Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Resource and Energy Economics 29 (2007) 17–39
RESOURCE and ENERGY ECONOMICS
www.elsevier.com/locate/ree

Sustainability of an economy with an exhaustible resource: A viable control approach

V. Martinet [a,*], L. Doyen [b]

[a] *EconomiX (UMR 7166 CNRS-UPX), Université Paris X-Nanterre, 200 Avenue de la République, 92001 Nanterre Cedex, France*
[b] *CERSP (UMR 5173 CNRS-MNHN), Muséum National d'Histoire Naturelle, 55 rue Buffon, 75005 Paris, France*

Received 25 November 2003; received in revised form 23 September 2005; accepted 7 March 2006

Abstract

We examine the conditions for the sustainability of a production–consumption system based on the use of an exhaustible natural resource. Instead of studying the environmental and economic interactions in terms of optimal control, we focus on the viability of the system, defined by a set of constraints combining guaranteed consumption and a stock of resources to be preserved at all times, which refers to a Rawlsian intergenerational equity perspective. Using the mathematical concept of viability kernel, which makes it possible to deal with the consistency between constraints and controlled dynamics, we exhibit the sustainable technological configurations and, whenever possible, the policy options and environmental-economic states required to obtain a perennial system. We point out the flexibility of the sustainable "extraction–consumption" choices and we show how they are neither reduced to constant consumption paths nor to Hartwick's rule. Numerical simulations illustrate the general results.

JEL Classification: Q01; Q32; O13; C61

Keywords: Exhaustible resource; Sustainability; Viability kernel

---

1. Introduction

A basic issue in sustainable development is the reconciliation of environmental and economic requirements with an intergenerational equity perspective. In the present paper, we are interested in the economic interpretation of sustainability and more specifically in the viable inter-temporal

---

* Corresponding author. Tel.: +33 2 98 22 44 80; fax: +33 2 98 22 47 76.
E-mail addresses: vincent.martinet@u-paris10.fr (V. Martinet), lucdoyen@mnhn.fr (L. Doyen).

doi:10.1016/j.reseneeco.2006.03.003

use of exhaustible resources. First, let us recall the main economic models and statements referring to this issue, as in Heal (1998). The modeling on this topic is often derived from the classic "cake-eating" economy first studied by Hotelling (1931). Such a model deals with the optimal allocation over an infinite horizon of an exhaustible resource considered as the only good of the economy. The problem is to maximize the sum of the discounted utilities derived from the consumption of this resource, i.e.

$$\max_{c(\cdot)} \int_0^{\infty} U(c(t))\,\mathrm{e}^{-\delta t}\mathrm{d}t$$
$$\text{s.t.}\, \dot{S}(t) = -c(t), \quad S(t) \geq 0,$$

where $S(t)$ is the resource stock, $c(t)$ the consumption and δ is the discount rate. Under reasonable assumptions on the utility function U, it turns out[1] that both the optimal consumption and the resource stock decline towards 0. In this model, the utility depends only on consumption. However, it can be expanded into a stronger sustainability context by considering the resource stock as an argument of the utility function, as in Heal (1998). The problem then becomes

$$\max_{c(\cdot)} \int_0^{\infty} U(c(t), S(t))\,\mathrm{e}^{-\delta t}\mathrm{d}t,$$

under the same constraints as before. In this case, if the marginal utility of consumption is infinite at zero, the resource stock is totally depleted. Otherwise, it is optimal to preserve a positive resource stock which depends both on the utility function and the discount rate.[2]

Nevertheless, in these two models, the optimal consumption $c(t)$ declines towards zero. No strictly positive consumption level can be guaranteed for any time. One way to avoid such a situation, where the only "guaranteed" consumption is zero, is to assume that capital accumulation and/or technical change can compensate for the diminution in resource extraction. Important contributions in this area are Dasgupta and Heal (1974, 1979), Hartwick (1977), Solow (1974) and Stiglitz (1974). In particular, Dasgupta and Heal (1974) present an economy where the social good is produced by a technology $f(K, r)$ using the resource r and a "human-made" reproducible capital K. The optimal dynamic control model then becomes

$$\max_{c(\cdot), r(\cdot)} \int_0^{\infty} \mathrm{e}^{-\delta t} U(c(t))\mathrm{d}t$$
$$\text{s.t.}\, \dot{K}(t) = f(K(t), r(t)) - c(t), \quad \dot{S}(t) = -r(t).$$

For a so-called "essential" resource,[3] an optimal path with a positive consumption exists if the discount rate δ is high enough.[4] In this case, the declining use of the resource is compensated for by capital accumulations. Nevertheless, if the discount rate is positive and the marginal productivity of capital is decreasing, the optimal path displays a decreasing consumption after a finite time period. Stiglitz (1974) presents a similar approach and introduces exogenous exponential technical progress in the production function. He determines the conditions for this technical progress to make a sustainable optimal growth possible.

---

[1] For instance, if the elasticity of substitution of marginal utility $\eta(c) = -cU''(c)/U'(c) > 0$ remains constant, the consumption decreases exponentially at the rate $-\delta/\eta$, namely $c(t) = c_0\,\mathrm{e}^{-(\delta/\eta)t}$.

[2] Such a sustainable stock S_U satisfies the relation $\delta U_c(0, S_U) = U_S(0, S_U)$.

[3] Namely if $\lim_{r \to 0} f_r(K, r) = \infty$.

[4] Namely if $\delta > \rho(1 - \eta)$ with $\rho = \lim_{x \to \infty} f(x)/(x)$ where $x \equiv K/r$.

These studies use a discounted utilitarian approach and, with no technological progress, the consumption decreases because of the discount factor.[5] Other criteria can be proposed to cope with the sustainability issues. For instance, Solow (1974) examines the same model using the maximin or Rawls' criterion (Rawls, 1971). The criterion is then to maximize the utility of the poorest generation, in the following sense:

$$\max_{\text{set of feasible allocations}} \{\min_t U(c(t))\}.$$

It turns out that the solution is the greatest constant consumption possible for all generations. With Cobb–Douglas technology, a sustainable positive level of consumption exists if the elasticity of output with respect to capital is greater than that of the resource used, and if there is no capital depreciation. Such a constant sustained consumption path requires the investment of the Hotelling resource rents, which is the Hartwick investment rule (Hartwick, 1977).

The discounted utilitarian criterion is often considered as a ‘‘dictatorship of the present’’ because the distant future is not properly taken into account. In the same way, the ‘‘green golden rule’’ criterion,[6] introduced by Chichilnisky et al. (1995) and discussed in Heal (1998), leads to preserve the whole resource stock with a zero consumption path. It thus considers only the distant future, neglects the present needs and is qualified of ‘‘dictatorship of the future’’. Chichilnisky (1996) has developed a criterion that makes it possible to avoid both dictatorships of the present and the future.[7] With this criterion, the consumption decreases towards zero and a positive resource stock is preserved.[8]

The sustainability issues are also addressed in the overlapping generations (OLG) framework. It is an interesting way of coping with intergenerational equity and altruism questions. OLG is also a relevant approach to show how the distribution of resource rights affects the sustainability issue. For instance, an interesting alternative to the ‘‘Rawlsian’’ approaches to equity is provided by Burton (1993) who considers two different discount factors: a personal discount factor that represents impatience, and an inter-generational discount factor. This work offers the analysis of the optimal allocation of an exhaustible resource among overlapping generations, without capital and for a finite time horizon only. Moreover, in this OLG approach, Gerlagh and Keyser (2003) show that conservationist policies can be Pareto efficient, and that strict resource conservation is equivalent to non-dictatorship of the present, as defined by Chichilnisky (1996).[9] Howarth and

---

[5] Heal (1998) describes how the discount factor impacts the inter-temporal optimal consumption. Constant discount rate jeopardizes future consumption. Other discount factors can be used. For example, decreasing discount rates (hyperbolic discounting) lead to higher consumption for future generations, but the consumption still decreases throughout time.

[6] It consists in maximizing the utility at an infinite time i.e. $\max_{c(\cdot)} \lim_{t\to\infty} U(c(t), S(t))$.

[7] The criterion is

$$W = \theta \int_0^\infty \Delta(t) U(c(t), S(t)) \mathrm{d}t + (1-\theta) \lim_{t\to\infty} U(c(t), S(t)),$$

where $\Delta(t)$ is the discount factor that can be exponentially decreasing or hyperbolic.

[8] We can compare the optimal resource stocks preserved under the four criteria. The ‘‘green golden rule’’ is the most conservative one, maintaining the whole initial stock $S_0 = S_{\text{GGR}}$. Chichilnisky's criterion leads to positive consumption and preserves more resources, S_{CHI}, than S_U, guaranteed by the utilitarian criterion involving the resource stock. The classical Utilitarian criterion does not preserve any part of the stock. To summarize, the order reads $S_{\text{GGR}} > S_{\text{CHI}} > S_U > 0$.

[9] They argue that ‘‘strict conservationist policies that impose explicit exploitation constraints ensure sustainability, and are far simpler to implement, compared to the more complex resource management rules that aim at a careful balancing of costs and benefits’’ (Gerlagh and Keyser, 2003, p. 312).

Norgaard (1995) also examine the sustainability issue in the overlapping generations framework. More specifically, Howarth (1991a) considers an exhaustible resource with production and investment, and Howarth (1991b) discusses the problem of technological change under uncertainty. These authors have written subsequent papers on the issue, but mainly concerning a renewable resource or pollution framework (see Toman et al., 1995, for a survey).

To summarize, most of the works dealing both with sustainability issues and exhaustible resources, including OLG, discounted, maximin and green golden rule criteria, rely on an optimality approach. Such an approach can be criticized because (a) the results are highly dependent on the criteria considered and (b) the optimal decisions generally follow a unique path.

The viability approach (Aubin, 1991) or weak invariance (Clarke et al., 1995) and viable control framework are another way of exploring sustainability problems. Such an approach focuses on inter-temporal feasible paths. It first requires the identification of a set of constraints that represents the "good health" or the safety and, by extension, the effectiveness of the system. Then, conditions which allow these constraints to be fulfilled at any time, including both present and future, are studied. So, by focusing specifically on the constraints associated with the controlled dynamic systems, the controversial choice of a criterion is avoided. We refer for instance to Bene et al. (2001); Bonneuil (1994); Doyen and Bene (2003) for some similar analyses in other contexts. The tolerable windows approach (Bruckner et al., 1999; Schellnhuber and Wenzel, 1998) proposes a similar framework on climatic change issues. In the environmental context, viability may allow for the satisfaction of both economic and environmental constraints in a multi-criteria perspective. Moreover, since the viability constraints are the same at any moment of an infinite time horizon, intergenerational equity is naturally integrated within this framework.

In this paper, the question of the intergenerational allocation of an exhaustible resource involved in the production process is examined with the viable control approach. Our purpose is to determine feasible economic paths, including states and control variables, which satisfy a set of constraints characterizing the sustainability of the system. Here, the constraints mainly include a guaranteed consumption level together with a guaranteed level of an exhaustible resource. We use the mathematical concept of viability kernel to characterize the sustainability of the system. This kernel is the set of initial resources and capital levels from which it is possible to define acceptable regimes of exploitation and consumption paths satisfying all of the constraints throughout time. Therefore, the viability kernel refers to some *ex post* viability features. It provides the "true" constraints of the system in the sense that if an economic state is out this kernel, there are no decisions that make it possible to satisfy the constraints throughout time. In particular, if the viability kernel is the empty set, there is no economic state that makes it possible to satisfy these constraints. More specifically, by using the viability kernel, we address the following questions:

- What are the technology and production configurations which allow for sustainable consumption and extraction paths? In particular, do economies based on non-essential resources display better sustainability features?
- Given a sustainable technology, what are the initial resource conditions for which such sustainable policies exist?
- What are the possible controls associated with these sustainable states? Is there uniqueness? In particular, are the viable choices reduced to constant consumption levels and the Hartwick investment rule?

The paper is organized as follows. In Section 2, we describe the dynamics of the system and we identify the viability constraints. We define the sustainability of the economy through the viability kernel. Section 3 proposes a taxonomy of sustainable and unsustainable economies depending on the technology. In particular, we detail the Cobb-Douglas production function case. The main result is given in Proposition 3. Section 4 provides some numerical simulations and we conclude on the interest and limits of the method in Section 5. To restrict the mathematical content in the core of the text, proofs of the formal results are exposed in Appendix A. A short mathematical guide entitled "viable control approach at a glimpse" is also there provided.

2. The model

2.1. The dynamics

Following Dasgupta and Heal (1974) and Solow (1974), the economy is subject to dynamics:

$$\dot{S}(t) = -r(t), \qquad \dot{K}(t) = f(K(t), r(t)) - c(t) - \lambda K, \tag{1}$$

where $S(t)$ is the exhaustible resource stock, $r(t)$ stands for the extraction flow, $K(t)$ represents the accumulated capital, $c(t)$ stands for the consumption and function f corresponds to the technology of the economy. Parameter λ is the rate of capital depreciation which can vanish. The decisions or controls of this economy are the levels of consumption $c(t)$ and extraction $r(t)$.

2.2. The sustainability constraints

As mentioned above, the viability approach is based on a set of constraints. First, the extraction $r(t)$ is assumed to be irreversible in the sense that

$$0 \leq r(t). \tag{2}$$

Taking the resource scarcity into account reads

$$0 \leq S(t).$$

More generally, we consider a stronger conservation constraint for the resource as follows:

$$S_\flat \leq S(t), \tag{3}$$

where $S_\flat \geq 0$ stands for some guaranteed resource target, referring to a strong sustainability concern whenever it has a strictly positive value.

Consumption does not exceed the production level:

$$0 \leq f(K(t), r(t)) - c(t). \tag{4}$$

In other words, investment in reproducible capital is assumed to be irreversible, which ensures the growth of capital if there is no depreciation. Furthermore, the capital is non-negative:

$$0 \leq K(t). \tag{5}$$

The most important requirement imposed is related to some guaranteed consumption level $c_\flat$ throughout time:

$$0 < c_\flat \leq c(t). \tag{6}$$

This constraint refers to a sustainability and intergenerational equity concern since it can be enounced equivalently[10] in utilitarian terms in a form close to that of the maximin criterion, namely $U(c(t)) \geq U_\flat$.

2.3. The viability kernel as an indicator of sustainability

A question that arises now is whether the dynamics of the resource and capital (1) are compatible and consistent with all of the constraints (2)–(6) at every period $t \geq 0$. In particular, one objective is to identify the initial states (S_0, K_0) that are associated with decisions $(c(\cdot), r(\cdot))$ and trajectories $(S(\cdot), K(\cdot))$ satisfying all the conditions for any time $t \in \mathbf{R}^+$. The set of such initial states is called the viability kernel associated with the dynamics and the constraints (see Appendix A.1 for a more general description of the viability kernel and possible extensions). Here, since it depends on both the production function f and the basic needs $c_\flat$ together with the resource minimum standard $S_\flat$, we denote this kernel by $\text{Viab}(f, c_\flat, S_\flat)$:

$$\begin{aligned}\text{Viab}(f, c_\flat, S_\flat) = \{&(S_0, K_0) | \text{there exists decisions } (c(\cdot), r(\cdot)) \text{ and states } (S(\cdot), K(\cdot)) \\ &\text{starting from } (S_0, K_0) \text{ satisfying conditions } (1)-(6) \text{ for any time } t \in \mathbf{R}^+\}.\end{aligned}$$

In the general mathematical framework, this set can alternatively be empty, the whole state constraint set, or even a strict part of the initial state constraint domain.

The viability kernel captures an irreversibility mechanism. Indeed, from the very definition of this kernel, every state lying outside the viability kernel violates the constraints in finite time, no matter what the decisions applied. This situation means that crisis is unavoidable. For instance, the extreme case where the viability kernel is empty corresponds to a hopeless configuration.[11] We will consider that economies with empty viability kernels are not sustainable, in the sense that no strictly positive consumption can be guaranteed along with a resource preservation constraint.

3. A taxonomy of sustainable and unsustainable economies

In this section, we analyze the sustainability of the economy with respect to different technological structures. To achieve this analysis, we compute the viability kernel $\text{Viab}(f, c_\flat, S_\flat)$ for several function f. The Cobb–Douglas case is particularly scrutinized.

3.1. Assumptions and notations

The production function f is obviously assumed to increase with both its arguments, capital K and extraction r.

We introduce the extraction indicator denoted by $r_\flat(f, K)$ and defined by

$$r_\flat(f, K) = \inf\,(r \geq 0 | f(K, r) \geq c_\flat), \tag{7}$$

---

[10] Assuming an increasing utility function $U(\cdot)$.

[11] As suggested by an anonymous referee, an empty kernel allows us to eliminate "red herrings", objective functions that could never have a feasible solution. An empty kernel means that there are no sustainable paths for the described economy, i.e. no paths that respect the set of constraints representing sustainability. It is thus necessary to "modify" the problem to be able to define some sustainable decisions or states. Appendix A.1 proposes several strategies in such non-viable situations.

along with the minimal extraction indicator $r_\flat(f)$ defined by

$$r_\flat(f) = \inf_{K \geq 0} r_\flat(f, K). \tag{8}$$

The value of this minimal extraction indicator $r_\flat(f)$ plays an important role for the sustainability concern. Some illustrative cases are examined below. However, we will see that the most interesting Cobb–Douglas case requires the introduction of further indicators.

3.2. A non-sustainable economy: a "strongly" essential resource

We first consider an economy where the resource is strongly essential to the production in the sense that $r_\flat(f) > 0$. Such is the case for a production function like $f(K, r) = \min(aK, br)$, as $r_\flat(f) = c_\flat/b > 0$ for $K \geq c_\flat/a$. It represents a "complementary" point of view (Daly, 1990) where capital and natural resources are complements rather than substitutes.

Not surprisingly, it turns out that this situation of a strongly essential resource is not sustainable with an infinite time horizon. For any initial state, the quantity of resource extracted at each time is strictly positive since $r(t) \geq r_\flat(f) > 0$; consequently the stock $S(t)$ is depleted in a finite time and the production stops. Thus, the viability kernel is empty and, in this sense, the economy is not sustainable. The formal proof of this result stated in the proposition below is given in Appendix A.

Proposition 1. *If the resource is strongly essential for the technology f in the sense that* $r_\flat(f) > 0$ *then the economy is unsustainable, i.e.* $\mathrm{Viab}(f, c_\flat, S_\flat) = \emptyset$.

3.3. A sustainable economy: a non-essential resource

We now pay attention to a "non-essential" resource where $r_\flat(f) = 0 = r_\flat(f, K^+)$: a stock of capital K^+ makes it possible to produce $c_\flat$ using no natural resource. Such is the case, for example, if the production function has an additive form, i.e. $f(K, r) = aK + br$ where $r_\flat(f) = 0 = r_\flat(f, c_\flat/a)$. If $K \geq K^+$, we can produce the quantity $c_\flat$ without using any part of the resource stock. So, any state such that $K \geq K^+$ and $S \geq S_\flat$ is viable and, therefore, the viability kernel is not empty. We then obtain the following proposition which is proven in Appendix A.

Proposition 2. *If the resource is non-essential for the technology f in the sense that there is a capital level* K^+ *with* $r_\flat(f) = 0 = r_\flat(f, K^+)$ *then the economy is sustainable in the sense that the viability kernel is not empty or* $\mathrm{Viab}(f, c_\flat, S_\flat) \neq \emptyset$.

3.4. The Cobb–Douglas case: a "weakly" essential resource

We now focus on "essential resources" in the sense of Dasgupta and Heal (1974), namely resources needed in production but with an unbounded potential production. For such a purpose, we use the Cobb–Douglas production function. This case is interesting because an imperfect substitution effect occurs between the natural resource and capital, and an infinite capital accumulation can offset decreasing use of the resource.

3.4.1. No capital depreciation, no technical progress

We consider the usual case of a Cobb–Douglas production function $f(K, r) = K^\alpha r^\beta$. In this case, the technology f is such that $r_\flat(f) = 0 = \lim_{K \to \infty} r_\flat(f, K)$. The following proposition characterizes the viability kernel.

Proposition 3. *Consider a Cobb–Douglas technology* $f(K,r) = K^{\alpha} r^{\beta}$ *with* $\beta < 1$. *Then the viability kernel depends on parameters* α *and* β *as follows*:

$$\mathrm{Viab}(f, c_\flat, S_\flat) = \begin{cases} \varnothing & \text{if } \alpha \leq \beta, \\ \{(S,K) \text{ such that } S \geq V(K, c_\flat, S_\flat)\}, & \text{if } \alpha > \beta. \end{cases}$$

where V is a function defined by

$$V(K, c_\flat, S_\flat) = \frac{1}{\alpha - \beta}\left(\frac{c_\flat}{1-\beta}\right)^{(1-\beta)/\beta} K^{(\beta-\alpha)/\beta} + S_\flat. \tag{9}$$

Results of Proposition 3 present interesting features with respect to irreversibility, flexibility, robustness and equity issues.

3.4.1.1. Substitutability. Proposition 3 claims that the economy is sustainable if and only if resource elasticity β is smaller than capital elasticity α. In that case, capital accumulation makes it possible to compensate for the decline in the extraction in the production system. Otherwise, the viability kernel is empty and no feasible inter-temporal path exists. This situation is reminiscent of Solow (1974) results on the substitutability issues.

3.4.1.2. Crisis and irreversibility. Proposition 3 also claims that whenever capital elasticity α is strong enough ($\alpha > \beta$), a sustainability condition linking the resource and capital levels is required to achieve the viability of the system. Resource stock S has to be larger than a threshold $V(K, c_\flat, S_\flat)$ depending on capital stock K. From the very definition of the viability kernel, if the initial state lies outside this viability kernel, or if the current state leaves it, then viability constraints are violated in a finite time whatever the admissible decisions applied. In the present case, this means that, whenever the resource stock $S(t)$ declines below the viability threshold $V(K(t), c_\flat, S_\flat)$, then, for any consumption $c(\cdot)$ and extraction $r(\cdot)$ applied in the future, either the basic need constraint or the guaranteed resource stock constraint will not be satisfied in some future:

$$\text{If } S(t) < V(K(t), c_\flat, S_\flat) \text{ then } \exists\, T > t \text{ such that } S(T) < S_\flat \text{ or } c(T) < c_\flat.$$

In this case, a crisis cannot be avoided and some future generations will not meet their needs. In fact, no generation after T will fulfill all of the constraints in an irreversible way. In other words, the complementary of the viability kernel represents the situations where the economy is no longer sustainable. In this sense, the viability kernel represents the "true" sustainability constraints.

3.4.1.3. Flexibility. Whenever they are possible, relevant policies consist in maintaining the state within the viability kernel, avoiding the crisis mentioned above. Within the interior of the viability kernel, there is no difficulty since every control (a priori admissible) is relevant. More specifically, for any state (K,S) within the interior of the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$, the appropriate viable controls belong to the feasible set:

$$\mathcal{C}(K,S) = \{(r,c)\,|c_\flat \leq c \leq f(K,r),\ r_\flat(f,K) \leq r\}. \tag{10}$$

Consequently, in the interior of the viability kernel, the sustainable paths are not reduced to constant consumption paths, and the extraction does not necessarily satisfy the Hartwick rule, or the dynamic efficiency rule of Hotelling. However, on the boundary of the viability kernel $S = V(K, c_\flat, S_\flat)$, a specific path must be followed. As explained in Appendix A.1, given a viable current state $(K, S) \in \mathrm{Viab}(f, c_\flat, S_\flat)$, relevant viable controls (r, c) ensure that the velocities $(\dot{K}, \dot{S})$ are tangent or inward to the viability kernel. Indeed, it turns out that viable decisions are reduced to

$$r^*(K) = \left(\frac{c_\flat}{1-\beta}\right)^{1/\beta} K^{-(\alpha/\beta)}, \qquad c^*(K) = c_\flat. \tag{11}$$

The consumption then remains constant and the Hartwick investment rule holds true.[12] This path exhibits dynamic efficiency.

There is not, therefore, uniqueness of the sustainable controls in the viability kernel except on its boundary $S = V(K, c_\flat, S_\flat)$. The viability approach does not provide a unique policy but rather the set of all viable policies. At this stage, let us point out that the multiplicity of sustainable controls allows for distinct viable strategies. They include policies based on resource conservation goals (examined below in the paragraph on strong sustainability) or optimal strategies such as maximin (explored below in the paragraph related to equity) or the discounted utility approach. Some choices are also illustrated by the various simulations of Section 4.

Note that if the discounted approach is adopted, the result can be associated with Chichilnisky's approach. According to Gerlagh and Keyser (2003), strict resource conservation induces Chichilnisky's non-dictatorship of the present. Then, maximizing the discounted sum of utility within the viability kernel should lead to a similar solution as Chichilnisky's criterion.[13] At least, the non-emptiness of a viability kernel for some constraint on the resource stock means that there is a solution to Chichilnisky's criterion.

At this stage, two general remarks are in order. First, previous comments indicate that the weak invariance framework makes it possible to study sub-optimal economies (paths that are not dynamically efficient) while still facing the sustainability issue. Second, an interesting issue is to determine, given current conditions, how to place the economic path on the boundary of some viability kernel to get a unique efficient path. This last point is developed below, by considering equity and preservation issues.

3.4.1.4. Sensitivity and robustness. If we examine the sensitivity of the viability kernel with respect to constraints $c_\flat$ and $S_\flat$ together with parameters α and β, we obviously deduce from the definition of the boundary $V(K, c_\flat, S_\flat)$ that:

- Sustainability increases with the elasticity of production with respect to capital: The greater α is, the greater the viability kernel is

  $$\alpha_1 < \alpha_2 \Rightarrow \mathrm{Viab}(f^1, c_\flat, S_\flat) \subset \mathrm{Viab}(f^2, c_\flat, S_\flat),$$

  where f^i means the technology $f^i(K, r) = K^{\alpha_i} r^\beta$. This sensitivity statement is illustrated in Fig. 1.
- Sustainability decreases with the guaranteed consumption: The size of the viability kernel diminishes when $c_\flat$ increases, i.e.

  $$c_\flat^1 > c_\flat^2 \Rightarrow \mathrm{Viab}(f, c_\flat^1, S_\flat) \subset \mathrm{Viab}(f, c_\flat^2, S_\flat). \tag{12}$$

---

[12] We have $r = (c/1-\beta)^{1/\beta} K^{-\alpha/\beta}$ or equivalently $c = (1-\beta) K^\alpha r^\beta$ which is also the Hartwick investment rule in the Cobb–Douglas case where $\dot{K} = \beta K^\alpha r^\beta = r f_r'$.

[13] As suggested by an anonymous referee of the paper. This is part of future research to quantify the links between the two approaches more explicitly.

- Sustainability decreases with the guaranteed resource stock: The size of the viability kernel diminishes when $S_\flat$ increases, i.e.

$$S_\flat^1 > S_\flat^2 \Rightarrow \mathrm{Viab}(f, c_\flat, S_\flat^1) \subset \mathrm{Viab}(f, c_\flat, S_\flat^2). \tag{13}$$

In other words, all the initial states that belong to the viability kernel for a minimal consumption $c_\flat$ (respectively a guaranteed resource level $S_\flat$) are also in the viability kernel for all smaller levels of guaranteed consumption (respectively level of protected resource). Therefore, stronger sustainability requirements induce stronger viability conditions.

3.4.1.5. Equity. Let us now discuss the equity question from the maximin criterion point of view as in Solow (1974). In the set of constraints, we treat every generation equally with respect to the minimal level of consumption but we do not cope with actual consumption. We now seek the maximal level of consumption sustainable forever given an initial state. For this purpose, we compute the maximal guaranteed consumption $c_\flat^+$ for which a given initial state (K_0, S_0) belongs to the viability kernel:

$$c_\flat^+ = \max\,(c_\flat | (K_0, S_0) \in \mathrm{Viab}(f, c_\flat, S_\flat)).$$

From Proposition 3 and property (12), this "maximum sustainable" level of consumption solves the equation:

$$V(K_0, c_\flat^+, S_\flat) = S_0,$$

where $V(K_0, c_\flat, S_\flat)$ is given by Eq. (9). We obtain

$$c_\flat^+ = (1-\beta)((S_0 - S_\flat)(\alpha - \beta))^{\beta/(1-\beta)} K_0^{(\alpha-\beta)/(1-\beta)}, \tag{14}$$

which is the result given by (Solow, 1974, p. 39) for $S_\flat = 0$. Therefore, our study extends this statement to the resource conservation case. Furthermore, Solow notes (Solow, 1974, p. 35) that maximizing the constant level of consumption "is an unusual sort of maximum problem and [he] do[es] not see any obvious direct approach." Let us stress the fact that the use of the viability kernel is a straightforward way of dealing with this kind of maximin concern: maximizing the guaranteed consumption so that the current state is on the boundary of the viability kernel reduces the viable choices to the unique decisions defined by (11); the path is then the standard Solow path, which is intertemporally efficient.

3.4.1.6. A step towards "strong" sustainability?. It is often argued in the debate on sustainability that part of the resource has to be protected. The question is generally "How much of the resource has to be conserved ?" Our concern was "How much of the resource could be conserved ?" We have seen in (13) that the viability kernel is smaller when the guaranteed level of resource $S_\flat$ rises. We can seek the maximal quantity of resource that can be preserved keeping the initial state (K_0, S_0) in the viability kernel:

$$S_\flat^+ = \max\,(S_\flat | (K_0, S_0) \in \mathrm{Viab}(f, c_\flat, S_\flat)).$$

From Proposition 3 and property (13), this quantity is given by

$$S_\flat^+ = S_0 - V(K_0, c_\flat, 0),$$

which yields:

$$S_\flat^+ = S_0 - \frac{1}{\alpha - \beta}\left(\frac{c_\flat}{1-\beta}\right)^{(1-\beta)/\beta} K_0^{(\beta-\alpha)/\beta}.$$

Thus, given initial conditions, any quantity $S_\flat$ such that $0 \leq S_\flat \leq S_\flat^+$ can be maintained. Moreover, if $c_\flat$ represents the "needs" of each generation, the "extra" consumption may represent what it wants. There exists arbitrage between consumption and conservation for the quantity $S_\flat^+ - S_\flat$. The resource can be totally consumed or conserved according to one or another criterion of intergenerational equity. It depends on the interpretation of the "sustainable development", weak or strong, as described in Dobson (1996). Note that maximizing the stock conservation so that the current state touches the boundary of the viability kernel implies that the viable decisions are unique and reduced to efficient ones. Thus, we can reduce the viable choices to a unique inter-temporally efficient path close to the standard Solow path, but with a resource preservation, by increasing the consumption or resource preservation constraint. Some efficient paths combining various constraint levels are described by the simulations in Section 4.

3.4.2. What happens with capital depreciation and technological change?

Now we introduce a depreciation term in the dynamics of capital, i.e.

$$\dot{K} = K^\alpha r^\beta - \lambda K - c.$$

3.4.2.1. No sustainability of an economy with capital depreciation. It turns out that this depreciation term λK condemns the sustainability of the economy as proven in another context by Solow (1974). This result implicitly means that taking into account a population or labor growing at a constant rate jeopardizes the viability of the economy in per capita reasoning.

Proposition 4 (*Solow, 1974*).
If $\alpha < 1$ and $\lambda > 0$, the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ *is empty.*

3.4.2.2. Technological change as a solution. Thus, in the presence of a capital depreciation term, there is no sustainable path for a Cobb–Douglas production function. We now wonder whether a technological progress term integrating the production function makes it possible to restore viable paths. To achieve this, we now consider the model:

$$\dot{K}(t) = A(K)K^\alpha r^\beta - c(t) - \lambda K(t)\dot{S}(t) = -r(t),$$

where $A(K)$ stands for technological progress depending on accumulated capital K. It turns out that a sufficient condition for sustainability to occur refers to technological progress greater than the form $K^{1-\alpha}$. We focus on an endogenous form $A(K) = K^{1-\alpha+\varepsilon}$. The proof of the proposition is given in Appendix A.

Proposition 5. *If there is a capital depreciation term at a positive constant rate λ and if the production function has the form $f(A, K, r) = A(K)K^\alpha r^\beta$ with $\beta < \alpha$, $\beta < 1$ and $A(K) = K^{1-\alpha+\varepsilon}$ with $\varepsilon > 0$, then the viability kernel is not empty.*

We extend the result of Proposition 3 and express the viability kernel as the epigraph[14] of the function $V(K)$ defined by

$$V(K) = V(c_\flat) - \int_{c_\flat}^{K} \frac{1}{A(x)\beta}\left(\frac{c_\flat + \lambda x}{A(x)(1-\beta)}\right)^{(1-\beta)/\beta} x^{-\alpha/\beta}\mathrm{d}x$$

with

$$V(c_\flat) = S_\flat + \int_{c_\flat}^{\infty} \frac{1}{A(x)\beta}\left(\frac{c_\flat + \lambda x}{A(x)(1-\beta)}\right)^{(1-\beta)/\beta} x^{-\alpha/\beta}\mathrm{d}x.$$

Similarly, viable controls on the boundary of the viability kernel are given by

$$r^*(K) = \left(\frac{c_\flat + \lambda K}{A(K)(1-\beta)}\right)^{1/\beta} K^{-\alpha/\beta}, \qquad c^*(K) = c_\flat.$$

4. Numerical simulations

We here present some simulations and illustrative results for the Cobb–Douglas case.

4.1. Shape of the viability kernel

Fig. 1 represents the sensitivity of the viability kernel with respect to α and β. We compute numerical illustrations for the function $S = V(K)$ defining the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ as the epigraph of Vas enounced analytically in Proposition 3. We use different shares α of capital to represent such a sensitivity. Other parameters are set to $c_\flat = 1$, $S_\flat = 0$ while $\beta = 1-\alpha$. Upper curves correspond to smaller α (greater β). We show that the greater α is, the lower the value function V is and the greater the viability kernel is.

4.2. Viable paths

Fig. 2 represents some temporal paths associated with several sustainable policies. It specifically illustrates the flexibility in the decision choices within the viability kernel.

We assume that $\alpha = 2/3 > \beta = 1/3$. Thus, from Proposition 3, the kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ is not empty. We start from the interior of the viability kernel which means that the initial resource $S(0)$ is strictly greater than $V(K(0))$. We define the following extraction rule:

$$\tilde{r}(c(t), K(t)) = \left(\frac{c(t)}{1-\beta}\right)^{1/\beta} K(t)^{-\alpha/\beta}. \tag{15}$$

Let us recall that such an extraction associated with a constant consumption path leads to the optimal use of the resource as defined by the Hartwick–Solow rule (intertemporal efficient paths).

---

[14] The epigraph of the function $V : R \rightarrow R$ is defined by

$$\mathrm{Epi}(V) = \{(K, S), S \geq V(K)\}$$

.

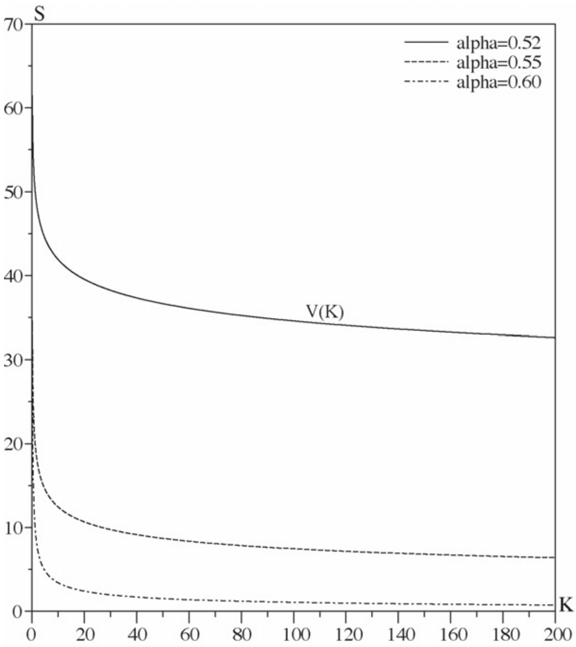

Fig. 1. Sensitivity of the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ and the function $V(K)$ with respect to the capital share α. The viability kernel is the epigraph (the set above the function) of the function $V(\cdot)$ defined in Proposition 3. The viability kernel growths with α.

We also define the maximal sustainable consumption after time t by

$$c^{+}(t) = (1-\beta)((S(t) - S_\flat)(\alpha - \beta))^{\beta/(1-\beta)} K(t)^{(\alpha-\beta)/(1-\beta)}. \tag{16}$$

This equation represents the maximal consumption that could be sustained for an endless time, starting from stocks $K(t)$ and $S(t)$ at time t.

We focus on consumption paths $c(t)$. In Fig. 2(c), we see that there is diversity in the sustainable inter-temporal consumptions. We distinguish the following viable evolutions:

- A first sustainable trajectory is obtained with the consumption fixed at $c(t) = c_\flat$ and the viable extraction set to $r(t) = \tilde{r}(c_\flat, K(t))$. It appears that a part $S_\flat^{+} > S_\flat$ of the resource stock is preserved, providing a strong sustainability perspective. It corresponds to the lower constant consumption path in Fig. 2(c) and to the upper curve of resource stock $S(t)$ in Fig. 2(a).
- Another viable path is plotted from the consumption $c(t) = c^{+}(t)$ and $r(t) = \tilde{r}(c^{+}(t), K(t))$. It turns out that consumption remains constant $c(t) = c_\flat^{+} > c_\flat$ as displayed by the upper constant path in Fig. 2(c), and only the part $S_\flat$ of the resource stock is preserved.
- Another sustainable consumption is defined as a convex combination of the previous constant consumptions $c(t) = c(0) = \lambda c_\flat^{+} + (1-\lambda)c_\flat$ with $\lambda \in [0, 1]$ while the extraction path is defined as in Eq. (15). The consumption is constant ($c_\flat \leq c(t) \leq c_\flat^{+}$) and part of the resource S_∞ is preserved ($S_\flat^{+} \geq S_\infty \geq S_\flat$).
- A viable trajectory with a decreasing consumption: the consumption is $c(t) = \lambda c^{+}(t) + (1-\lambda)c_\flat$ but the extraction is lower than $\tilde{r}(c(t), K(t))$ defined in (15). We take $r(t) = \tilde{r}(c_\flat, K(t))$. In

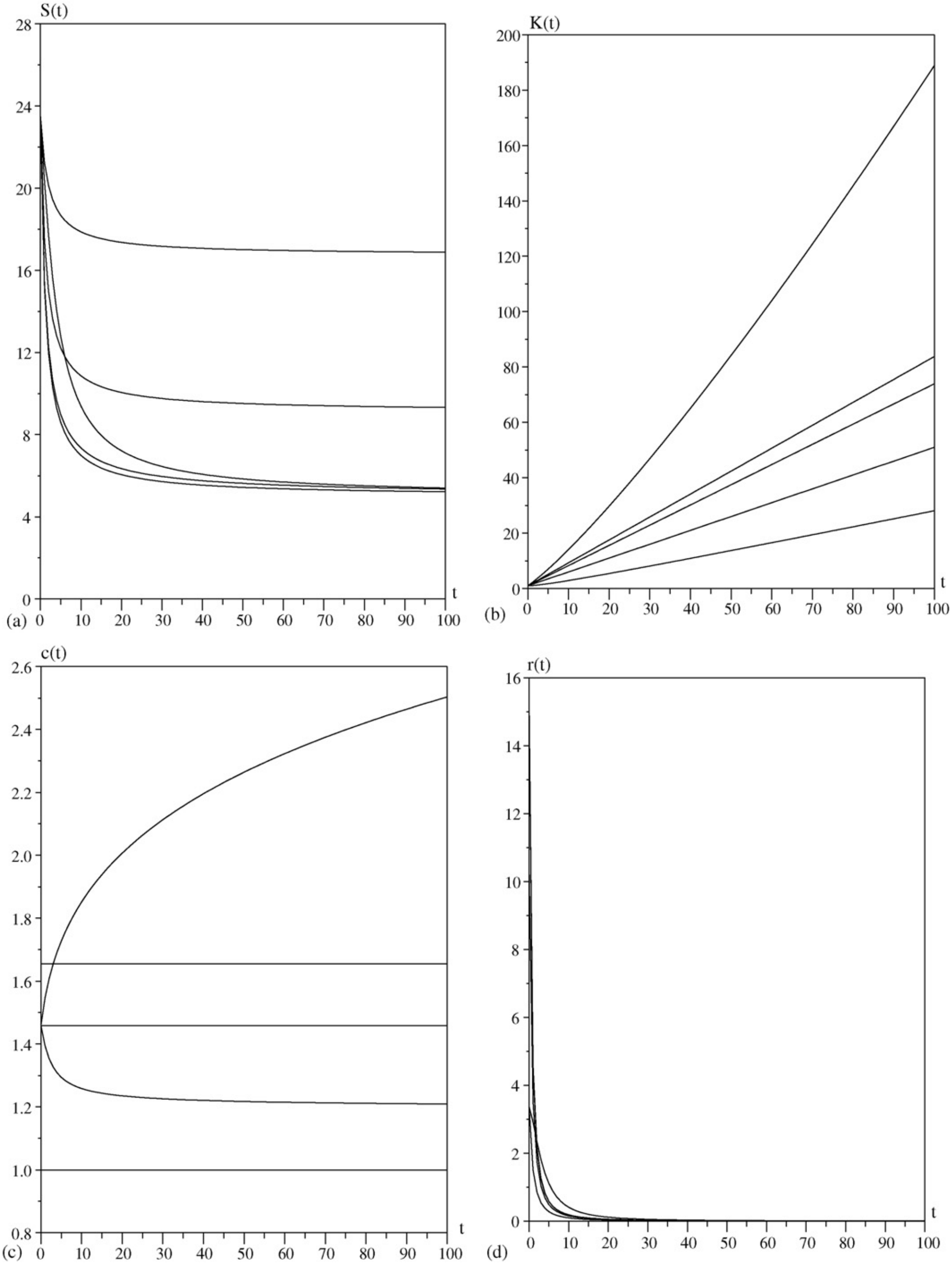

Fig. 2. Different viable evolutions (states and controls) over time $[0, 100]$. Other parameters are $c_\flat = 1$, $S_\flat = 4$ while $\alpha = 2/3$ and $\beta = 1 - \alpha$. Initial viable state conditions are $K_0 = 1$ and $S_0 = 15.5$. (a) Resource stock $S(t)$; (b) capital stock $K(t)$ (c) consumption $c(t)$; (d) extraction $r(t)$.

this configuration, the production and eventually the investment $\dot{K}(t)$ are not high enough to sustain a constant consumption path, and the consumption decreases towards the guaranteed threshold $c_\flat$. In this case, the preserved resource stock is also lower than in the previous constant consumption case even if the extraction is lower at the beginning. As insufficient capital is accumulated to offset the decreasing use of the resource, more resource has to be extracted after some time.

- A viable trajectory with an increasing consumption: here the consumption again is set to $c(t) = \lambda c^+(t) + (1 - \lambda)c_\flat$ but the extraction is greater than $\tilde{r}(c(t), K(t))$. We take for instance $r(t) = \tilde{r}(c^+(t), K(t))$. More capital is accumulated than in the Maximin path, so $c^+(t)$ increases and $c(t)$ rises as it is defined as a part of $c^+(t)$. Doing so, less resource is preserved than in the constant consumption case starting from $c(0)$ since only a part $S_\flat$ of the stock is conserved.

  This last case illustrates an interesting situation. Suppose that the reference path is given by the maximin criterion: the consumption is constant and equals $c^+(0)$. If the first generation consumes less than this level and accumulates the difference, then all future generations can have a constant consumption $c^+(1) > c^+(0)$. This situation corresponds to the sacrifice of the present generation. Moreover, if every generation consumes only a fraction of the maximum sustainable $c^+(t)$, a growing sustainable consumption path is possible despite the exhaustibility of the resource.

Note that for a constant consumption path $c(t) = c(0)$, if the extraction is $\tilde{r}(c(t))$, the capital accumulation $\dot{K}(t)$ is also constant. It is increasing if the extraction is greater than the reference one defined by Eq. (15), and otherwise decreasing (see Fig. 2(b)). We also see in Fig. 2(d) that the extraction $r(t)$ quickly declines towards zero in every case.

5. Conclusion

In this study, we address the problem of the inter-temporal allocation of an exhaustible resource with concerns to intergenerational equity and sustainability. Most of the formal research and analyses on this subject have adopted the optimal control approach. We here use another framework based on the viable control and weak invariance theory. Instead of maximizing an objective function, the main purpose of this approach is to focus on the role of constraints and to characterize the admissible paths and decisions.

We deal with the question of sustainability when an exhaustible resource is involved in the production. We define a set of constraints including a guaranteed level of consumption for each generation and a guaranteed stock of resource to be preserved, and we study various technological structures. Hence, the discussions on the sustainable level of consumption, the natural capital to preserve, and the substitutability between "man-made" and natural capital are addressed. Thus, our approach encompasses both the weak and strong sustainability points of view. The analysis highlights the role of the technology in the sustainability of the economy. We distinguish non-essential and essential resources to reveal the sustainability and especially focus on the Cobb–Douglas case.

The use of the viability kernel as an indicator of sustainability allows us to characterize the set of policies and states that do not drive the system into crisis situations outside the domain of constraints. We match this approach with the definition of sustainable development as a development "that meets the needs of the present without compromising the ability of future generations to meet their own needs". The paths lying within the viability kernel are sustainable in the sense defined above. Conversely, if the state variables leave the viability kernel, the constraints will be violated in a finite time and future generations will not be able to meet their needs, meaning that the viability kernel reveals the true or *ex post* sustainable constraints of the problem. Note that when the viability kernel is empty, it is impossible to insure a sustainable path to future generations. In this case, it is not necessary to study any optimal control problem under the involved constraints in their present forms.

First, it turns out that an economy with a strongly essential resource is not sustainable since the viability kernel is empty in this case. On the other hand, economies with non-essential and weakly essential resources (Cobb–Douglas) display better sustainability features. We also show how an economy with a depreciation of capital is not sustainable unless a significant technological change occurs.

The Cobb–Douglas case makes it possible for a detailed picture from the viewpoints of the substitutability, equity and flexibility, respectively. Thus, we extend to a strong sustainability context the major results of Solow (1974) and Dasgupta and Heal (1979) who ignore the guaranteed resource constraint within the Maximin framework. In particular, the economy is sustainable if and only if the elasticity of production with respect to the resource extraction is smaller than that with respect to capital. An important interest and originality of the approach refers to the flexibility of viable decisions. Instead of revealing the (generally unique) optimal allocation, the approach points out all the sustainable choices as soon as any one exists. This result naturally induces a multiplicity of relevant decisions. Here, the flexibility in the decisions is associated with all possible consumptions and extractions that make it possible to remain within the viability kernel. At this stage, a significant contribution of the paper is to prove how the sustainable consumptions are not reduced to constant ones. Similarly, sustainable extraction decisions can escape Hartwick's rule. Sensitivity analysis sheds light on the robustness of the results, again emphasizing the positive influence of the capital share in the production process.

Moreover, using an efficiency perspective, we determine the maximal level of resource that can be preserved and the maximal sustainable consumption. These two levels are linked and it is impossible to increase one without decreasing the other. In this sense, a choice must be made in the sustainability perspectives: consumption or resource preservation.

To illustrate the formal assertions, numerical computations are performed for specific functional forms and parameter values.

Further research needs to be carried out. In particular, one possible extension is to include uncertainties in the dynamics, needs and preferences as in Butterfield (2003) which may raise questions related to the value of information, the resolution of uncertainties and the precautionary principle. Combining overlapping generations and viable control frameworks, by extending the Olson and Knapp (1997) model to include capital, for example, is also a challenging task. More generally, we think that the weak invariance and viable control approach provides an interesting analytical framework to cope with sustainable development issues.

Appendix A

A.1. Weak invariance and viable control approach at a glimpse

We refer to Aubin (1991) or Clarke et al. (1995) for more details.

A.1.1. Viable control problem

Let us consider the following dynamic system under control and state constraints:

$$\dot{x}(t) = g(x(t), \quad u(t)), u(t) \in U(x(t)), \quad x(t) \in M, \tag{17}$$

where $x \in \mathbf{R}^n$ is the state of the system and $u \in \mathbf{R}^p$ represents the control. The set $U(x)$ stands for the domain of admissible controls. The set M corresponds to the domain of admissible states.

A.1.2. The viability kernel

It is the set of initial states x_0 from which a feasible path $(x(\cdot), u(\cdot))$ respecting the constraints (staying in M) at any positive time starts:

$$\begin{aligned}\mathrm{Viab}(g, U, M) = \{x_0 \in M \,|\, \exists\, u(\cdot)\ \exists\, x(\cdot) \text{ satisfying, } (17)\ \forall\, t \geq 0 \\ \text{with the initial condition } x(0) = x_0\}.\end{aligned}$$

A.1.3. Viable or weakly invariant set

A set M is said to be viable for the dynamics (g, U) if the viability kernel $\mathrm{Viab}(g, U, M)$ coincides with the initial constraint set M. This means that from any state in M feasible control starts that yields a trajectory remaining within M. Such a favorable situation occurs at state $x \in M$ whenever a control u leads the velocities $\dot{x} = g(x, u)$ to be tangent or inward to domain M. For closed sets M, under adequate assumptions on the dynamics[15] (g, U), this can be written through an Hamiltonian formulation. Let us consider the Hamiltonian:

$$H(x, p, u) = \sum_{i=1}^{n} p_i \cdot g_i(x, u).$$

The following assertions are equivalent:

$$\begin{array}{ll} \text{(i)} & M \text{ is viable or weakly invariant for } (g, U), \\ \text{(ii)} & \mathrm{Viab}\,(g, U, M) = M, \\ \text{(iii)} & \inf\limits_{u \in U(x)} H(x, p, u) \leq 0, \quad \forall\, x \in M,\ \forall\, p \in N_M(x), \end{array} \tag{18}$$

where $N_M(x)$ stands for the normal cone to M at point x. Consider the simple case where the constraint set M is characterized by m inequalities:

$$M = \{x \in \mathbf{R}^n \text{ such that } h_j(x) \leq 0,\ j = 1, \ldots, m\},$$

where the functions h_j are regular enough. If some constraint qualification[16] is fulfilled at point x, then $N_M(x)$ the normal cone to M at point $x \in M$ reads

$$N_M(x) = \left\{ p,\ \sum_{i=1}^{n} p_i \frac{\partial h_j}{\partial x_i}(x) \geq 0 \text{ and } p_j h_j(x) = 0,\ j = 1, \ldots, m \right\}. \tag{19}$$

A.1.4. Viable controls

Under the same adequate assumptions on the dynamics (g, U) and the state constraint M, the viability kernel of M for dynamics (g, U) is the largest viable domain contained in M. In particular, this means that the viability kernel is a viable domain. The viable strategy consists in staying within the viability kernel. Thus, for any $x \in \mathrm{Viab}(g, U, M)$, the relevant controls satisfy

$$u \in U(x) \text{ such that } H(x, p, u) \leq 0,\ \forall\, p \in N_{\mathrm{Viab}(g,U,M)}(x).$$

---

[15] For instance, if $U(x)$ is convex, closed and bounded together with g and U regular enough.

[16] There are some v^* such that $\sum_{i=1}^{n} (\partial h_j / \partial x_i)(x) v_i^* < 0$ for every j such that $h_j(x) = 0$.

For any x within the interior of $\mathrm{Viab}(g, U, M)$, this condition is satisfied for every $u \in U(x)$ since the normal cone is reduced to the origin.

A.1.5. *Empty kernel* $\mathrm{Viab}(g, U, M) = \varnothing$

An empty kernel means that there are no viable paths. Several strategies are available in such non-viable situations. A first possibility is to relax some constraints. A second possibility is to "relax" the dynamics. For instance, one can transform some fixed parameter of the dynamics into a new control.

A.1.6. *Other non-viability cases*

One important case occurs if the initial state does not belong to the viability kernel $x_0 \notin \mathrm{Viab}$. An option is to study the minimal time of crisis as in Bene et al. (2001) namely the minimal time spent outside of the viability kernel which corresponds to a specific optimal control problem. Another option requires the constraints to be satisfied only after some finite time t^* and thus to consider the transitions paths towards viable states. Then we face time-dependent constraints $M(t)$.

A.1.7. *Complexity and numerical schemes*

In complex problems with high dimensions of control or state, it is sometimes very difficult or impossible to specify the analytical form of the kernel. However, algorithms can be found for instance in Saint-Pierre (1994) to approximate numerically the viability kernel of the problem, and the associated viable policies. We must confess that these numerical schemes are time and space consuming.

A.2. *The proofs*

Proof of Proposition 1. Let us consider a technology f such that $0 < r_\flat(f) = \inf_{K \geq 0} r_\flat(f, K)$, with $r_\flat(f, K)$ defined in (7). Assume for the moment that the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ is not empty and consider any initial state $(S_0, K_0) \in \mathrm{Viab}(f, c_\flat, S_\flat)$. Then, from the very definition of the viability kernel in Section 2.3, there are controls $(r(t), c(t))$ and states $(S(t), K(t))$ starting from (S_0, K_0) such that the constraints and dynamics are satisfied throughout time. In particular, we can write

$$r(t) \geq r_\flat(f, K(t)) \geq r_\flat(f) > 0.$$

Consequently, we have

$$S(t) = S_0 - \int_0^t r(s)\mathrm{d}s \leq S_0 - t r_\flat(f).$$

Therefore, there is a time[17] T such that $S(T) < 0 \leq S_\flat$. This result does not respect the scarcity resource constraint (3) and we derive a contradiction. In other words, the viability kernel is empty or $\mathrm{Viab}(f, c_\flat, S_\flat) = \varnothing$. □

Proof of Proposition 2. Let us consider a capital K^+ such that $r_\flat(f) = 0 = r_\flat(f, K^+)$ or equivalently $f(K^+, 0) = c_\flat$. We prove that all the states lying in the set $A = \{K \geq K^+, S \geq S_\flat\}$ are

---

[17] with $(S_0/r_\flat(f)) < T < +\infty$.

viable.[18] Since $f(K,0) \geq f(K^+,0) = c_\flat$, we can choose feasible controls $r=0$ and $c \in [c_\flat, f(K,0)]$. For such controls, the dynamics is governed by $\dot{S}(t)=0$ and $\dot{K}(t) \geq 0$. This situation implies that set A is weakly invariant or viable. Thus, following Aubin (1991), we deduce that A is a subset of the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$. □

Proof of Proposition 3 for $\alpha > \beta$. Now assume that $1 > \beta$ and $\alpha > \beta$ and consider V defined by Eq. (9). We need to prove that the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ coincides with $\mathrm{Epi}(V)$, the epigraph of the function V:

$$\mathrm{Epi}(V) = \{(K,S),\ V(K) \leq S\}.$$

To achieve this, let us introduce the following Hamiltonian:

$$\mathcal{H}(p,K,r,c) = p(f(K,r)-c)+r.$$

We proceed in three steps:

(a) We first prove that the function V is the solution of the Hamilton–Jacobi–Bellman equation:

$$\min_{(r,c)\,\in\,\mathcal{C}(K,S)} \mathcal{H}(V'(K),K,r,c) = 0 \tag{20}$$

with constraints $\mathcal{C}$ previously defined by Eq. (10).

(b) We deduce from (20) that the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ contains $\mathrm{Epi}(V)$.

(c) From (20), we also deduce that the complementary of $\mathrm{Epi}(V)$ contains the complementary of $\mathrm{Viab}(f, c_\flat, S_\flat)$, making it possible to conclude with the equality:

(a) Since $V'(K)$ is negative, we obtain

$$\inf_{(r,c)\,\in\,\mathcal{C}(K,S)} \mathcal{H}(V'(K),K,r,c) = \inf_{r \geq r_\flat(f,K)} \{V'(K)(f(K,r)-c_\flat)+r\} \tag{21}$$

$$= \inf_{r \geq r_\flat(f,K)} \mathcal{H}(V'(K),K,r,c_\flat). \tag{22}$$

Consider r^* as defined in (11) by

$$r^*(K) = \left(\frac{c_\flat}{1-\beta}\right)^{1/\beta} K^{-\alpha/\beta}. \tag{23}$$

We prove that r^* is the solution to the previous optimality problem (21). A straightforward computation provides the first and second order optimality conditions:

$$0 = \frac{\partial \mathcal{H}}{\partial r}(V'(K),K,r^*(K),c_\flat), \qquad 0 \leq \frac{\partial^2 \mathcal{H}}{\partial^2 r}(V'(K),K,r,c_\flat).$$

Moreover, it is clear that $r^*(K)$ is an admissible control since $\beta < 1$ and

$$f(K,r^*(K)) = K^\alpha \frac{c_\flat}{1-\beta} K^{-\alpha} = \frac{c_\flat}{1-\beta} \geq c_\flat.$$

---

[18] The viability kernel contains every state with resource stock large enough to allow for sufficient capital accumulation to reach K^+ without violating the constraint on the resource stock. The expression of the whole viability kernel is not given here for sake of simplicity. It implies solving a Hamilton–Jacobi–Bellman equation close to the one given below in Eq. (24) together with the boundary condition $V(K^+) = S_\flat$. One can refer to the method and proof used below in the Cobb–Douglas case.

Thus, $r^*(K)$ is the relevant control solution of (20) and consequently for any $K \in \mathbf{R}_*^+$:

$$\min_{(r,c) \in \mathcal{C}(K,S)} \mathcal{H}(V'(K), K, r, c) = \mathcal{H}(V'(K), K, r^*(K), c) = 0. \tag{24}$$

(b) We need to prove that the previous HJB equation implies that the epigraph $\mathrm{Epi}(V)$ of the function V is a viable domain for the dynamics (1). A desired viability condition (18iii) reads

$$\inf_{(r,c) \in \mathcal{C}(K,S)} (p_1(f(K,r) - c) - p_2 r) \leq 0, \tag{25}$$

for any $p = (p_1, p_2)$ in the normal cone of $\mathrm{Epi}(V)$. Using characterization (19), we deduce that such a normal p satisfies

$$p = \begin{cases} \mu(V'(K), -1) & \text{if } S = V(K),\ (\mu \geq 0) \\ (0,0) & \text{if } S > V(K). \end{cases}$$

Thus, if $S > V(K)$, condition (25) is clearly fulfilled. On the boundary $S = V(K)$, the Hamiltonian condition (25) yields:

$$\inf_{(r,c) \in \mathcal{C}(K,S)} \mu(V'(K)(f(K,r) - c) + r) \leq 0.$$

Since μ is positive, this is equivalent to the inequality:

$$\min_{(r,c) \in \mathcal{C}(K,S)} \{V'(K)(f(K,r) - c) + r\} \leq 0,$$

which holds true from (20). In other words, Epi(V) is a viable domain for the dynamics under concern. Thus, for any $(K_0, S_0) \in \mathrm{Epi}(V)$, there are admissible controls $(r(t), c(t))$ and states $(S(t), K(t))$ starting from (S_0, K_0) such that

$$S(t) \geq V(K(t)).$$

Furthermore, since $\alpha - \beta > 0$, for any positive K, we deduce

$$S_\flat \leq V(K(t)) \leq S(t).$$

Thus, all constraints are satisfied throughout time and we claim that $(K_0, S_0) \in \mathrm{Viab}(f, c_\flat, S_\flat)$. We thus deduce that

$$\mathrm{Epi}(V) \subset \mathrm{Viab}(f, c_\flat, S_\flat).$$

(c) It remains to be proven that the viability kernel and the epigraph of V coincide. Now assume for a moment that this situation does not hold true. Then we find $(K_0, S_0) \in \mathrm{Viab}(f, c_\flat, S_\flat)$ and $(K_0, S_0) \notin \mathrm{Epi}(V)$. First, the condition $(K_0, S_0) \in \mathrm{Viab}$ means that there are viable controls $(r(\cdot), c(\cdot))$ and states $(S(\cdot), K(\cdot))$ starting from (K_0, S_0). Now, consider the difference $e(t) = V(K(t)) - S(t)$. By assumption, $0 < e(0) = e_0 = V(K_0) - S_0$. On the other hand, using Hamiltonian formulation (20), we write that

$$\begin{aligned} \dot{e}(t) &= V'(K(t))(f(K(t), r(t)) - c(t)) + r(t) \\ &\geq \inf_{(r,c) \in \mathcal{C}(K(t),S(t))} (V'(K(t)), K(t), r, c) \geq 0. \end{aligned}$$

Thus, $e(\cdot)$ is increasing. Using the assumption $0 \leq e_0 = V(K_0) - S_0$, we infer that $e(t) \geq e(0) = e_0$ or equivalently $V(K(t)) - S(t) \geq e_0$. Since $(S(\cdot), K(\cdot))$ is a viable trajectory, we have $S(t) \geq 0$ which implies that $V(K(t)) \geq e_0 > 0$, thus making it possible to infer that $K(t)$ is bounded, namely

$$K(t) \leq K_\sharp.$$

Since $r(t)$ is admissible, this implies that $r(t)$ is bounded from below, namely

$$r(t) \geq r_\flat(f, K(t)) \geq r_\flat(f, K_\sharp) > 0,$$

where $r_\flat(f,K) = \left(\frac{c_\flat}{K^\alpha}\right)^{1/\beta}$. Consequently, the stock of resource satisfies

$$S(t) = S_0 - \int_0^t r(s)\mathrm{d}s \leq S_0 - t r_\flat(f, K_\sharp).$$

Therefore, there is a time T where $S(T) < 0 \leq S_\flat$ and we derive a contradiction. We conclude that $\mathrm{Viab}(f, c_\flat, S_\flat) = \mathrm{Epi}(V)$. □

Proof of Proposition 3 for $\alpha \leq \beta$. If the viability kernel $\mathrm{Viab}(f, c_\flat, S_\flat)$ were not empty, there would be at least one extraction–consumption $(r(t), c(t))$ path for which the set of constraints is respected. But, the same extraction path with a constant consumption limited to $c_\flat$ is also relevant. Yet, Solow proved (Solow, 1974, Appendix B) that no constant positive consumption path exists for such a system if $\alpha \leq \beta$. Then, the viability kernel is empty. □

Proof of Proposition 5. We present the proof for the smallest endogenous technological change, $A(K) = K^{1-\alpha+\varepsilon}$ where $\varepsilon > 0$. We show that, in this case, the viability kernel is the epigraph of a function $V(K)$ with

$$V(K) = V(c_\flat) + \int_{c_\flat}^{K} -\frac{1}{\beta}\left(\frac{\lambda + c_\flat/x}{1-\beta}\right)^{(1-\beta)/\beta} x^{-(\varepsilon+\beta)/\beta}\mathrm{d}x.$$

Furthermore, we require $\lim_{K\to\infty} V(K) = S_\flat$, which yields:

$$V(c_\flat) = S_\flat + \int_{c_\flat}^{\infty} \frac{1}{\beta}\left(\frac{\lambda + c_\flat/x}{1-\beta}\right)^{(1-\beta)/\beta} x^{-(\varepsilon+\beta)/\beta}\mathrm{d}x.$$

Let us prove that $V(c_\flat)$ is finite. First, we write

$$\begin{aligned}(V(c_\flat) - S_\flat)\beta(1-\beta)^{(1-\beta)/\beta} &= \int_{c_\flat}^{\infty} \left(\lambda + \frac{c_\flat}{x}\right)^{\frac{1-\beta}{\beta}} x^{-(\varepsilon+\beta)/\beta}\mathrm{d}x \\ &\leq \int_{c_\flat}^{\infty} (\lambda+1)^{(1-\beta)/\beta} x^{-(\varepsilon+\beta)/\beta}\mathrm{d}x.\end{aligned}$$

We deduce that

$$(V(c_\flat) - S_\flat)\beta\left(\frac{1-\beta}{\lambda+1}\right)^{(1-\beta)/\beta} \leq \int_{c_\flat}^{\infty} x^{-(\varepsilon+\beta)/\beta}\mathrm{d}x \leq \left[-\frac{\beta}{\varepsilon}x^{-\varepsilon/\beta}\right]_{c_\flat}^{\infty} = \frac{\beta}{\varepsilon}c_\flat^{-\varepsilon/\beta}.$$

$V(c_\flat)$ is thus bounded. It remains to be proven that the viability kernel is the epigraph of the function $V(K)$; we use the same steps as for Proposition 3:

- $V(K)$ is defined on $\mathbf{R}_*^+$.
- $V(K) \geq S_\flat$.
- The main step is to demonstrate the Hamiltonian conditions:

$$\inf_{(r,c)\,\in\,\mathcal{C}(K,S)} \{V'(K)(A(K)K^\alpha r^\beta - \lambda K - c) + r\} = 0.$$

With

$$V'(K) = -\frac{1}{\beta}\left(\frac{\lambda + c_\flat/K}{1-\beta}\right)^{(1-\beta)/\beta} K^{-(\varepsilon+\beta)/\beta},$$

taking specific feedback controls $c^* = c_\flat$ and $r^* = ((\lambda + c_\flat/K)/(1-\beta))^{1/\beta} K^{-\varepsilon/\beta}$, we obtain the desired result. □

References

Aubin, J.P., 1991. Viability Theory. Springer Verlag, Birkhäuser.

Bene, C., Doyen, L., Gabay, D., 2001. A viability analysis for a bio-economic model. Ecological Economics 36, 385–396.

Bonneuil, N., 1994. Capital accumulation, inertia of consumption and norms of reproduction. Journal of Population Economics 7, 49–62.

Bruckner, T., Petschell-Held, G., Toth, F.L., Fussel, H.-M., Helm, C., Leimbach, M., Schellnhuber, H.J., 1999. Climate change decision-support and the tolerable windows approach. In: Schellnhuber, H.-J., Toth, F.L. (Eds.), Environmental Modeling and Assessment, Earth System Analysis and Management, vol. 4. pp. 217–234.

Burton, P.S., 1993. Intertemporal preferences and intergenerational equity considerations in optimal resource harvesting. Journal of Environmental Economics and Management 24, 119–132.

Butterfield, D.W., 2003. Resource depletion under uncertainty: implications for mine depreciation, Hartwick's rule and national accounting. Resource and Energy Economics 25, 219–238.

Chichilnisky, G., 1996. An axiomatic approach to sustainable development. Social Choice and Welfare 13, 219–248.

Chichilnisky, G., Heal, G., Beltratti, A., 1995. The green golden rule. Economics Letters 49, 175–179.

Clarke, F.H., Ledyaev, Y.S., Stern, R.J., Wolenskip, R., 1995. Qualitative properties of trajectories of control systems: a survey. Journal of Dynamical Control Systems 1, 1–48.

Daly, H., 1990. Toward some operational principles of sustainable development. Ecological Economics 2, 1–6.

Dasgupta, P., Heal, G., 1974. The optimal depletion of exhaustible resources, review of economic studies 41. In: Proceedings of the Symposium on the Economics of Exhaustible Resources. pp. 1–28.

Dasgupta, P., Heal, G., 1979. Economic Theory and Exhaustible Resources. Cambridge University Press.

Dobson, A., 1996. Environment sustainabilities: an analysis and a typology. Environmental Politics 5, 401–428.

Doyen, L., Bene, C., 2003. Sustainability of fisheries through marine reserves: a robust modeling analysis. Journal of Environmental Management 69, 1–13.

Gerlagh, R., Keyser, M.A., 2003. Efficiency of conservationist measures: an optimistic viewpoint. Journal of Environmental Economics and Management 46, 310–333.

Hartwick, J., 1977. Intergenerational equity and the investing of rents from exhaustible resources. American Economic Review 67, 972–974.

Heal, G., 1998. Valuing the Future, Economic Theory and Sustainability. Columbia University Press, New York.

Hotelling, H., 1931. The economics of exhaustible resources. Journal of Political Economy 39, 137–175.

Howarth, R.B., 1991a. Intertemporal equilibria and exhaustible resources: an overlapping generations approach. Ecological Economics 4, 237–254.

Howarth, R.B., 1991b. Intergenerational competitive equilibria under technological uncertainty and an exhaustible resource constraint. Journal of Environmental Economics and Management 21, 225–243.

Howarth, R.B., Norgaard, R.B., 1995. Intergenerational choices under global environmental change. In: Bromley, D.W. (Ed.), The Handbook of Environmental Economics. Blackwell, pp. 112–138.

Olson, L., Knapp, K., 1997. Exhaustible resource allocation in an overlapping generations economy. Journal of Environmental Economics and Management 32, 277–292.

Rawls, J., 1971. A theory of Justice. Clarendon, Oxford, UK.

Saint-Pierre, P., 1994. Approximation of viability kernel. Applied Mathematics and Optimization 29, 187–209.

Schellnhuber, H.J., Wenzel, V., 1998. Earth System Analysis, Integrating Science for Sustainability. Springer.

Solow, R.M., 1974. Intergenerational equity and exhaustible resources, review of economic studies 41. In: Proceedings of the Symposium on the Economics of Exhaustible Resources. pp. 29–45.

Stiglitz, J., 1974. Growth with exhaustible natural resources: efficient and optimal growth paths, review of economic studies 41. In: Proceedings of the Symposium on the Economics of Exhaustible Resources. pp. 123–137.

Toman, A., Pezzey, J., Krautkraemer, J., 1995. Neoclassical economic growth theory and ‘‘sustainability’’. In: Bromley, D.W. (Ed.), The Handbook of Environmental Economics. Blackwell, pp. 139–165.