must be unique. QED

Since the facets of $Z_{n_1}(\bar{b})$ are generated through the sequence of sets $Z_j(\bar{b})$, $j \in J_1$, the LP that generates columns for the directional derivative LP (defining $\rho_i^-(\bar{b})$, or $\rho_i^+(\bar{b})$) may require the solution of $O(n_1)$ cut generation linear programs (see Balas et al (1993) or Sherali and Shetty (1980)) to determine whether all necessary columns have been generated.

5. Conclusions

Shadow prices are important tools for economists and managers in creating market mechanisms for resource allocation problems. Finding a price system in resource allocation models with indivisibilities that has attributes of shadow prices has remained long-standing unresolved problem in economic theory. In this paper, we resolve this issue for binary MILP problems. We characterize a new class of two-sided shadow prices, and develop an LP-based methodology for calculating them. These prices have the shadow price interpretations as in classic linear programming. We also provide economic interpretation of the process of forming implied constraints due to indivisibility restrictions.

As for future research, there are several directions that one might pursue. Perhaps, the most pressing need on the computational side requires studying methods that will generate the implied inequalities (and hence the start-up and shadow prices) after an optimal solution to the binary MILP is obtained using some other algorithm (e.g. branch-and-bound). On the economic side of the research agenda, one might consider studying models that provide estimates of shadow prices for carbon constraints, electricity generation or others. Finally, in the area of computational economics, one might consider studying extensions of our approach to general MILP problems (not simply binary), as well as generalizations required for linear-quadratic type problems (commonly studied in economics) involving integral activities.

APPENDIX: Proofs of Theorem 1 and Proposition 4.

<u>*Proof of Theorem 1*:</u> : Let the optimal solution x^* be written as $x^* = (x_1^*, x_2^*)$ where $x_1^* \in B^{n_1}$ and $x_2^* \in \Re_+^{n_2}$. Suppose we solve the restricted problem

$$\begin{aligned} &\min c_2^{\mathrm{T}} x_2 \\ &s.t. \quad A_2 x_2 \geq b - A_1 x_1^*, \ \ x_2 \geq 0 \end{aligned}$$

where $c^{\mathrm{T}} = [c_1^{\mathrm{T}}, c_2^{\mathrm{T}}]$ and $A = [A_1, A_2]$. Then x_2^* is an optimal solution to this restricted problem. Then there exists a dual optimum y^* such that

$$c_2^{\mathrm{T}} x_2^* = (y^*)^{\mathrm{T}} (b - A_1 x_1^*) \,. \tag{A.0}$$

Next consider the following LP,

$$\max \sum_{j\in J_1} (\Pi_{0j}^{\mathrm{T}}\theta_j - \sigma_{1j}) \qquad \text{(A.1a)}$$

$$s.t. \;\; \sum_{j\in J_1} \Pi_{1j}^{\mathrm{T}}\theta_j - E_1\sigma_1 + E_0\sigma_0 = c_1 - A_1^{\mathrm{T}} y^* \qquad \text{(A.1b)}$$

$$\sum_{j\in J_1} \Pi_{2j}^{\mathrm{T}}\theta_j \le 0 \qquad \text{(A.1c)}$$

$$\theta_j \ge 0,\; \sigma_{1j} \ge 0,\; \sigma_{0j} \ge 0,\; \forall j \in J_1 \qquad \text{(A.1d)}$$

Let $(\{\theta_j^*\}_{j\in J_1}, \sigma_1^*, \sigma_0^*)$ denote an optimal solution to the above LP. Note that (x_1^*, x_2^*) is dual feasible to this LP, and since $x_2^* \ge 0$, (A.1c) implies that

$$\sum_{j\in J_1} (\Pi_{2j}^{\mathrm{T}}\theta_j^*)^{\mathrm{T}} x_2^* \le 0 \qquad \text{(A.2)}$$

By assumption $\Pi_{2j} x_2^* \ge 0,\; \forall j \in J_1$. Hence (A.2) implies that

$$\sum_{j\in J_1} (\Pi_{2j}^{\mathrm{T}}\theta_j^*)^{\mathrm{T}} x_2^* = 0 \qquad \text{(A.3).}$$

It follows that (x_1^*, x_2^*) satisfies complementary slackness for the restricted dual LP (A.1) and therefore

$$(c_1 - A_1^{\mathrm{T}} y^*)^{\mathrm{T}} x_1^* = \sum_{j\in J_1} (\Pi_{0j}^{\mathrm{T}}\theta_j^* - \sigma_{1j}^*) \qquad \text{(A.4)}$$

Combining (A.0) and (A.4) we have

$$c_1^{\mathrm{T}} x_1^* + c_2^{\mathrm{T}} x_2^* = b^{\mathrm{T}} y^* + \sum_{j\in J_1} [\Pi_{0j}^{\mathrm{T}}\theta_j^* - \sigma_{1j}^*].$$

It follows that $(y^*, \{\theta_j^*\}_{j\in J_1}, \sigma_1^*, \sigma_0^*)$ is optimal for the dual to the cut enhanced LP. Moreover (A.3), and the fact that $\Pi_{2j} x_2^* \ge 0$, imply that $(\Pi_{2j} x_2^*)^{\mathrm{T}} \theta_j = 0,\; \forall j \in J_1$. Consequently, complementary slackness of the cut enhanced LP implies that $(\theta_j^*)^T \Pi_{0j} = (\theta_j^*)^T \Pi_{1j} x_1^* \ge 0,\; \forall j \in J_1$. This proves the result. QED

It is interesting to recognize that the method of proof is constructive and is intimately tied to decomposition methods normally used in stochastic programming.

Proof of Proposition 4: Consider a row index i as defined in Proposition 4. Let $I(\bar{b}) = \{j \in J_1 \mid x_j(\bar{b}) = 1\}$ and $N(\bar{b}) = J_1 \setminus I(\bar{b})$. Next define subsets $I^+(\bar{b}) = \{j \in I(\bar{b}) \mid a_{ij} > 0\}$ and $I^-(\bar{b}) = \{j \in I(\bar{b}) \mid a_{ij} < 0\}$. Similarly, define $N^+(\bar{b}) = \{j \in N(\bar{b}) \mid a_{ij} > 0\}$ and $N^-(\bar{b}) = \{j \in N(\bar{b}) \mid a_{ij} < 0\}$. Since at least one of the variables in these subsets must change values in a solution (x) that satisfies $a_i^T x(\bar{b}) \geq \bar{b}_i + \tau$, $\tau > 0$, it follows that *at least one* of the following inequalities must hold:

$$\sum_{j \in I^+(\bar{b})} a_{ij} x_j \leq \sum_{j \in I^+(\bar{b})} a_{ij} - \underset{j \in I^+(\bar{b})}{Min}\, a_{ij} \tag{A.5a}$$

$$\sum_{j \in I^-(\bar{b})} a_{ij} x_j \geq \sum_{j \in I^-(\bar{b})} a_{ij} - \underset{j \in I^-(\bar{b})}{Max}\, a_{ij} \tag{A.5b}$$

$$\sum_{j \in N^+(\bar{b})} a_{ij} x_j \geq \underset{j \in N^+(\bar{b})}{Min}\, a_{ij} \tag{A.5c}$$

$$\sum_{j \in N^-(\bar{b})} a_{ij} x_j \leq \underset{j \in N^-(\bar{b})}{Max}\, a_{ij} \tag{A.5d}$$

Combining these inequalities, we require the following.

$$\sum_{j \in I^+(\bar{b})} a_{ij} x_j - \sum_{j \in I^-(\bar{b})} a_{ij} x_j - \sum_{j \in N^+(\bar{b})} a_{ij} x_j + \sum_{j \in N^-(\bar{b})} a_{ij} x_j \leq \sum_{j \in I^+(\bar{b})} a_{ij} - \sum_{j \in I^-(\bar{b})} a_{ij} - \zeta \tag{A.6}$$

where,

$$\zeta = Min\left\{ \underset{j \in I^+(\bar{b})}{Min}\, a_{ij},\ \underset{j \in N^-(\bar{b})}{Min} - a_{ij},\ \underset{j \in I^-(\bar{b})}{Min} - a_{ij},\ \underset{j \in N^+(\bar{b})}{Min}\, a_{ij} \right\}$$

From the definition of $I^+(\bar{b}), I^-(\bar{b}), N^+(\bar{b}), N^-(\bar{b})$, we have $\zeta > 0$. Moreover, for $x = x(\bar{b})$ the expression $\sum_{j \in I^+(\bar{b})} a_{ij} x_j - \sum_{j \in I^-(\bar{b})} a_{ij} x_j - \sum_{j \in N^+(\bar{b})} a_{ij} x_j + \sum_{j \in N^-(\bar{b})} a_{ij} x_j$ attains the value $\sum_{j \in I^+(\bar{b})} a_{ij} - \sum_{j \in I^-(\bar{b})} a_{ij}$. Hence the point $x = x(\bar{b})$ violates (A.6). As for other mixed-integer solutions of (A.2), note that at least one of the four inequalities (A.5a-A.5d) must hold. Hence the expression $\sum_{j \in I^+(\bar{b})} a_{ij} x_j - \sum_{j \in I^-(\bar{b})} a_{ij} x_j - \sum_{j \in N^+(\bar{b})} a_{ij} x_j + \sum_{j \in N^-(\bar{b})} a_{ij} x_j$, which attains the value of $\sum_{j \in I^+(\bar{b})} a_{ij} - \sum_{j \in I^-(\bar{b})} a_{ij}$ at the point $x(\bar{b})$ must reduce by at least the smallest non-zero change among the four inequalities (A.5a-A.5d). Since ζ measures this change, the result follows. QED

Acknowledgments: We thank Simge Kucukyavuz for her comments on an earlier version of this paper. We also thank Binyuan Chen for his work on computational aspects which has provided us some insights into the results reported in this paper. The first author gratefully acknowledges research support from the Air Force Office of Scientific

Research (MURI grant F49620-03-1-0447 and Discrete Mathematics and Optimization Program grant FA9620-03-1-0477). The second author gratefully acknowledges the research support of the Social Sciences and Humanities Research Council of Canada.

REFERENCES

Alcaly, R.E., A.K. Klevorick. 1966. A note on the dual prices of integer programs. *Econometrica* **34** 206-214.

Balas, E. 1979. Disjunctive programming. *Annals of Discrete Mathematics* **5** 3-51.

Balas, E., S. Ceria, G. Cornuéjols. 1993. A lift- and project-cutting plane algorithm for mixed 0–1 programs. *Mathematical Programming* **58** 295–324.

Caroe, C.C., J. Tind. 1998. L-shaped decomposition of two-stage stochastic programs with integer recourse. *Mathematical Programming* **83** 451-464.

Clarke, F.H. 1983. *Optimization and Non-smooth Analysis,* Wiley-Interscience Publications, New York.

Crema, A.C. 1995. Average shadow price in a mixed integer linear programming problem. *European Journal of Operational Research* **85** 625-635.

Fuller, D. 2008. Market equilibrium models with continuous and binary variables. Working paper, University of Waterloo.

Gal, T. 1997. Sensitivity analysis and parametric programming under degeneracy. Appeared in *Advances in sensitivity analysis and parametric programming* edited by Gal and Greenberg. Kluwer Law International, Hague-London-Boston.

Geoffrion, A.M. 1974. Lagrangian relaxation for integer programming. *Mathematical Programming Studies* **2** 82-114.

Gomory, R.E. 1958. Outline of an algorithm for integer solutions to linear programs. *Bulletin of the American Mathematical Society* **64**.

Gomory, R.E., W.J. Baumol. 1960. Integer programming and pricing. *Econometrica* **28** 521-550.

Kim, S., S. Cho. 1988. A shadow price in integer programming for management decision. *European Journal of Operational Research* **37** 328-335.

Klabjan, D. 2006. Subadditive approaches in integer programming, to appear in *European Journal of Operational Research*

Lasdon, L.S. 1970. *Optimization Theory for Large Systems*. Dover Publications, New York.

Martin, R.K. 1998. *Large Scale Linear and Integer Optimization: A Unified Approach*. Springer.

Nemhauser, G., L. Wolsey. 1988. *Integer and Combinatorial Optimization*. Wiley-Interscience Publication, New York, Toronto.

Ntaimo, L., S. Sen. 2005. The Million Variable "March" for Stochastic Combinatorial Optimization. *Journal of Global Optimization* **32**.

O'Neill, R., P. Sotkiewicz, B. Hobbs, M. Rothkoph, W. Stewart. 2005. Efficient market-clearing prices in markets with nonconvexities. *European Journal of Operational Research* **164** 269-285.

Scarf, H.E. 1990. Mathematical programming and economic theory. *Operations Research* **38** 377-385.

Scarf, H.E. 1994. The allocation of resources in the presence of indivisibilities. *Journal of Economic Perspectives* **8** 111-128.

Sen, S. 2005. Algorithms for Stochastic Mixed-Integer Programming Models. *Handbook of Discrete Optimization*, (K. Aardal, G.L. Nemhauser, and R. Weismantel eds.), North-Holland Publishing Co., pp. 515-558.

Sherali, H. D., W.P. Adams. 1990. A Hierarchy of Relaxations Between the Continuous and Convex Hull Representations for Zero-One Programming Problems, *SIAM Journal on Discrete Mathematics*, Vol. 3, No. 3, pp. 411-430,

Sherali, H.D., C.M. Shetty 1980. *Optimization with Disjunctive Constraints*, Lecture Notes in Economics and Math. Systems, Vol. 181, Springer-Verlag, Berlin.

Williams, H.P. 1996. Duality in mathematics and linear and integer programming. *Journal of Optimization Theory and Applications* **90** 257-278.

Wolsey, L.A. 1981. Integer programming duality: price functions and sensitivity analysis. *Mathematical Programming* **20** 173-195.

Yuan, Y., S. Sen. 2007. Enhanced cut generation methods for decomposition-based branch-and-cut algorithms for two-stage stochastic mixed-integer programs. accepted for publication in *INFORMS Journal on Computing*.